Some Characterizations For Some Sporadic Simple Groups

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Abstract

We identify Co_1, $M(24)'$ and the Monster group from their 3-local information. Let G be a finite group and H_1 and H_2 be two subgroups of G such that H_1 is the normalizer of a 3-central element in G and H_2 is the normalizer of a maximal elementary abelian 3-group in G. In this thesis we show that if H_1 has shape $3^{1+12}.2Suz:2$, H_2 has shape $3^8:\Omega_8^-(3)$ and $H_1 \cap H_2$ has shape $3^8.3^6.2U_4(3):2$, then G is isomorphic to the Monster group. If H_1 has shape $3^{1+10}.U_5(2):2$, H_2 has shape $3^7\Omega_7(3)$ and $H_1 \cap H_2$ has shape $3^7.3^5.U_4(2):2$, then $G \cong M(24)'$. If H_1 has shape $3^{1+4}.Sp_4(3):2$, H_2 has shape $3^6:2M_{12}$ and $H_1 \cap H_2$ has shape $3^6.3^2.(GL_2(3)\times 2)$, then $G \cong Co_1$. Also we identify the group $M(24)'$ by the structure of the normalizer of a 3-central element.

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Contents

Chapter 1

Introduction

The classification of the finite simple groups is announced around 1981. Let me give some notations to state the classification theorem. Set $K_1 := \{Z_p; p \text{ is a prime}\}$, $K_2 := \{A_n; n \geq 5\}$ (the alternating groups of degree at least 5) and let K_3 be the set of all simple groups of Lie type and K_4 be the set of all 26 sporadic simple groups. Set $K := \bigcup_{i=1}^{4} K_i$. Now we can state the classification theorem.

Theorem 1.1 *(Classification Theorem) Each finite simple group is isomorphic to one of the groups in K.*

The first proof for theorem 1.1 was so complicated. So, soon after the classification was announced, Gorenstein, Lyons and Solomon started a program whose aim was to give a better proof for theorem above. We refer the reader to ([GLS1]-[GLS6]) for more details of this program.

Let me give the following definition. In what follows p is a prime.

Definition 1.2 *Let G be a finite group and let p be a prime.*

i) $O_p(G)$ is the largest normal p-subgroup of G.

ii) We say that G has characteristic p if $C_G(O_p(G)) \leq O_p(G)$.

iii) A local p-subgroup of G is a subgroup of the form $N_G(T)$ for some p-subgroup $1 \neq T$ of G. If each p-local subgroup of G has characteristic p we say that G has local characteristic p.

iv) G is called K_p-group if any simple section in any p-local of G is isomorphic to one of the groups in K.

Many of the groups in K are of local characteristic p, for some prime p. For example, each group of Lie type defined over a field of characteristic p is of local characteristic p. There are also some examples of the groups in K_4, such as the groups J_4, M_{24} and TH for $p = 2$, Ly for $p = 5$ and ON for $p = 7$. Also each group A_p is of local characteristic p and so there are some examples in the groups in K_2. It seems that the classification of all finite K_p-groups of local characteristic p will give a new proof for theorem 1.1. This is the main idea of a recent program led by Meierfrankenfeld, Stellmacher and Stroth (see [MS] and [MSS]). The main idea of this program is to classify the finite groups of local characteristic p, for p a prime, and this thesis has application in this project. In the next sections of this chapter we close to the classification of the finite groups of local characteristic p to see where this thesis is applied.

1.1 Groups of local characteristic p and H-Structure Theorem

In this section at first, we give some information about the classification of the finite groups of local characteristic p, in general. Then we will state the H-structure theorem where this thesis is applied. Our information in this section come from [MSS] and [MS]. In what follows p is a prime.

Assumption *: G is a K_p-group and each local subgroup of G which contains a Sylow p-subgroup of G is of characteristic p.

In [MSS], they split the project into three major parts:

1) Modules.

2) Local Analysis.

3) Global Analysis.

To see the connection between these steps we invite the reader to see ([MS] and [MSS], specially example 1.2 in [MSS]). In fact, the main results of the steps 1 and 2 for a finite group G which satisfies the assumption *, are the structures of some p-locals of the group G (an amalgam for G), and in the next step (step 3 (global analysis)) they need to identify the group G with these p-local information. This thesis has application in the final step of this program (step 3) when the groups $M(24)'$, Co_1 and F_1 are appeared. In the remainder of this section we state the H-theorem where this thesis is applied. We note that H-structure theorem is proved in [MS].

To state the H-structure theorem precisely, we need more definitions, but here we just give some definitions and we state the H-structure theorem to see an application of this thesis. We recall that H-structure theorem is proved in [MS] and we invite the reader to see [MS] for an exact exposition of H-structure theorem. In what follows, p is a prime, G is a finite K_p-group and $S \in Syl_p(G)$.

Notations: 1) $\ell(S)$ is the set of all subgroups $X \leq G$ of characteristic p and containing S.

2) For $M \in \ell(S)$, by ([MSS], lemma 2.0.1) we denote by Y_M the unique maximal elementary abelian normal p-subgroup of M such that $O_p(M/C_M(Y_M)) = 1$.

3) $C = N_G(\Omega_1(Z(S)))$ and $\widetilde{C} \in \ell(S)$ with $C \leq \widetilde{C}$ and $\widetilde{C}$ is maximal. Set $Q = O_p(\widetilde{C})$.

4) Set $E = O^p(F_p^\star(C_{\widetilde{C}}(Y_{\widetilde{C}})))$, where $F_p^\star(X)$ is the inverse image of $F^\star(X/O_p(X))$.

5) For $M \in \ell(S)$, set $M^0 = \langle Q^M \rangle$ and $M_0 = M^0 S$.

E-uniqueness: If X is some p-local of G with $E \leq X$, then $X \leq \widetilde{C}$.

Assume that $M \in \ell(S)$ and we have E-uniqueness. Then by ([MSS], lemmas 2.4.1 and 2.4.2) we get that $M = M^0(M \cap \widetilde{C})$. Therefore M^0 determins the structure of M. We recall that a subgroup $P \in \ell(S)$ is called *minimal parabolic* if S is contained in a unique maximal subgroup of P and S is not normal in P. Now we can state the H-structure theorem.

Theorem 1.3 *(H-Structure Theorem)*

*Let G be a finite group which satisfies the assumption *. Assume that $O_p(\langle X | X \in \ell(S) \rangle) = 1$ and we have E-uniqueness. Let $M \in \ell(S)$ with M^0 be maximal and Y_M is not contained in Q. Then one of the following holds.*

(1) There is a subgroup H of G with $M_0 \leq H$ and $O_p(H) = 1$, such that for $F^\star(H)$ the parabolics containing S are as in one of the folowing groups.

i) A group of Lie type in characteristic p and of rank at least three.

ii) $p = 2$ and we have He, Co_2, Co_1, $M(24)'$, J_4, Suz, F_2, F_1 or $U_4(3)$.

iii) $p = 3$ *and we have* Co_1*,* $M(24)'$ *or* F_1*.*

(2) $p = 2$ *and* M *is an extension of an elementary abelian group of order 16 by* $L_3(2)$*,* $\widetilde{C}$ *is an extension of an extraspecial group of order 32 by* $S_3 \times S_3$*. Further there are minimal parabolics* P_1 *and* P_2 *with* $P_1/O_2(P_1) \cong P_2/O_2(P_2) \cong S_3$ *and* $O_2(\langle P_1, P_2\rangle = 1$*.*

(3) $p = 3$*,* M *and* $\widetilde{C}$ *are as in* Co_3*. There are two minimal parabolics* P_1 *and* P_2 *with* $P_1/O_3(P_1) \cong L_2(9)$*,* $P_2/O_3(P_2) \cong SL_2(9)$ *and* $O_3(\langle P_1, P_2\rangle = 1$*.*

(4) M_0 *is a minimal parabolic.*

Let me look at the results of theorem 1.3, in details. In case (3) by [KPR] we get that $G \cong Co_3$ and in case (2) it seems that by [As4] the group G is known and $F^\star(G) \cong G_2(3)$. In case (1) of theorem 1.3 we have the structure of a subgroup H of G and naturally the following question arises:

Question 1.4 *Is* H *a proper subgroup of* G*?.*

Our attempt in this thesis is to give a negative answer to question 1.4 in case (1)(iii). In fact in this case (case (1)(iii)), by theorem 1.3 we have the structures of two 3-local subgroups of G and in this thesis we will identify the group G with these two 3-locals of G. Cases (1)(i) and (1)(ii) are still under investigations. But we heard from Professor Stroth in [OR] that for case (1)(i) and when p is odd, it would be interesting to prove the following theorem:

Theorem 1.5 *Let* G *be a finite* K_p*-group containing a subgroup* H *which is a group of Lie type in characteristic* p *and of rank at least three. If* H *is strongly* p*-embedded (i.e* $N_G(P) \leq H$ *for each nontrivial* p*-subgroup* P *of* H*), then* $G = H$

I heard that Professor Parker and Professor Stroth have recently proved the theorem above. In the next sections we will state our main results and we will give an outline of the proofs.

1.2 Main results

As we said in section 1.1, in this thesis we identify three finite known simple groups Co_1, $M(24)'$ and F_1 from their 3-local information. Before we state our main theorems we need some definitions.

Definition 1 *Let G be a finite group and $S \in Syl_3(G)$. We say that G is of Co_1-type, if there are two subgroups H_1 and H_2 in G containing S such that;*

i) $H_1 = N_G(Z(O_3(H_1)))$, $O_3(H_1)$ is an extraspecial group of order 3^5, $H_1/O_3(H_1) \cong Sp_4(3) : 2$ and $C_{H_1}(O_3(H_1)) = Z(O_3(H_1)) = \langle t \rangle$.

ii) $O_3(H_2)$ is an elementary abelian group of order 3^6 and $H_2/O_3(H_2) \cong 2M_{12}$.

iii) $(H_1 \cap H_2)/O_3(H_2)$ is an extension of an elementary abelian group of order 9 by $GL_2(3) \times Z_2$.

Definition 2 *Let G be a finite group, $\tau \in G$ be of order three and $H_1 = N_G(\langle \tau \rangle)$. We say that G is of $M(24)'$-type, if*

i) $O_3(H_1)$ is extraspecial group of order 3^{11} and exponent 3, $H_1/O_3(H_1) \cong U_5(2) : 2$ and $C_G(O_3(H_1)) = Z(O_3(H_1))$.

ii) Let U be an elementary abelian subgroup in $O^2(H_1)$ of order 16 such that $N_{H_1}(U)O_3(H_1)/O_3(H_1)$ is an extension of a special group of order 2^8 with center $UO_3(H_1)/O_3(H_1)$ by $(3 \times A_5).2$. Then $\langle \tau \rangle$ is not weakly closed in $C_{H_1}(A)$ with respect to $C_G(A)$ for some subgroup A of U of order 4 such that all involutions in A are non 2-central involutions in $O^2(H_1)$.

Definition 3 *Let G be a finite group and $S \in Syl_3(G)$. We say that G is of Monster-type if there are subgroups H_1 and H_2 in G containing S such that:*

i) $H_1 = N_G(Z(O_3(H_1)))$, $O_3(H_1)$ is extraspecial group of order 3^{13}, $H_1/O_3(H_1) \cong 2Suz : 2$ and $C_{H_1}(O_3(H_1)) = Z(O_3(H_1))$.

ii) $O_3(H_2)$ is an elementary abelian group of order 3^8 and $H_2/O_3(H_2) \cong \Omega_8^-(3)$ with the natural action.

iii) $(H_1 \cap H_2)/O_3(H_2)$ is an extension of an elementary abelian group of order 3^6 by $2U_4(3) : 2$.

In this thesis we shall prove theorems 4, 5 and 6.

Theorem 4 *A group of Co_1-type is isomorphic to Co_1.*

Theorem 5 *A group of $M(24)'$-type is isomorphic to $M(24)'$.*

Theorem 6 *A group of Monster-type is isomorphic to the largest sporadic simple group, the Monster.*

Also we shall prove the following corollary.

Corollary 7 *Let D be a finite group and $S \in Syl_3(G)$. Let D_1 and D_2 be two subgroups of D containing S such that:*

i) $D_1 = N_D(Z(O_3(D_1)))$, $O_3(D_1)$ is extraspecial group of order 3^{11} and exponent 3, $D_1/O_3(D_1) \cong U_5(2) : 2$ and $C_{D_1}(O_3(D_1)) = Z(O_3(D_1)) = \langle\alpha\rangle$.

ii) $O_3(D_2)$ is an elementary abelian group of order 3^7 and $D_2/O_3(D_2) \cong O_7(3)$ with natural action.

iii) $(D_1 \cap D_2)/O_3(D_2)$ is an extension of an elementary abelian group of order 3^5 by $U_4(2) : 2$.

Then D is isomorphic to $M(24)'$.

Theorem 4 is proved in chapter 3, theorem 5 and corollary 7 are proved in chapter 4 and theorem 6 is proved in chapter 5. We note that we have used of theorem 4 and corollary 7 in the proof of theorem 6 and corollary 7 is a consequence of theorem 5. In the next section we give an outline of the proofs. We note that the results of chapters 3 and 4 are published in [Sa1] and [Sa2], respectively. But the results of chapter 5 still are not published elsewhere. Also we notice that, as a personal interest we have identified the group $M(24)$ from its 3-local information in [Sa3].

We follow [As3] for notations for Fischer's groups. We have used the atlas [AT] notations for group extensions and other simple groups except for orthogonal groups and symplectic groups. By notations in [AT], we use of notations $\Omega_n^\epsilon(q)$ and $PSp_n(q)$ instead of $O_n^\epsilon(q)$ and $S_n(q)$, respectively. The other notations follow [As1]. For a finite group G, $O(G)$ is the largest normal subgroup of G of odd order. We denote by p^{1+2n} and p^{m+2n} ($m \geq 2$) an extraspecial group of order p^{1+2n} and a special group of order p^{m+2n} with center of order p^m, respectively. We say that H has shape $A.B.C.....Z$ when H has a normal series with factors of shape $A, B, C, ..., Z$.

1.3 An outline of the proofs

In this section we give an ouline of the proofs of our main theorems. We recall that theorem 4 and corollary 7 are used in the proof of theorem 6.

1.3.1 Theorem 4

This theorem is proved in chapter 3. Assume that G is of Co_1-type and we keep the notations in definition 1. The main idea is to show that G has an involution z such that the group G and the involution z satisfy the conditions of theorem 2.3.3, then the theorem 4 holds. We start by the group H_2. Lemma 2.4.2 will give us more information about the group H_2. Set $E = O_3(H_2)$. By 2.4.2, under the action of H_2/E on $P(E)$ (the set of the subgroups of order 3 in E) we have three orbits L, I and J such that $|L| = 12$, $|I| = 132$ and $|J| = 220$. Also by the structure of $H_1 \cap H_2$ and 2.4.2 we have $\langle t \rangle \in J$ and $t = abc$ where $\langle a \rangle$, $\langle b \rangle$ and $\langle c \rangle$ are in L and $\langle ab \rangle$ and $\langle ab^{-1} \rangle$ are in I. Set $U = \langle a, b \rangle \leq E$. The major part of the proof is to find the structure of the centralizer of each element of order three of U in the group G. We explain the proof in four steps.

Setp 1: The structure of $C_G(U)$

Set $M = C_G(U)$. Of course the structure of $N_M(\langle t \rangle)$ and $N_M(E)$ are known. This and lemma 2.4.4 will able us to select a suitable involution $z \in C_M(t)$ and show that there is an element $1 \neq h \notin U$ of order three in $C_M(z)$ such that h is conjugate to t in $C_M(z)$. Further $C_E(z)$ is of order 81 and contains a Sylow 3-subgroup of $C_M(z)$. From the structure of H_2 we could find a suitable involution $\alpha \in N_{H_2}(U) \setminus M$ such that $C_{H_1 \cap M}(\alpha)$ and $C_{H_2 \cap M}(\alpha)$ will satisfy the conditions of theorem 2.3.6. Then theorem 2.3.6 will give us that $C_M(\alpha) \cong U_4(2)$ and this will help us to show that $Y = O_2(C_M(t,z))O_2(C_M(h,z))$ is an extraspecial group of order 32. Then the structure of $N_M(Y)/\langle U, Y \rangle \cong S_3 \times S_3$ will be known. We will invoke theorem 2.3.8 to show that $N_M(Y) = C_M(z)$ and then theorem 2.3.5 will give us that $M/U \cong U_4(3)$.

Step 2: The structure of $C_G(a)$

Of course, from step 1 we have the structure of $C_G(U) = M$ and by lemma 2.4.2 we have the structure of $C_{H_2}(a)$ as well. Set $C = C_G(z, a)$ and $W = C_E(z)$ (remember z from step 1). By lemma 2.4.2 we get that $|P(W) \cap I| = 4$. Let $u \in W$ be conjugate to a in $C \cap H_2$ and $u \notin \langle U, c \rangle$. The structure of M and lemma 2.4.2 will able us to show that $K = O_2(C_C(u))O_2(C_C(c))$ is an extraspecial group of order 2^7 and then the structure of $N_C(K)/\langle a, K \rangle \cong U_4(2)$ will be known. Again we will invoke theorem 2.3.8 to show that $N_C(K) = C$ and then theorem 2.3.4 will give us that $C_G(a) \cong 3Suz$.

Step 3: The structure of $C_G(ab)$

Set $s = ab$. From the structure of H_2 we are able to find an involution $\alpha \in (C_G(s) \cap N_{H_2}(U)) \setminus (C_G(U))$ such that $C_{H_1}(\alpha, s)/\langle s, \alpha\rangle$ and $C_{H_2}(s, \alpha)/\langle s, \alpha\rangle$ satisfy the conditions of theorem 2.3.6. Then theorem 2.3.6 will give us that $C_G(s, \alpha)/\langle s, \alpha\rangle \cong U_4(2)$. This will be useful to show that $C_G(U)$ is of index 2 in $C_G(s)$ and then $C_G(s)$ is known. In fact $C_G(s) \cong 3U_4(3) : 2$.

Final step:

Set $D = C_G(z)$ (remember z from steps 1 and 2). From the structures of $C_G(a)$, H_1 and $C_G(s)$ we will be able to show that $Q = O_2(C_D(a))O_2(C_D(s))O_2(C_D(t))$ is an extraspecial group of order 2^9 and then the srructure of $N_D(Q)/Q \cong \Omega_8^+(2)$ will be known. Finally, we will invoke theorem 2.3.8 to show that $D = N_G(Q)$ and theorem 4 will follow from theorem 2.3.3.

1.3.2 Theorem 5 and Corollary 7

Theorem 5 and Corollary 7 are proved in chapter 4. Of course, Corollary 7 is a consequence of theorem 5. Assume that G is of $M(24)'$-type. The strategy is to show that there is an elementary abelian subgroup M of order 2^{11} in G such that G and M satisfy the conditions of theorem B in [Re]. Then the theorem will follow from ([Re], lemma 9) and ([As3], theorem 34.1). By notations in definition 2 we shall show that $C_{H_1}(A)$ and $C_G(A)$ satisfy the conditions of theorem 1 in [Pa]. Then by ([Pa], theorem 1) we have that $C_G(A)/A \cong U_6(2)$. This and ([SD], theorem 3.1) will give us that $C_G(z) \cong 2M(22) : 2$ for each involution $z \in A$. Of course, the structure of $C_G(z)$ shows that there is an elementary abelian subgroup M of order 2^{11} containing z such that $N_{C_G(z)}(M)/M \cong M_{22} : 2$ and $C_G(z, M) = M$. This and since all involutions of A are conjugate in G will able us to show that $N_G(M)/M \cong M_{24}$ and G and $N_G(M)$ satisfy the conditions of theorem B in [Re] as desired.

1.3.3 Theorem 6

This theorem is proved in chapter 5 and so we could use of theorem 4 and corollary 7. Assume that G is of Monster-type and we keep the notations in definition 3. We start by choosing an involution $z \in H_1$ such that $zO_3(H_1) \in Z(H_1/O_3(H_1))$. Our attempt is to show that there is an involution $t \in G$, $t \neq z$ such that $C_G(z)$ and $C_G(t)$ satisfy the conditions of theorem 2.3.1. Then theorem 6 follows from theorem 2.3.1. Set $\langle \tau \rangle = Z(O_3(H_1))$.

Let $\langle r \rangle \in (O_3(H_2) \cap O_3(H_1))$ be a non isotropic element under the action of $H_2/O_3(H_2)$ on $P(O_3(H_2))$. Then we shall show that $C_{H_1}(r)/\langle r \rangle$, $C_{H_2}(r)/\langle r \rangle$, $C_G(r)/\langle r \rangle$ and $C_{H_1 \cap H_2}(r)/\langle r \rangle$ satisfy the conditions of corollary 7. Thus by corollary 7 we get that $C_G(r) \cong 3M(24)'$. Hence r is not conjugate to τ in G.

The actions of z on $O_3(H_1)$ and $O_3(H_2)$ show that $L = C_{O_3(H_2)}(z) = \langle \tau, \epsilon \rangle$ is of order 9. Also, in $C_{H_2}(z)$ we have that τ is conjugate to ϵ and $\tau\epsilon$ is conjugate to $\tau^{-1}\epsilon$. Further $\tau\epsilon$ is conjugate to r in G. Set $s = \tau\epsilon$. Then $C_G(s, z)$ and $C_{H_1}(z)$ will able us to show that either $O_2(C_G(z)) = \langle z \rangle$ or $O_2(C_G(z))$ is an extraspecial group of order 2^{25}. Set $\overline{W} = C_G(z)/O_2(C_G(z))$. The next step is to show that $\overline{W}$ is of Co_1-type. Then by theorem 4 we will get that $\overline{W} \cong Co_1$.

From the structures of $C_G(z, \tau)$ and $C_{H_2}(z)$ and orbit calculations we can show that there is an elementary abelian subgroup $\overline{E}$ of order 3^6 in $\overline{W}$ such that $N_{\overline{W}}(\overline{E})/\overline{E} \cong 2M_{12}$. To have that $\overline{W}$ is of Co_1-type, we need to find another 3-local subgroup $\overline{X}$ in $\overline{W}$ such that $N_{\overline{W}}(\overline{E})$, $\overline{X}$ and $\overline{X} \cap N_{\overline{W}}(\overline{E})$ satisfy the conditions of definition 1. An easy observation shows that such a 3-local $\overline{X}$ of $\overline{W}$ is not contained in any 3-local of $\overline{W}$ which is found up to now. So we need to generate such a 3-local subgroup $\overline{X}$ of $\overline{W}$. This has been done by using of theorem 2.4.2 about the structure of $N_{\overline{W}}(\overline{E})$ and 3-local information of the group $\overline{C_G(z, \tau)} \cong 3Suz$. Then we have that $\overline{W} \cong Co_1$ by theorem 4. This will give us that $C_G(z)$ is as desired. The next step is to find the involution t and the structure of $C_G(t)$.

The structure of $C_G(s, z)$ will able us to select an involution t in $O_2(C_G(z, s))$ such that $C_G(z, t)$ and $C_G(t)$ satisfy the conditions of theorem 2.3.2. Thus by theorem 2.3.2 we get that $C_G(t) \cong 2F_2$ and then theorem 6 follows from theorem 2.3.1.

1.4 Recent characterizations for the sporadic simple groups

As we said in the last sections, in the final stage of the classification of the finite groups of local characteristic p, p a prime, they need to identify a K_p-group G from its p-local information. This is the idea behind of many recent identifications for the sporadic simple groups, and here we will mention some of them. Ly and the Monster are characterized from their 5-local information in [PR] and [PW1], respectively. A 3-local identification for $M(22)$ and $U_6(2)$ has been done in [Pa]. Co_3 is characterized in [KPR] from two of its 3-local subgroups and in [PW2], the Monster group is identified by its 7-local subgroups.

Chapter 2

Preliminaries

In this chapter we give some preliminary lemmas and theorems which are required in the next chapters.

2.1 Elementary definitions and results

Definition 2.1.1 *Let G be a finite group and H be a subgroup of G. Let $X \subset H$. Then X is said to be weakly closed in H with respect to G if $X^G \cap H = \{X\}$.*

Definition 2.1.2 *Let p be a prime, a p-element x of a group G is called p-central, if $C_G(x)$ contains a Sylow p-subgroup of G. If $C_G(x)$ does not contain a Sylow p-subgroup of G, we say that x is non p-central.*

Lemma 2.1.3 *(Frattini Argument) Let H be a normal subgroup of G and $P \in Syl_p(H)$, for a prime p. Then $G = N_G(P)H$.*

Proof: See ([As1], 6.2).□

Lemma 2.1.4 *(Three subgroup lemma) Let X, Y and Z be three subgroups of a finite group G with $[X, Y, Z] = [Y, Z, X] = 1$. Then $[Z, X, Y] = 1$.*

Proof: See ([As1], 8.7).□

Definition 2.1.5 *Let $1 \neq T \leq T_1 \leq H$ be groups, then T is called strongly closed in T_1 with respect to H, if $T^h \cap T_1 \leq T$ for each $h \in H$.*

Definition 2.1.6 *For a prime p, a subgroup $1 \neq T$ of a group H is called strongly p-embedded in H if p divides $|T|$ and p does not divide $|T \cap T^g|$ for all $g \in H$ such that $g \notin T$.*

2.2 p-Groups

Definition 2.2.1 *Let G be a finite p-group, p a prime.*

i) Let $K(G)$ be the set of all elementary abelian subgroups of G of maximal order. Set $J(G) = \langle K(G) \rangle$. $J(G)$ is called the Thompson subgroup of G.

ii) $\Omega_1(G) = \langle g \in G; g^p = 1 \rangle$.

iii) $\Phi(G)$ is the intersection of all maximal subgroups of G. $\Phi(G)$ is called the Frattini subgroup of G.

iv) If $Z(G) = [G, G] = \Phi(G)$, then G is called a special group.

v) G is called an extraspecial group if G is special and $Z(G)$ is cyclic.

Theorem 2.2.2 *Suppose that p is a prime, X is a finite group and $P \in Syl_p(X)$. Let $x, y \in Z(J(P))$ be X-conjugate. Then x and y are $N_X(J(P))$-conjugate.*

Proof: Let X be a finite group and $P \in Syl_p(X)$. Let $x, y \in Z(J(P))$ be X-conjugate. Then $J(P) \leq (C_X(x) \cap C_X(y))$. Therefore there are $T \in Syl_p(C_X(x))$ and $R \in Syl_p(C_X(y))$ such that $J(P) \in T \cap R$. By definition 2.2.1(i) we get that $J(P) = J(T) = J(R)$. We have $T^g \leq C_X(y)$ for some $g \in X$. Therefore by Sylow's theorem we get that $T^{gx} = R$ for some $x \in C_X(y)$. This gives us that

$$J(P)^{gx} = J(T)^{gx} = J(T^{gx}) = J(R) = J(P).$$

Hence x is conjugate to y in $N_X(J(P))$ and the lemma holds.□

Theorem 2.2.3 *Let p be odd and P be an extraspecial p-group of order p^{1+2n} and exponent p. Then*

i) $P/Z(P)$ is a $2n$-dimensional symplectic space over $GF(p)$.

ii) $Aut(P)/Inn(P) \cong GSp_{2n}(p) : 2$.

Proof: ([GLS2], theorem 10.5).□

Theorem 2.2.4 *(Coprime action) Let G be a finite p-group and V be a finite group of order coprime to p. Assume that G acts on V then*

i) $V = C_V(G)[V, G]$.

ii) $[V, G, G] = [V, G]$.

iii) Let N be a G-invariant normal subgroup of V, then $C_{V/N}(G) = C_V(G)N/N$.

iv) Let r be a prime divisor of $|V|$. Then there is a G-invariant Sylow r-subgroup in V.

v) If G is elementary abelian and noncyclic then $V = \langle C_V(g)|g \in G^\sharp \rangle$.

Proof: i), ii), iii), iv) follow from ([As1], 24.4, 24.5, 18.7(1) and 18.7(4)) respectively, and v) follows from ([BH], X.1.9).□

2.3 Some basic theorems

In this section we give some known theorems which are used in the next chapters. We need the following theorem for identifying the Monster group in chapter 5.

Theorem 2.3.1 *([GMS]) Let G be a finite group containing two involutions z and t such that $F^*(C_G(z)) = K$ is an extraspecial 2-group of order 2^{25}, $C_G(z)/K \cong Co_1$ and $C_G(t) \cong 2F_2$, where F_2 is the baby monster group. Then G is the Monster group.*

We make use of the next theorem for identifying F_2, the baby monster group when we use of theorem 2.3.1.

Theorem 2.3.2 *([Bi]) Let G be a finite group and z be an involution in G such that*

1) $F^*(C_G(z))$ is an extraspecial 2-group of width 11, and

2) $C_G(z)/F^*(C_G(z))$ is isomorphic to the second conway group Co_2.

Then one of the following holds:

a) $G = O(G)C_G(z)$.

b) G is isomorphic to F_2, the baby monster group.

The next theorem identifies Co_1 by the structure of the centralizer of a 2-central involution. We will apply this theorem in chapter 3 for identiying Co_1.

Theorem 2.3.3 *([As2], lemma 49.15) Let G be a finite group containing an involution z such that $F^*(C_G(z)) = K$ is an extraspecial 2-group of order 2^9, $C_G(z)/K \cong \Omega_8^+(2)$ and z is not weakly closed in K with respect to G. Then G is isomorphic to Co_1.*

In chapter 3 for identifying Co_1 we need to determine the structures of the centralizers of two non 3-central elements. Therefore we shall use of the following two theorems for the groups Suz and $U_4(3)$.

Theorem 2.3.4 *([As2], lemma 48.17) Let G be a finite group containing an involution z such that $F^*(C_G(z)) = K$ is an extraspecial 2-group of order 2^7, $C_G(z)/K \cong \Omega_6^-(2)$ and z is not weakly closed in K with respect to G. Then G is isomorphic to Suz.*

Theorem 2.3.5 *([Ph]) Let t_0 be an involution in $U_4(3)$. Denote by H_0 the centralizer of t_0 in $U_4(3)$. Let G be a finite group with the following properties:*

a) G has no subgroup of index 2.

b) G has an involution z such that $H = C_G(z)$, the centralizer of z in G is isomorphic to H_0.

Then G is isomorphic to $U_4(3)$.

We shall use of the following theorem in chapter 3.

Theorem 2.3.6 *([Ha]) Suppose that X is isomorphic to the centralizer of a non-trivial 3-central element in $PSp_4(3)$ and that H is a group with an element d such that $C_H(d) \cong X$. Let $P \in Syl_3(C_H(d))$ and E_2 be the elementary abelian subgroup of P of order 27. If E_2 does not normalize any $3'$-subgroup of H and d is not H-conjugate to its inverse, then either H has a normal subgroup of index 3 or $H \cong PSp_4(3)$.*

The next theorem will be required when we apply the theorem 2.3.6.

Theorem 2.3.7 *([Pa],lemma 6) Suppose that X is a group such that $O_3(X)$ is an extraspecial group of order 27 and exponent 3, $X/O_3(X) \cong SL_2(3)$, $O_2(X) = 1$*

and that a Sylow 3-subgroup of X contains an elementary abelian subgroup of order 27. Then X is isomorphic to the centralizer of a non-trivial 3-central element in $PSp_4(3)$.

The following theorem will play a crucial role in the final stage of the proof of theorem 1.4. This theorem is due to Goldschmidt.

Theorem 2.3.8 *(Goldschmidt's theorem) Let G be a finite group, $S \in Syl_2(G)$ and A be an abelian subgroup of S such that A is strongly closed in S with respect to G. Set $M = \langle A^G \rangle$. For $X \leq G$, define $\overline{X} = MO(M)/O(M)$. Then*

i) $\overline{A} = O_2(\overline{M})\Omega_1(\overline{S})$.

ii) $\overline{M}$ is a central product of an abelian group and groups isomorphic to one of: $L_2(2^n)$, $n \geq 3$, $Sz(2^{n+1})$, $n \geq 1$, $U_3(2^n)$, $n \geq 2$, $L_2(q)$ for $q \equiv 3, 5$ (mod 8), J_1 or a group of Ree type $({}^2G_3(3^n), n > 1)$.

Proof: See ([Go], theorem A).□

Theorem 2.3.9 *([Go], theorem A and lemma 3.2) Let N be a finite simple group isomorphic to $L_2(2^n)$, $n \geq 3$, $Sz(2^{n+1})$, $n \geq 1$ or $U_3(2^n)$, $n \geq 2$. Let $T \in Syl_2(N)$ and $B = N_N(T)$, then there is an abelian subgroup A of T such that A is strongly closed in T with respect to N. Also we have*

i) B is a semi-direct product TH where $C_H(T) = 1$ and $[T, H] = T$.

ii) If $N \cong L_2(2^n)$, then H is cyclic of order $2^n - 1$, $A = T$ and A is elementary abelian.

iii) If $N \cong Sz(2^{n+1})$, then H is cyclic of order $2^n - 1$, $A = Z(T)$, $|A| = 2^n$, $|T| = 2^{2n}$ and A is elementary abelian.

iv) If $N \cong U_3(2^n)$, then H is cyclic of order $2^n - 1/d$ where $d = 1$ if n is even and $d = 3$ if n is odd, $A = Z(T)$, $|A| = 2^n$, $|T| = 2^{3n}$ and A is elementary abelian.

Theorem 2.3.10 *([Go], theorem A and lemma 3.4) Let N be a finite simple group isomorphic to $L_2(q)$ for $q \equiv 3, 5$ (mod 8), J_1 or a group of Ree type. Let $T \in Syl_2(N)$, then there is an abelian subgroup A of T such that A is strongly closed in T with respect to N. Also we have*

i) $A = T$.

ii) If $N \cong L_2(q)$, then $|T| = 4$ and $N_N(A) \cong A_4$.

iii) If N is isomorphic to J_1 or a group of Ree type, then $O(C_N(A)) = Z(N_N(A))$ is cyclic and $N_N(A)/C_N(A)$ is a Frobenius group of order 21.

2.4 Modules

Definition 2.4.1 *i) Let V be a vector space, $P(V)$ is the set of 1-dimensional subspaces of V.*

ii) Let H be a finite group, V a $GF(p)H$-module and $A \leq H$, then we say that A acts cubic on V if $[V, A, A, A] = 1$.

Lemma 2.4.2 *Let $X \cong 2M_{12}$ and E be a faithful irreducible 6-dimensional $GF(3)X$-module. Then X has three orbits L, I and J on $P(E)$:*

i) $|L| = 12$ and X is 5-transitive on L. For an element $\langle x\rangle$ of L, we have $C_X(x) \cong M_{11}$ and $N_X(\langle x\rangle) \cong 2 \times M_{11}$.

ii) $|I| = 132$ and for an element $\langle x\rangle$ of I, we have $N_X(\langle x\rangle) \cong A_6.2 \times 2$ and $x = yz$ where $\langle y\rangle$ and $\langle z\rangle$ are two distinct elements of L.

iii) $|J| = 220$ and for an element $\langle x\rangle$ of J, we have $N_X(\langle x\rangle)$ is an extension of an elementary abelian group of order 9 by $GL_2(3) \times 2$ and $x = ryz$ where $\langle r\rangle$, $\langle y\rangle$ and $\langle z\rangle$ are three distinct elements of L.

iv) Let $\langle \tau\rangle$ be an element of the orbit L and U_1 be a Sylow 3-subgroup of $N_X(\langle \tau\rangle)$. Then $|C_E(U_1)| = 3^3$, the action of U_1 on E is cubic and $[E, U_1] : U_1$ is a special 3-group of order 3^7 and exponent 3 with center of order 27.

v) Let $x = x_1x_2x_3$ where $\langle x_1\rangle$, $\langle x_2\rangle$ and $\langle x_3\rangle$ are three distinct elements of L. Then $|O_3(C_X(x_1) \cap C_X(x))| = 9 = |O_3(C_X(x_3x_2) \cap C_X(x))|$ and $[E, X_1] : X_1$ is the unique extraspecial 3-group of order 3^5 in $E : C_X(x)$ where $X_1 = O_3(C_X(x))$. Also, $O_3(C_X(x))$ does not centralize any element of the orbit L and acts cubic on E.

vi) Let $T \in Syl_3(E : X)$, then E is a characteristic subgroup of T.

vii) Let $x \in X$ and $\langle x_1x_2\rangle \in I$ where $\langle x_1\rangle$ and $\langle x_2\rangle$ are two distinct elements of L. If x centralizes x_1x_2, then x^2 centralizes each x_i, $i = 1, 2$.

Proof: Let K and Y be two non conjugate subgroups in X isomorphic to M_{11} (see ([AT],page 32)). Then every subgroup of index at most 12 in $X/Z(X) \cong M_{12}$ is conjugate to the image of either K or Y. Moreover $X = \langle Y, K\rangle$ and $Y \cap K \cong L_2(11)$.

Let Z be one of the subgroups K, Y and $K \cap Y$. By [JLPW] a faithful irreducible $GF(3)Z$-module of dimension less or equal to 6 is 5-dimensional. This means that Z normalizes in E a 1-subspace or a 5-subspace. Suppose that E contains a 1-subspace normalized by K and a 1-subspace normalized by Y. Then both these 1-spaces are normalized by $K \cap Y$ and hence they are the same and so normalized by the whole $X = \langle K, Y \rangle$, a contradiction to the irreducibility of E. Applying the same argument we obtain that subspaces in E normalized by K and Y have different dimensions and we can choose our notation such that Y normalizes a 1-space D and K normalizes a 5-space H in E. Set $L = \{D^X\}$ and $L_1 = \{H^X\}$. Then L is an orbit of X on $P(E)$ of length 12 and L_1 is an orbit of X on hyperplanes in E of Length 12 and X is 5-transitive on both orbits L and L_1. We note that for each element $M \in L_1$ we have that $P(M) \cap L = \varnothing$.

Let $\langle x \rangle$ and $\langle y \rangle$ be two distinct elements of L and $M \in L_1$. Then $\langle x, y \rangle \cap M \neq 1$. Let $F = \langle xy \rangle$ and $C = \langle xy^{-1} \rangle$. Since $P(M) \cap L = \varnothing$, either $F \leq M$ or $C \leq M$. Since X is 5-transitive on L_1, we get that the intersection of any five elements of L_1 is conjugate to either F or C. Let S be the set of all elements of L_1 contaning F and $S_1 = L_1 \setminus S$. Assume that $|S| \geq 7$, then by 5-transitivity of X on L_1 we get that there is an element $g \in X$ such that $|S^g \cap S| \geq 5$ and $S^g \neq S$. This gives us that the interection of the elements in S, S^g and $S \cap S^g$ all are equal to F, a contradiction to $S^g \neq S$. Hence $|S| \leq 6$. A same argument shows that $|S_1| \leq 6$. Therefore both F and C are contained in exactly six elements of L_1. Since X is 5-transitive on L_1 we get that F is conjugate to C. Hence X is transitive on the set I of all $\langle x_1 x_2 \rangle$ where $\langle x_1 \rangle$ and $\langle x_2 \rangle$ are two distinct elements of L and $|I| = 2\binom{12}{2} = 132$. Since X is 3-transitive on L, the remaining $\binom{12}{3} = 220$ elements are in another orbit J and the normalizer of any element of J is an extension of an elementary abelian group of order 3^2 by $GL_2(3) \times 2$ and the normalizer of any element of I is $2 \times A_6.2$. So i),ii) and iii) hold. We note that as E is a faithful irreducible X-module, for involutions $x \in Z(X)$ we have that x acts fixed point freely on E.

Let $x \in X$ and x centralize an element $\langle \delta\delta_1 \rangle$ from the orbit I where $\langle \delta \rangle$ and $\langle \delta_1 \rangle$ are in the orbit L. Then $(\delta\delta_1)^x = \delta^x \delta_1^x = \delta\delta_1$. It gives that $\delta^x = \delta\delta_1(\delta_1^x)^{-1}$. As δ is not conjugate to $\delta\delta_1$ and $\delta\delta_1\delta_2$ in X for each $\delta_2 \in L$ and $\delta_2 \notin \{\delta_1, \delta\}$, we get that x^2 centralizes δ and δ_1 and vii) holds. Let $\langle \tau \rangle \in L$, $U_1 \in Syl_3(C_X(\tau))$ and $T = \langle \tau\delta\delta_1 \rangle \in J$. Set $N_1 = N_X(T)$, $U = O_3(N_1)$ and let $P \in Syl_3(N_1)$. By i) and ([AT], page 18) we get that $C_X(\tau, \delta\delta_1)$ is an extension of an elementary abelian group of order 9 by a cyclic group of order 4. Hence by vi) and conjugations in $C_X(\tau)$ we may assume that $\langle \delta, \delta_1 \rangle \leq C_E(U_1)$. Therefore $\langle \tau, \delta, \delta_1 \rangle \leq C_E(U_1)$ and $|C_E(U_1)| \geq 3^3$. By ([AT],page 32) we get that $N_X(P)/P$ is of order 8 and by iii) we have that $N_1/U \cong GL_2(3) \times 2$. Hence $N_X(P) = N_{N_1}(P)$. This and i),ii)and iii) give us that $C_E(P) = T$. We have $C_E(U_1)$ is $N_X(U_1)$-invariant and by ([AT], page 32) we get that $N_X(U_1)/U_1 \cong GL_2(3) \times 2$. Therefore if $|C_E(U_1)| \geq 3^4$, we get that $|C_E(P)| \geq 9$ which is a contradiction. Hence $|C_E(U_1)| \leq 3^3$. This gives us that $C_E(U_1)$ is of order 27 and $\langle \tau, \delta, \delta_1 \rangle = C_E(U_1)$. We assume that U is conjugate to U_1 in X, then $|C_E(U)| = 27$. By iii) we have $N_1 = U(B_1 \times Z_2)$, where $B_1 \cong GL_2(3)$. Let $x \in B_1$ be of order three and $\langle U, x \rangle = P$. Under the action of B_1 we have $C_E(U) = T \oplus E_1$ where $|E_1| = 3^2$. Since E_1 is a natural B_1-module, x centralizes one element of order three in E_1. But this is a contradiction to $C_E(P) = T$. This contradiction shows that U and U_1 are not conjugate in X. As T is the only subgroup of order three of E which is B_1-invariant, we have $C_E(U) = T$.

By ([AT],page 18), $N_{C_X(\tau)}(U_1) = U_1 B$ where $B \cong SD_{16}$. By ([Ch], theorem A) we get that U and U_1 do not act quadratically on E. Let $i \in Z(B_1)$ be an involution. Then i is a 2-central involution in X. Let $z \in Z(B)$ be an involution. Then by ([AT],page 18) $C_{C_X(\tau)}(z) \cong GL_2(3)$. As z is of determinant 1, either $\left|C_{E/\langle\tau\rangle}(z)\right| = 3$ or $\left|C_{E/\langle\tau\rangle}(z)\right| = 3^3$. By ([AT],page 18)$N_{C_X(\tau)}(\widehat{r})$ is a subgroup of $M_{10} \times 2$ for each element $\widehat{r} \in E/\langle\tau\rangle$ of order three. As there is no subgroup isomorphic to $C_{C_X(\tau)}(z)$ in $M_{10} \times 2$, we get that $\left|C_{E/\langle\tau\rangle}(z)\right| \neq 3$ and then $|C_E(z)| = 3^4$. Since each involution z in $C_X(\tau)$ is conjugate to i in X and $|C_E(z)| = 3^4$, we have $|C_E(i)| = 3^4$. Let E_1 and E_2 be two subgroups of E such that $E_1/T = C_{E/T}(U)$ and $E_2/C_E(U_1) = C_{E/C_E(U_1)}(U_1)$. Then E_1 is B_1-invariant and E_2 is B-invariant. Therefore $R = \langle E_1, U \rangle$ is an extraspecial group. So $|E_1| = 3^3$. Since i acts fixed point freely on E_1/T and $|C_E(i)| = 3^4$, we get that i acts trivially on E/E_1. Let $\overline{x}$ be an element in E/E_1, then $\overline{x}^i = \overline{x}$ and $[\overline{x}, u]^i = [\overline{x}, u]$ for each $u \in U$. This gives that $[\overline{x}, u] = 1$. Therefore $[E, U] \leq E_1$. Hence $[E, U, U] \leq [E_1, U] = T$. Obviously $[E, U] \neq T$. So $[E, U, U] = T$ and U acts cubic on E. Therefore R is the unique extraspecial normal subgroup in EN_1 of order 3^5. We have $\langle \tau, \delta, \delta_1 \rangle = C_E(U_1)$. As $C_E(z, U_1)$ is B-invariant and $z \in Z(B)$, by vi) we get that $C_E(U_1) \leq C_E(z)$. This gives us that $\langle E_2, U_1 \rangle$ is a special group of order 3^7. Hence $[E, U_1]$ is of order 3^5 and

U_1 acts cubic on E.

Let $F = E : P$ then $F \in Syl_3(E : N_1)$. Let A be an elementary abelian subgroup of P of order 3^2. Then $|U \cap A| \geq 3$. As for each element $x_1 \in U$ we have $|C_E(x_1)| \leq 3^3$, we get that $|C_E(A)| \leq 3^3$. So for each element $y \in P$ of order three we have $|C_E(y)| \leq 3^4$. Hence E is the unique maximal abelian subgroup of F. Now iv),v),vi) hold and the lemma is proved. □

By 2.4.2 and ([AT],page 18) we have the following lemma.

Lemma 2.4.3 *Let $X \cong M_{11}$ and E be a faithful irreducible 5-dimensional $GF(3)X$-module. Then under the action of X on $P(E)$ we have two orbits J_1 and J_2:*

i) $|J_1| = 11$, X is 4-transitive on J_1 and for an element $\langle x\rangle \in J_1$, we have $C_X(x) \cong A_6$ and $N_X(\langle x\rangle) \cong M_{10}$.

ii) $|J_2| = 110$. For an element $\langle x\rangle \in J_2$, we have $C_X(x) \cong 3^2 : 4$ and $x = x_1x_2$ where $\langle x_1\rangle$ and $\langle x_2\rangle$ are two distinct elements of J_1.

The next lemma follows from ([AT],page 4) and 2.4.3.

Lemma 2.4.4 *Let $X \cong A_6$ and E be a faithful irreducible 4-dimensional $GF(3)X$-module. Then under the action of X on $P(E)$ we have three orbits Y_1, Y_2 and Y_3 such that:*

i) $|Y_1| = 10$ and for $\langle x\rangle \in Y_1$ we have $C_X(x)$ is an extension of an elementary abelian group of order 9 by Z_2. Further X is 3-transitive on Y_1 and $C_X(x)$ has index 2 in $N_X(\langle x\rangle)$.

ii) $|Y_2| = |Y_3| = 15$ and for $\langle x\rangle \in Y_i$ we have $C_X(x) \cong A_4$.

iii) Let $T \in Syl_3(E : X)$, then E is a characteristic subgroup of T.

Lemma 2.4.5 *Suppose that $X \cong S_4$ and V is a faithful 3-dimensional $GF(3)X$-module. Then*

i) There is a set of 1-dimensional subspaces $\beta = \{\langle v_1\rangle, \langle v_2\rangle, \langle v_3\rangle\}$ such that $X/O_2(X)$ acts as S_3 on β and each subspace in β is inverted by $O_2(X)$.

ii) X has orbits of length 3,6 and 4 on $P(V)$ with representatives $\langle v_1\rangle$, $\langle v_1 + v_2\rangle$ and $\langle v_1 + v_2 + v_3\rangle$ respectively.

Proof: Let $Q = O_2(X)$ and $Q^\sharp = \{q_1, q_2, q_3\}$. Then, as V is a faithful irreducible $GF(3)X$-module and X acts transitively on $Q^\sharp$ by conjugation, we get that $V = C_V(q_1) \oplus C_V(q_2) \oplus C_V(q_3)$ and that X permutes the subspaces $\{C_V(q_i)|1 \leq i \leq 3\}$ transitively. Setting $\langle v_i \rangle = C_V(q_i)$, we have that i) holds.

Obviously $\{\langle v_i \rangle\, |1 \leq i \leq 3\}$ is an orbit of length 3 on $P(V)$. The subspaces $\langle v_1 \pm v_2 \pm v_3 \rangle$ form an orbit of length 4 and the subspaces $\langle v_i \pm v_j \rangle$ with $i \neq j$ give an orbit of length 6. This proves ii). □

The following lemma is well-known.

Lemma 2.4.6 *Suppose that X is a group, V is an elementary abelian normal 2-subgroup of X and $x \in X$ is an involution. Set $C := C_X(x)$. Then there is a one to one correspondence between VC-orbits on the involutions in the coset Vx and the C-orbits on the elements of $C_V(x)/[V,x]$. Furthermore, for vx an involution in Vx, $|(vx)^{VC}| = |(v[V,x])^C||[V,x]|$.*

Proof: The map $(vx)^{VC} \longmapsto (v[V,x])^C$, where $vx \in Vx$ is an involution, is the required bijection.□

Lemma 2.4.7 *Suppose that $X \cong \Omega_6^-(2) \cong PSp_4(3)$ and V is the natural $GF(2)X$-module of dimension 6. Then*

i) X has two classes of involutions. Let $x \in X$ be an involution, then 3 divides $|C_X(x)|$.

ii) X has two orbits V_1 and V_2 on $P(V)$ such that $|V_1| = 27$, for $\langle v \rangle \in V_1$ we have that $C_X(v)$ is an extension of an elementary abelian group of order 16 by A_5, $|V_2| = 36$ and for $\langle v \rangle \in V_2$ we have that $C_X(v) \cong S_6$.

iii) Let $x \in X$ be an involution, then $|C_V(x)| = 16$ and $|[V,x]| = 4$.

iv) Suppose that $A \leq X$ is of order 32, $r \in Z(A)$ and A contains an elementary abelian group of order 16, then $C_V(A) \leq [C_V(r), A]$.

v) Let $Y \leq X$ be an elementary abelian subgroup of X. If $|Y| = 16$, then $|C_V(Y)| = 2$. There is no elementary abelian group of order 8 in X all of whose non trivial elements are non 2-central.

vi) Let x and y be two distinct 3-central elements in X and $D \leq X$ be an elementary abelian group of order 27 containing $\langle x, y \rangle$. Then $C_X(xy)/D$ is isomorphic to a subgroup of D_8.

vii) Let $r \in X$ be a 2-central involution, then $C_X(r)/O_2(C_X(r))$ is an extension of an elementary abelian group of order 9 by a group of order 2, $O_2(C_X(r))$ is an extraspecial group of order 32 and $C_X(r)/O_2(C_X(r))$ acts irreducibly on $O_2(C_X(r))/\langle r\rangle$. Let $N \in Syl_3(C_X(r))$, then any N-invariant nontrivial subgroup of $O_2(C_N(r))$ is isomorphic to Q_8.

viii) Let N be the quasisimple group $Sp_4(3)$. Then N has two classes of involutions. Let z and r be two involutions in N such that z is a 2-central involution and r is a non 2-central involution, then $O_2(C_N(r)) \cong Q_8 \times Q_8$, $C_N(r)/O_2(C_N(r))$ is an elementary abelian group of order 9 and $z \in Z(N)$. Let $X_1 \in Syl_3(C_N(r))$ and $Y < O_2(C_N(r))$ be X_1-invariant and $|Y \cap Z(O_2(C_N(r)))| = 2$, then either $Y = Y \cap Z(O_2(C_N(r)))$ is of order 2 or $Y \cong Q_8$.

Proof: Parts i), ii), iii), vii) and viii) follow from the atlas of finite groups ([AT],page 26) or by easy calculation. Part iv) follows from v) and so we just prove v) and vi). Let $Y \leq X$ be an elementary abelian 2-group, if $|Y| = 16$, then by ii) we get that $|C_V(Y)| = 2$. We note that if $|Y| = 16$, then $N_X(Y)/Y \cong A_5$ and $N_X(Y)$ is a maximal subgroup of X([AT],page 26). Let Y be of order 8 such that all of whose non trivial elements are non 2-central . Let $T \in Syl_2(X)$ which contains Y and $A \leq T$ be an elementary abelian group of order 16. By ([AT],page 26) we have $N_X(A)/A \cong A_5$ and the extension splits. Let $F_1 \cong A_5$ be a subgroup of $N_X(A)$, $F \in Syl_2(F_1)$ and $F \leq T$, then F is an elementary abelian group of order four and $T = A : F$. By ([AT],page 2) we have that all involutions in F are conjugate. By ([AT],page 26) we get that $Z(T)$ is of order 2 and hence $C_A(f)$ is of order 4 for each involution $f \in F$. Now if Y is not a subgroup of A, then we get that $Z(T) \leq Y \cap A$ and this gives us that there is a 2-central involution in Ywhich is a contradiction. Hence $Y \leq A$. By ([AT],page 26) under the action of $N_X(A)/A$ on $P(A)$ we have two orbits B and C such that $|B| = 5$, the involutions in B are 2-central involutions in X, $|C| = 10$ and the involutions in C are non 2-central involutions in X. Also $N_X(A)/A$ is 3-transitive on B. This gives us that $B = \{\langle x_1\rangle, \langle x_2\rangle, \langle x_3\rangle, \langle x_4\rangle, \langle x_1x_2x_3x_4\rangle\}$ and $C = \{\langle x_ix_j\rangle, \langle x_ix_jx_r\rangle$ where $i \neq j \neq r$, $i = 1, ...4$, $j = 1, ...4$, $r = 1, ..., 4$ and $\langle x_i\rangle$, $\langle x_j\rangle$ and $\langle x_r\rangle$ are in $B\}$. By the representations of the elements in the orbit C we get that $P(Y)$ is not a subset of C and v) is proved.

Let x and y be two 3-central elements in X and D an elementary abelian group of order 27 in X containing $\langle x, y\rangle$. By ([AT], page 26) each element of order three in X is conjugate to an element of D and $N_X(D)/D \cong S_4$. Also by ([AT],page 26) we get that $N_X(D)/D$ has three orbits D_1, D_2 and D_3 on $P(D)$ such that $|D_1| = 3$, $|D_2| = 4$ and $|D_3| = 6$. If $D_1 = \{\langle x_i \rangle, i = 1, 2, 3\}$, then $D_2 = \{\langle x_1 \pm x_2 \pm x_3 \rangle\}$ and $D_3 = \{\langle x_i \pm x_j \rangle, i \neq j, \}$. Since x and y are two 3-central elements , we get that $\langle x \rangle$ and $\langle y \rangle$ are in the orbit D_2 and we can see that $\langle xy \rangle$ is in $D_1 \cup D_3$. By ([AT],page 26) we get that $C_X(xy)/D$ is of order at most 4 and $C_X(xy) = C_{N_X(D)}(xy)$. Since a Sylow 2-subgroup of S_4 is isomorphic to D_8 we have proved vi) and the lemma is proved. □

The following lemma follows from the atlas ([AT],page 85) or by easy calculation.

Lemma 2.4.8 *Suppose that $X \cong \Omega_8^+(2)$ and V is the natural $GF(2)X$-module of dimension 8. Then*

i) Let $x \in X$ be an involution, then 3 divides the order of $C_X(x)$. Further X has 5 classes $2A, 2B, 2C, 2D$ and $2E$ of involutions. If x is in one of the classes $2B, 2C$ or $2D$, then $|C_X(x)| = 2^{10} \cdot 3^2 \cdot 5$, if x is in class $2A$, then x is a 2-central involution and 27 divides the order of $C_X(x)$. If x is in class $2E$, then $|C_X(x)| = 2^{10} \cdot 3$.

ii) X has two orbits N_1 and N_2 on $P(V)$, the elements of N_1 are isotropic elements and the elements of N_2 are non isotropic elements. Let $\langle v \rangle \in N_i$, then 3 divides the order of the stabilizer of $\langle v \rangle$ in X. Let $\langle x \rangle \in N_1$, then $C_X(x)$ is an extenstion of an elementary abelian group of order 2^6 by A_8. For $\langle y \rangle \in P(O_2(C_X(x)))$ we have that either y is a 2-central involution or y is in class $2B$.

iii) Let $x \in X$ be in one of the classes $2C, 2D$ or $2E$, then $C_X(x) = [V, x]$.

iv) Let $x \in X$ be in one of the classes $2A$ or $2B$, then $|C_V(x)| = 64$. If x is in class $2A$, then $C_X(x)$ has 3 orbits of lengths 1,6 and 9 on $C_V(x)/[V, x]$ and if x is in class $2B$, then 9 does not divide the lengths of orbits of $C_X(x)$ on $C_V(x)/[V, x]$. Let x be in class $2B$ and $T \in Syl_2(C_X(x))$, then $C_V(T) \leq [C_V(x), T]$.

v) Let $Y \leq X$ be an elementary abelian group of order 81, then any element of order three of X is conjugate to an element of Y. Further $N_X(Y)/Y$ is an extension of an extraspecial group of order 32 by S_3 and under the action of $N_X(Y)/Y$ on $P(Y)$, we have 5 orbits L_i, $i = 1, 2, ..., 5$ such that $|L_1| = |L_2| = |L_3| = 4$ and for $\langle x \rangle \in L_1 \cup L_2 \cup L_3$, we have $C_X(x) \cong 3 \times U_4(2)$, $|L_4| = 16$ and for $\langle x \rangle \in L_4$ we have $|C_X(x)| = 2^3 \cdot 3^5$, $|L_5| = 12$ and for $\langle x \rangle \in L_5$ we have $|C_X(x)| = 2^3 \cdot 3^4$. Also for

$\langle x\rangle \in L_i$, $i = 1, 2, 3, 4, 5$, we have that $\langle x\rangle$ is not conjugate to any element of L_j in X for $j \neq i$ and $j = 1, 2, ..., 5$.

Proof: The lemma follows from the atlas ([AT],page 85) and the natural action of X on V. Let $x \in X$ be an involution, then as $|V| = 2^8$, we have $|C_V(x)| \geq 2^4$ and $|[V, x]| \geq 2^2$. Also it is known that $dim([V, x]) + dim(C_V(x)) = dim(V)$ (identify $[V, x]$ by $1 - x$). We remark that as V is a natural module for X, we have $|C_V(x)| = 2^{2\alpha}$ for some $1 \leq \alpha \leq 3$. By ([AT],page 85) X has 5 classes $2A$,$2B$,$2C$,$2D$ and $2E$ of involutions and by using the notations in [AS] we have that the involutions in class $2A$ are in orthogonal Suzuki form a_2, the involutions in class $2B$ are in orthogonal Suzuki form c_2, the involutions in class $2C$ are in orthogonal Suzuki form a_4, the involutions in class $2D$ are in orthogonal Suzuki form a'_4 and the involutions in class $2E$ are in orthogonal Suzuki form c_4. In fact involutions in the classes $2C$ and $2D$ are conjugate in $O_8^+(2) \cong X : 2$. We just remind that if $x \in X$ is an invoution and x is in orthogonal Suzuki form a_l, a'_l or c_l, then $l = dim[V, x]$. □

The next lemma follows from ([AT], page 141).

Lemma 2.4.9 *Let $X \cong \Omega_8^-(3)$ and V be a natural $GF(3)X$-module. Then X has three orbits A, B and C on $P(V)$ such that:*

i) The elements in A are isotropic points and for $\langle x\rangle \in A$ we have $N_X(\langle x\rangle)$ is an extension of an elementary abelian group of order 3^6 by $2U_4(3) : 2$. Further $C_X(x)$ is of index 2 in $N_X(\langle x\rangle)$.

ii) The elements in B and C are non isotropic points and for $\langle x\rangle \in B \cup C$ we have $N_X(\langle x\rangle) \cong \Omega_7(3) : 2$. Further $C_X(x)$ is of index 2 in $N_X(\langle x\rangle)$.

Chapter 3

Characterization of Co_1

In this chapter we prove theorem 4. So in this chapter G is a group of Co_1-type and we keep the notations H_1, H_2 and t in definition 1. In 1.3.1 we gave an outline of the proof for theorem 4. We said there that our strategy is to determine the structure of the centralizer of a 2-central involution in the group G. We find an involution z in G such that $C_G(z)$ is an extension of an extraspecial 2-group of order 2^9 by $\Omega_8^+(2)$ and the main result will follow by applying the theorem 2.3.3.

Let me give a sketch of the proof and say how this chapter is organized. This chapter has four sections. In section 3.1 we will select a suitable subgroup $U = \langle a, b \rangle$ of order 9 in $O_3(H_2)$ and we shall show that $N_G(U)/C_G(U) \cong D_8$ (lemma 3.1.4(i)). Our first step is to find the structure of the centralizer of each element of order three of $O_3(H_2)$ in the group G. We note that by 2.4.2 and 2.2.2 the elements of order three in $O_3(H_2)$ are from three conjugacy classes of the elements of order three in G. Also by 2.4.2 we are allowed to assume that $t = abc$ and we have in H_2 that, a, b and c are conjugate and ab and ab^{-1} are conjugate. Further t is not conjugate to a or ab in G. We will start by $C_G(U)$ in section 3.2. In section 3.2 we will find the structure of $C_G(U)$. We will prove that $C_G(U)/U \cong U_4(3)$ (theorem 3.2.14). In section 3.2 theorems 2.3.8 and 2.3.5 have played a crucial role. In section 3.3 we will give the structure of $C_G(a)$, we show that $C_G(a)/\langle a \rangle \cong Suz$ (theorem 3.3.14). Of course in section 3.3 we have used of theorem 2.3.8 in the final stage when we will show that $C_G(a)/\langle a \rangle$ satisfies the conditions of theorem 2.3.3. Then 2.3.3 will give us the structure of $C_G(a)$ as desired. In section 3.4, at first we find the structure of $C_G(ab)$, then we will select an involution z in $C_G(U)$ and we will give the structure of $C_G(z)$. To determine the structure of $C_G(ab)$ we will invoke Thompson's transfer lemma to show that $C_G(U)$ is of index 2 in $C_G(ab)$. Then the structure of $C_G(ab)$ will be known. In theorem 3.4.10 we shall prove that $C_G(ab)/U \cong U_4(3) : 2$. In lemmas 3.4.18 and 3.4.19, by using of the structures $C_G(a)$, $C_G(s)$, $C_G(U)$ and H_1 we will show that there is an extraspecial group K of order 2^9 in $C_G(z)$ which is $C_{H_2}(z)$-invariant. In lemma 3.4.22 we will determine the structure of $N_G(K)$ and we shall show that $N_G(K)/K \cong \Omega_8^+(2)$. In lemma 3.4.26, theorem 2.3.8 is used to show that $C_G(z) = N_G(K)$ and then G and $C_G(z)$ will satisfy the conditions of theorem 2.3.3. Then 2.3.3 will give us that $G \cong Co_1$ and theorem 4 will be proved.

3.1 Some first steps

Notations: By our assumption $H_2/O_3(H_2) \cong 2M_{12}$ and $O_3(H_2)$ is a 6-dimensional $H_2/O_3(H_2)$-module, so we adopt the notations L, I and J in 2.4.2 for orbits $H_2/O_3(H_2)$ on $O_3(H_2)$. Then we have $\langle t\rangle \in J$. We now fix a, b and c such that $t = abc$ where $\langle b\rangle$,$\langle a\rangle$ and $\langle c\rangle$ are in L. Set $R = O_3(H_1)$ and $E = O_3(H_2)$. Let $U = \langle b, a\rangle$, then U is a subgroup of E of order 9 such that $|P(U) \cap L| = |P(U) \cap I| = 2$. Set $C_a = C_G(a)$, we use the bar notation for $\overline{C_a} = C_a/\langle a\rangle$. Let $\overline{R_1} = O_3(C_{\overline{C_a}}(\overline{t}))$.

Lemma 3.1.1 *i)* $|E \cap O_3(C_a \cap H_1)| = 3^5$.

ii) $(C_a \cap H_1)/O_3(C_a \cap H_1) \cong SL_2(3) \times Z_2$ *where* $O_3(C_a \cap H_1)$ *is a special 3-group of order* 3^7 *and exponent 3 with center of order 27.*

iii) $O_3(C_a \cap H_1) \leq O_3(C_a \cap H_1 \cap H_2)$.

Proof: We note that $R \leq H_2$. Assume that $x = a$ or b or c and $x \in R$. Then as $x \notin Z(R)$, there is an element $1 \neq y \in R$ such that $[x, y] = t$ or t^{-1}. By this, the representations of the elements in the orbits I, L, J in 2.4.2 and since $t = abc$, we get that $\langle x^y\rangle \in I \cup J$, a contradiction. Therefore aR is an element of order three in $H_1/R \cong Sp_4(3) : 2$. By 2.4.2(v), $O_3(C_a \cap H_1 \cap H_2) = EU_1$ where U_1 is an elementary abelian group of order 9. As $|C_R(a)| \leq 3^4$, $|C_{H_1}(a)|_3 \geq |EU_1| = 3^8$ and $|Sp_4(3) : 2|_3 = 3^4$, we get that aR is a 3-central element in H_1/R and $|C_R(a)| = 3^4$. The structure of the centralizer of a 3-central element in $PSp_4(3)$ can be found in ([AT],page 26). By ([AT],page 26) we get that $(C_a \cap H_1)/O_3(C_a \cap H_1)$ is an extension of an element of order 2 by $SL_2(3)$, $O_3(C_a \cap H_1) = C_R(a)R_1$ where R_1 is an extraspecial 3-group of order 27 and exponent 3 and $|C_R(a)| = 3^4$. Since a Sylow 2-subgroup of the normalizer of a Sylow 3-subgroup in $Sp_4(3)$ is a four-group, we have $(C_a \cap H_1)/O_3(C_a \cap H_1) \cong SL_2(3) \times Z_2$. Since $O_3(C_a \cap H_1) \leq O_3(C_a \cap H_1 \cap H_2)$ and by 2.4.2 $C_a \cap H_1 \neq C_a \cap H_1 \cap H_2$, we have $|E \cap O_3(C_a \cap H_1)| = 3^5$. As U_1 acts trivially on $E/(E \cap O_3(C_a \cap H_1))$ and by 2.4.2(iv) $|[E, U_1]| = 3^5$, we get that $[E, U_1] = E \cap O_3(C_a \cap H_1)$. Now by 2.4.2(iv) we have $O_3(C_a \cap H_1)$ is a special 3-group of order 3^7 and exponent 3 with center of order 27. □

By 2.4.3, under the action of $C_{H_2/E}(a) \cong M_{11}$ on $P(\overline{E})$, we have two orbits J_1 and J_2 such that J_1 is of length 11 and J_2 is of length 110.

We have $\langle t, a \rangle \leq O_3(C_a \cap H_1)$ and by 3.1.1(iii) $O_3(C_a \cap H_1) \leq O_3(C_a \cap H_1 \cap H_2)$. We note that by 2.4.2(i) for each four distinct elements $\langle x_i \rangle$, $i = 1, ..., 4$, from the orbit L we have that $x_1, ..., x_4$ are linear independent. So $\langle a, b, c \rangle \leq Z(O_3(C_a \cap H_1))$ and by 3.1.1(ii) we get that $\langle a, b, c \rangle = Z(O_3(C_a \cap H_1))$ and $P(Z(O_3(C_a \cap H_1))) \cap L = \{\langle a \rangle, \langle b \rangle, \langle c \rangle\}$. Since $t = bac$, if $x \in (C_a \cap H_1)$, then x centralizes t as well. So $t \in Z(C_a \cap H_1)$ and then $\overline{C_a \cap H_1} = C_{\overline{C_a}}(\bar{t})$. We have that $Z(\overline{R_1}) = \langle \bar{b}, \bar{c} \rangle$ and $\overline{R_1} \leq O_3(C_{N_{\overline{C_a}}(\overline{E})}(\bar{t}))$. This and 3.1.1(ii) give us that $\overline{R_1}$ is a special 3-group of order 3^6. So by 3.1.1 we get the following lemma.

Lemma 3.1.2 *i)* $\overline{R_1}$ *is a special 3-group of order* 3^6.

ii) $C_{\overline{C_a}}(\bar{t})/\overline{R_1} \cong SL_2(3) \times Z_2$.

iii) $Z(\overline{R_1}) = \langle \bar{b}, \bar{c} \rangle$.

The following lemma follows from 3.1.2.

Lemma 3.1.3 *Let* x *be an element of order three in* E, $\langle x \rangle \in J$ *and* $x = x_1 x_2 x_3$ *where* $\langle x_i \rangle \in L$. *Let* $y = x_i x_j$ *or* $x_i^{-1} x_j$, $i \neq j$ *and* $i = 1, 2, 3$, $j = 1, 2, 3$. *Then*

i) $O_3(C_G(x, y))$ *is a special 3-group of order* 3^7.

ii) $C_G(x, y)/O_3(C_G(x, y)) \cong SL_2(3) \times Z_2$.

iii) $Z(O_3(C_G(x, y))) = \langle x_1, x_2, x_3 \rangle$.

Further notations: Set $\overline{M} = C_{\overline{C_a}}(\bar{b})$ and $\widetilde{M} = \overline{M}/\langle \bar{b} \rangle$. Then $\widetilde{M} \cong C_G(U)/U$ (we recall that $U = \langle b, a \rangle$).

Lemma 3.1.4 *i)* $N_G(U)/C_G(U) \cong D_8$.

ii) There is an involution $\alpha \in H_2$ *such that* $a^\alpha = b$ *and* αE *is a 2-central involution in* H_2/E.

Proof: We have that $N_G(U)/C_G(U)$ is isomorphic to a subgroup of $GL_2(3)$. As by 2.2.2 $\langle a\rangle$ and $\langle ab\rangle$ are not conjugate in G, we get that $N_G(U)/C_G(U)$ is isomorphic to a subgroup of D_8. By ([GLS3], table 5.3b) we get that the preimage of a 2-central involution in M_{12} is an involution in $2M_{12}$ and the preimage of a non 2-central involution in M_{12} is an element of order 4 in $2M_{12}$. Let $x \in H_2$ be an involution such that $xE \in Z(H_2/E)$, then as E is a faithful irreducible H_2/E-module, we get that x acts fixed point freely on E. Since M_{12} is a simple group and L is an orbit of H_2/E, there are elements α and β in H_2 such that α is of order 2 and $a^{\alpha} = b$ and β is of order 4 and $a^{\beta} = b^{-1}$ and $b^{\beta} = a$. Now $\langle \alpha,\beta\rangle / C_G(U) \cap \langle \alpha,\beta\rangle \cong D_8$ and the lemma is proved. □

3.2 Identifying $U_4(3)$

In this section we shall find the structure of $\overline{M}$. We will show that $\overline{M} \cong 3U_4(3)$. We recall our notations :

- $Z(O_3(H_1)) = \langle t\rangle$, $t = abc$ and $C_a = C_G(a)$ where $\langle b\rangle$, $\langle a\rangle$ and $\langle c\rangle$ are in L.
- We use the bar notation for $\overline{C_a} = C_a/\langle a\rangle$.
- We have that $\overline{R_1} = O_3(C_{\overline{C_a}}(\overline{t})) = O_3(\overline{C_a \cap H_1})$, $\overline{t} = \overline{b}\overline{c}$ and $\overline{M} = C_{\overline{C_a}}(\overline{b})$, where $\langle \overline{c}\rangle$ and $\langle \overline{b}\rangle$ are in the orbit J_1.
- We have $\widetilde{M} = \overline{M}/\langle \overline{b}\rangle$ and $\widetilde{R_1} = O_3(C_{\widetilde{M}}(\widetilde{t}))$.

By 2.4.3(i), we have $\widetilde{M_1}/\widetilde{E} \cong A_6 \cong \Omega_4^-(3)$ where $\widetilde{M_1} = \widetilde{M} \cap (\widetilde{C_a \cap H_2})$. By 2.4.4 under the action of $\widetilde{M_1}/\widetilde{E}$ on $P(\widetilde{E})$ we have three orbits Y_1, Y_2 and Y_3 such that Y_1 has length 10 and Y_2 and Y_3 have lengths 15.

Lemma 3.2.1 *i)* $C_{\widetilde{M}}(\widetilde{t})/\widetilde{R_1} \cong SL_2(3)$.
iii) $\widetilde{R_1}$ *is an extraspecial 3-group of order* 3^5.

Proof: By 3.1.2 $Z(\overline{R_1}) = \langle \overline{c}, \overline{b}\rangle$ and $C_{\overline{C_a}}(\overline{t})$ contains a subgroup $\overline{X}$ such that $\overline{X} = \langle \overline{Q}, \overline{y}\rangle$ where $\overline{Q} \cong Q_8$, $\overline{y} \in \overline{E}$, $\overline{X}/\overline{R_1} \cong SL_2(3)$ and $\overline{X}$ acts trivially on $Z(\overline{R_1})$. Therefore $(\overline{M} \cap C_{\overline{C_a}}(\overline{t}))/\overline{R_1}$ contains a subgroup isomorphic to $SL_2(3)$. Since $C_{\overline{C_a \cap H_2}}(\overline{t})$ contains an involution $\overline{x}$, such that $\overline{b}^{\overline{x}} = \overline{c}$, we get that $C_{\overline{M}}(Z(\overline{R_1}))/\overline{R_1} \cong SL_2(3)$. Since $\overline{t} = \overline{b}\overline{c}$, we get $(\overline{M} \cap C_{\overline{C_a}}(\overline{t}))/\overline{R_1} = C_{\overline{M}}(Z(\overline{R_1}))/\overline{R_1} \cong SL_2(3)$. We have that $(\overline{M} \cap C_{\overline{C_a}}(\overline{t}))/\langle \overline{b}\rangle \leq C_{\widetilde{M}}(\widetilde{t})$. Let $\overline{x}\langle \overline{b}\rangle$ be an element in $C_{\widetilde{M}}(\widetilde{t})$ with $\overline{x} \in \overline{M}$. Since $\overline{t} = \overline{b}\overline{c}$, we have $\overline{x} \in C_{\overline{C_a}}(\overline{t}))$. Hence $(\overline{M} \cap C_{\overline{C_a}}(\overline{t}))/\langle \overline{b}\rangle = C_{\widetilde{M}}(\widetilde{t})$. As there is no subgroup isomorphic to A_4 in $C_{\widetilde{M}}(\widetilde{t})$, we have $\langle \widetilde{t} \rangle \in Y_1$. By 3.1.2 we have that $\overline{R_1}$ is special 3-group of order 3^6 with $Z(\overline{R_1}) = \langle \overline{a}, \overline{b} \rangle$. So $\widetilde{R_1} = \overline{R_1} / \langle \overline{b} \rangle$ is an extraspecial 3-group of order 3^5. $\square$

Further notations: Let $\widetilde{z} \in C_{\widetilde{M}}(\widetilde{t})$ be an involution such that $\widetilde{z}\widetilde{R_1} \in Z(C_{\widetilde{M}}(\widetilde{t})/\widetilde{R_1})$. Set

$$\widetilde{W} = C_{\widetilde{E}}(\widetilde{z}),\ \widetilde{N} = C_{N_{\widetilde{M}}(\widetilde{E})}(\widetilde{z}),\ \widetilde{C_{\widetilde{z}}} = C_{\widetilde{M}}(\widetilde{z}).$$

We get the following lemma.

Lemma 3.2.2 *i)* $|\widetilde{R_1} \cap \widetilde{E}| = 3^3$.

ii) $|\widetilde{W}| = 9$.

iii) Let $\widetilde{T} \in Syl_3(N_{\widetilde{M}}(\widetilde{E}))$, *then* $\widetilde{E}$ *is a characteristic subgroup of* $\widetilde{T}$ *and* $\widetilde{T} \in Syl_3(\widetilde{M})$.

Proof: From 3.1.1(i) we get that $|\widetilde{R_1} \cap \widetilde{E}| = 3^3$. Since $\widetilde{z}$ acts fixed point freely on $\widetilde{R_1}/Z(\widetilde{R_1})$, ii) follows from i) and iii) follows from 2.4.2(iv). $\square$

Lemma 3.2.3 *i)* $\widetilde{N} = N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{W})$ *and* $\widetilde{N}/\widetilde{W} \cong D_8$.

ii) $\widetilde{W} \in Syl_3(\widetilde{C_{\widetilde{z}}})$.

iii) $P(\widetilde{W}) = \left\{ \langle \widetilde{t} \rangle, \left\langle \widetilde{h} \right\rangle, \left\langle \widetilde{h_1} \right\rangle, \left\langle \widetilde{h_2} \right\rangle \right\}$ *where* $\langle \widetilde{t} \rangle$ *is conjugate to* $\left\langle \widetilde{h} \right\rangle$, $\langle \widetilde{t} \rangle$ *is not conjugate to* $\left\langle \widetilde{h_i} \right\rangle$ *in* $\widetilde{C_{\widetilde{z}}}$, *for* $i = 1, 2$, *and* $\left\langle \widetilde{h_1} \right\rangle$ *is not conjugate to* $\left\langle \widetilde{h_2} \right\rangle$ *in* $\widetilde{C_{\widetilde{z}}}$.

Proof: By ([AT],page 4) we get that $\widetilde{N}/\widetilde{W} \cong D_8$, where D_8 is a dihedral group of order 8. As there is no subgroup isomorphic to D_8 in $N_{\widetilde{M_1}}(\langle \widetilde{t} \rangle)$, there is another element $\widetilde{h} \in \widetilde{W}$ of order three which is $\widetilde{N}$-conjugate to $\widetilde{t}$. As by ([AT],page 4), A_6 has only one class of involutions , $\widetilde{z}$ centralizes some element from the orbits Y_2 and Y_3. Let $\left\langle \widetilde{h_1} \right\rangle \in (Y_2 \cap P(\widetilde{W}))$ and $\left\langle \widetilde{h_2} \right\rangle \in (Y_3 \cap P(\widetilde{W}))$. By 2.2.2, we get that $\left\langle \widetilde{h} \right\rangle$ and $\left\langle \widetilde{h_i} \right\rangle$ are not $\widetilde{M}$-conjugate for $i = 1, 2$. As $\left\{ \langle \widetilde{t} \rangle, \left\langle \widetilde{h} \right\rangle \right\}$ is the orbit of $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{W})$ which contains $\langle \widetilde{t} \rangle$, we have $[N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{W}, \langle \widetilde{t} \rangle) : N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{W})] = 2$. Since $N_{N_{\widetilde{C_{\widetilde{z}}}}(\langle \widetilde{t} \rangle)}(\widetilde{W}) = N_{\widetilde{N}}(\widetilde{W}, \langle \widetilde{t} \rangle)$ and $N_{\widetilde{N}}(\widetilde{W}, \langle \widetilde{t} \rangle)$ is of index 2 in $\widetilde{N}$, we get $\widetilde{N} = N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{W})$. As $\widetilde{W}$ is a characteristic subgroup of $\widetilde{N}$ and $\widetilde{W} \in Syl_3(\widetilde{N})$, we have $\widetilde{W} \in Syl_3(\widetilde{C_{\widetilde{z}}})$ and the lemma is proved. $\square$

By 3.2.3 we have that

$$P(\widetilde{W}) = \left\{ \langle \widetilde{t} \rangle, \left\langle \widetilde{h} \right\rangle, \left\langle \widetilde{h_1} \right\rangle, \left\langle \widetilde{h_2} \right\rangle \right\}$$

where $\langle \widetilde{t} \rangle$ is conjugate to $\left\langle \widetilde{h} \right\rangle$ and $\langle \widetilde{t} \rangle$ is not conjugate to $\left\langle \widetilde{h_i} \right\rangle$ in $\widetilde{C_{\widetilde{z}}}$ for $i = 1, 2$. Further $\left\langle \widetilde{h_1} \right\rangle$ is not conjugate to $\left\langle \widetilde{h_2} \right\rangle$ in $\widetilde{C_{\widetilde{z}}}$. In the next lemma we show that $\widetilde{N} = N_{\widetilde{C_{\widetilde{z}}}}(\left\langle \widetilde{h_2} \right\rangle) = N_{\widetilde{C_{\widetilde{z}}}}(\left\langle \widetilde{h_1} \right\rangle)$.

Lemma 3.2.4 *i)* $\widetilde{N} = N_{\widetilde{C_{\widetilde{z}}}}(\left\langle \widetilde{h_2} \right\rangle)$.

ii) $\widetilde{N} = N_{\widetilde{C_{\widetilde{z}}}}(\left\langle \widetilde{h_1} \right\rangle)$.

Proof: We prove i) and the proof of ii) is similar. As $\widetilde{N} \leq N_{\widetilde{C_{\widetilde{z}}}}(\left\langle \widetilde{h_2} \right\rangle)$, it is enough for us to show that $C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{h_2})$ is of index 2 in $\widetilde{N}$. Let $\overline{b} \neq \overline{x} \in \overline{E}$ be an element of order three. Then as $\langle \overline{b} \rangle \in J_1$, by 2.4.3 we get that either $\langle \overline{x} \rangle \in J_2$ or $\langle \overline{x}\overline{b} \rangle \in J_2$. Hence $C_{\widetilde{M}}(\overline{x} \langle \overline{b} \rangle)$ is isomorphic to a subgroup of $\widetilde{C_{\overline{C_a}}(\overline{r})}$ for some element $\overline{r}$ such that $\langle \overline{r} \rangle$ is in the orbit J_2. We have the structure of $C_{\overline{C_a}}(\overline{y})$ for each element $\langle \overline{y} \rangle \in J_2$. Let $\widetilde{h} = \overline{c_1} \langle \overline{b} \rangle$, then $\widetilde{h_2} = \overline{cc_1} \langle \overline{b} \rangle$, where $\overline{c_1} \in \overline{E}$ is an element of order three. Let $\overline{x} \langle \overline{b} \rangle \in C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{h_2})$ and $\widetilde{z} = \overline{z} \langle \overline{b} \rangle$, where $\overline{x}, \overline{z} \in \overline{M}$ and $\overline{z}$ is an involution, then as $\overline{cc_1}$ is not conjugate to $\overline{cc_1}\overline{b}$ in $\overline{M}$, we have $\overline{x} \in C_{\overline{C_a}}(\overline{cc_1}, \overline{z}, \overline{b})$. Therefore $C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{h_2})$ is isomorphic to $C_{\overline{C_a}}(\overline{cc_1}, \overline{z}, \overline{b}) / \langle \overline{b} \rangle$. Let $\overline{X} = C_{\overline{C_a}}(\overline{cc_1})$. As by 2.4.3 $\overline{cc_1}$ is conjugate to $\overline{t}$ in $\overline{C_a}$, we get that $\overline{X}/O_3(\overline{X}) \cong SL_2(3) \times 2$, $O_3(\overline{X}) \cong \overline{R_1}$ is a special 3-group of order 3^6 and $Z(O_3(\overline{X})) = \langle \overline{c}, \overline{c_1} \rangle$. As $\widetilde{N} \leq N_{\widetilde{C_{\widetilde{z}}}}(\left\langle \widetilde{h_2} \right\rangle)$, by 3.2.3(ii) we get that $\widetilde{W}$ is a Sylow 3-subgroup of $C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{h_2})$. We have that $\overline{z} \in (C_{\overline{C_a}}(\overline{c}) \cap C_{\overline{C_a}}(\overline{c_1}))$ and $\overline{W} = \langle \overline{c}, \overline{c_1}, \overline{b} \rangle$ is a subgroup of $C_{\overline{X}}(\overline{z})$. Since $C_{\overline{X}}(\langle \overline{c}, \overline{c_1} \rangle)/O_3(\overline{X}) \cong SL_2(3)$, we have $\overline{z} \in O^2(\overline{X})$ and $C_{\overline{X}}(\overline{z}) / \langle \overline{c}, \overline{c_1} \rangle \cong SL_2(3) \times 2$. Since $\langle \overline{c}, \overline{c_1} \rangle$ is normal in $C_{\overline{X}}(\overline{z})$,we get that $\overline{x}$ normalizes $\overline{W}$. Therefore $C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{h_2})$ is isomorphic to $N_{C_{\overline{X}}(\overline{z})}(\overline{W}) / \langle \overline{b} \rangle \cong \widetilde{W} : 2^2$. Since $N_{C_{\overline{X}}(\overline{z})}(\overline{W}) / \langle \overline{b} \rangle \leq \widetilde{N}$, we have $C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{h_2}) \leq \widetilde{N}$ and the lemma holds. □

Lemma 3.2.5 *Let $\widetilde{K}$ be a $\widetilde{W}$-invariant $3'$-subgroup of $\widetilde{C_{\widetilde{z}}}$. Then*

i) Either $\widetilde{K} = \langle \widetilde{z} \rangle$ or $\widetilde{K}$ is a 2-group and $|\widetilde{K}| \leq 2^5$.

ii) Either for each element $\widetilde{x} \in \widetilde{W}^\sharp$ we have $C_{\widetilde{K}}(\widetilde{x}) = \langle \widetilde{z} \rangle$ and then $\widetilde{K} = \langle \widetilde{z} \rangle$ or for $\widetilde{x} = \widetilde{t}$ or $\widetilde{x} = \widetilde{h}$ we have $C_{\widetilde{K}}(\widetilde{x}) \cong Q_8$.

Proof: As $\widetilde{K}$ is $\widetilde{W}$-invariant, by coprime action we have

$$\widetilde{K} = \left\langle C_{\widetilde{K}}(\widetilde{x}), \widetilde{x} \in \widetilde{W}^\sharp \right\rangle.$$

Assume that $\widetilde{x} = \widetilde{t}$, then $\widetilde{K} \cap C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{x}) = \langle \widetilde{z} \rangle$ or $\widetilde{K} \cap C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{x}) = O_2(C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{x})) \cong Q_8$ by 3.2.1(ii). Suppose that $\widetilde{x} = \widetilde{h}_2$ or $\widetilde{x} = \widetilde{h}_1$, then by 3.2.4 $\widetilde{K} \cap C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{x}) = \langle \widetilde{z} \rangle$.

Therefore either for each element $\widetilde{x} \in \widetilde{W}^{\sharp}$ we have $C_{\widetilde{K}}(\widetilde{x}) = \langle \widetilde{z} \rangle$ and then $\widetilde{K} = \langle \widetilde{z} \rangle$ or for $\widetilde{x} = \widetilde{t}$ or $\widetilde{x} = \widetilde{h}$ we have $C_{\widetilde{K}}(\widetilde{x}) \cong Q_8$. So $\widetilde{K}$ is a 2-group. In the latter case by Wielandt's order formula ([BH],XI.12.4) we get that $\left|\widetilde{K}\right| \leq 2^5$ and hence the lemma is proved. □

Set

$$\widetilde{Y} = \left\langle O_2(C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{h})), O_2(C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{t})) \right\rangle.$$

Then we get the following lemma.

Lemma 3.2.6 *i) There is a subgroup $\widetilde{X} \cong PSp_4(3)$ in $\widetilde{M}$ containing $\widetilde{z}$ and $\widetilde{W}$.*

ii) $\widetilde{Y}$ is an extraspecial 2-group of order 2^5.

iii) $\widetilde{Y}$ is the unique maximal $\widetilde{W}$-invariant $3'$-subgroup of $\widetilde{C_{\widetilde{z}}}$.

iv) $\widetilde{N} \leq N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})$.

v) There is an involution $\alpha \in N_{H_2}(U)$ conjugate to z in H_2 such that $a^{\alpha} = b$, $\widetilde{W} \leq C_{\widetilde{M}}(\alpha)$ and α centralizes Y (z is a pre-preimage of $\widetilde{z}$).

Proof: By 3.1.4(i) we have that $N_G(U)/C_G(U) \cong N_{H_2}(U)/C_{H_2}(U) \cong D_8$, where $U = \langle a, b \rangle$. We have that $M_1 = C_{H_2}(U)$, where M_1 is the pre-preimage of $\widetilde{M_1}$. By 2.4.3(i) $M_1/E \cong \widetilde{M_1}/\widetilde{E} \cong A_6$. By ([GLS3] table 5.3b) the preimage of each 2-central involution in M_{12} is a 2-central involution in $2M_{12}$. So there is an involution $\alpha \in N_{H_2}(U)$ such that $\langle \alpha E, M_1/E \rangle \cong S_6$, $C_{\widetilde{M_1}/\widetilde{E}}(\alpha) \cong S_4$ and α is conjugate to z (pre-preimage of $\widetilde{z}$) in H_2 (there is such an involution in M_{12}). By 2.4.2(i) we have that $N_{H_2}(\langle a \rangle)/E \cong N_{H_2}(\langle b \rangle)/E \cong 2 \times M_{11}$. Since there is no subgroup isomorphic to S_6 in $N_{H_2}(\langle a \rangle)/E$, we get that α normalizes neither $\langle a \rangle$ nor $\langle b \rangle$ and hence α does not act fixed point freely on U. In fact α centralizes either ab or $a^{-1}b$. By 2.4.2(i), $C_{H_2}(b)/E \cong M_{11}$ and by 2.4.3(i) we get that ab and $a^{-1}b$ are conjugate in $N_{H_2}(U)$. So we may assume that α centralizes ab. Since α is conjugate to z in H_2, we have $|C_E(\alpha)| = 3^4$ and then $|C_{\widetilde{E}}(\alpha)| = 27$, as $\alpha \notin C_G(U)$ and does not act fixed point freely on U. Set $\widetilde{V} = C_{\widetilde{E}}(\alpha)$. Then $\widetilde{E} = \widetilde{V} \oplus \langle \widetilde{r} \rangle$, with $\langle \widetilde{r} \rangle \in P(\widetilde{E})$, which is $C_{\widetilde{M_1}/\widetilde{E}}(\alpha)$-invariant and then by 2.4.4 $N_{\widetilde{M_1}/\widetilde{E}}(\langle \widetilde{r} \rangle) \cong S_4$. So we get that $C_{\widetilde{M_1}/\widetilde{E}}(\alpha) = N_{\widetilde{M_1}/\widetilde{E}}(\langle \widetilde{r} \rangle)$. Therefore for an involution $\gamma \in O_2(C_{\widetilde{M_1}}(\alpha)/\widetilde{V})$ we have that $|C_{\widetilde{V}}(\gamma)| = 3$ and hence $\widetilde{V}$ is a faithful, irreducible $C_{\widetilde{M_1}}(\alpha)/\widetilde{V}$-module. By 2.4.4 $C_{\widetilde{M_1/E}}(\widetilde{r}) \cong A_4$. Now let $\widetilde{x} \in C_{\widetilde{M_1}}(\alpha)$ be an involution which inverts $\widetilde{r}$. Then $C_{\widetilde{E}}(\widetilde{x}) \leq \widetilde{V}$. Since $\widetilde{x}$ is conjugate to $\widetilde{z}$ in $\widetilde{M_1}$, we may assume that $\widetilde{z} \in C_{\widetilde{M_1}}(\alpha)$ and $\widetilde{W} = C_{\widetilde{E}}(\widetilde{z}) \leq \widetilde{V}$. We recall that $\widetilde{W} = \left\langle \widetilde{h}, \widetilde{t} \right\rangle$. Let $\widetilde{K_1} = C_{C_{\widetilde{M}}(\widetilde{t})}(\alpha)$, then since $\widetilde{W} \leq \widetilde{V}$ and by 3.2.1 $C_{\widetilde{M}}(\widetilde{t})/\widetilde{R_1} \cong SL_2(3)$, we get that $\widetilde{K_1}/O_3(\widetilde{K_1}) \cong SL_2(3)$. Now as $\widetilde{V} \leq C_{\widetilde{M}}(\widetilde{t})$, we get that $O_3(\widetilde{K_1})$ is an extraspecial 3-group of order 27. As $\widetilde{V}$ is elementary abelian of order 27, we get that $\widetilde{K_1}$ satisfies the conditions of 2.3.7. Set $\widetilde{K_2} = C_{N_{\widetilde{M}}(\widetilde{E})}(\alpha)$ and $\widetilde{X} = \left\langle \widetilde{K_1}, \widetilde{K_2} \right\rangle$. Then $O_3(\widetilde{K_2}) = \widetilde{V}$ and $\widetilde{K_1} = C_{\widetilde{X}}(\widetilde{t})$. Since $\widetilde{t} \in \widetilde{V}$ and $C_{\widetilde{K_1}}(\widetilde{V}) = \widetilde{V}$, we have $C_{\widetilde{X}}(\widetilde{V}) = \widetilde{V}$. By 2.4.5 under the action of $\widetilde{K_2}/\widetilde{V}$ on $P(\widetilde{V})$ we have three orbits of lengths 3,4 and 6. As $|N_{\widetilde{K_1}}(\widetilde{V})/\widetilde{V}| = 6$, we get that $\langle \widetilde{t} \rangle$ is in the orbit of length 4. By 3.2.3 we have that $\left\langle \widetilde{h_2} \right\rangle$ and $\left\langle \widetilde{h_1} \right\rangle$ are in $P(\widetilde{W})$ and they are not conjugate to $\langle \widetilde{t} \rangle$ in $\widetilde{M}$. Since $\widetilde{W} \leq \widetilde{V}$, we have $\left\langle \widetilde{h_2} \right\rangle$ and $\left\langle \widetilde{h_1} \right\rangle$ are in $P(\widetilde{V})$. Therefore under the action of $N_{\widetilde{X}}(\widetilde{V})/\widetilde{V}$ on $P(\widetilde{V})$, $\langle \widetilde{t} \rangle$ is in the orbit of length 4. This gives us that $|N_{\widetilde{X}}(\widetilde{V})/\widetilde{V}| = 24$. As $\widetilde{K_2} \leq N_{\widetilde{X}}(\widetilde{V})$ and $\widetilde{K_2}/\widetilde{V} \cong S_4$, we have $\widetilde{K_2} = N_{\widetilde{X}}(\widetilde{V})$.

Let $\widetilde{V_1} \leq \widetilde{V}$ be of order 9. If $\widetilde{V_1}$ contains $\widetilde{t}$, then since there is no $\widetilde{V}$-invariant $3'$-subgroup in $\widetilde{K_1}$, there is no $\widetilde{V}$-invariant $3'$-subgroup in $C_{\widetilde{X}}(\widetilde{V_1})$. Suppose that $\widetilde{V_1}$ does not have an element conjugate to $\widetilde{t}$ and let $\overline{V_1}$ and $\overline{V}$ be two subgroups of $\overline{E}$ such that $\overline{V}/\left\langle \overline{b} \right\rangle = \widetilde{V}$ and $\overline{V_1}/\left\langle \overline{b} \right\rangle = \widetilde{V_1}$. Since $\left\langle \overline{b} \right\rangle$ is in the orbit J_1 , by 2.4.3 for each element $\left\langle \overline{b} \right\rangle \neq \langle \overline{x} \rangle \in J_1$ we have $\left\langle \overline{xb} \right\rangle \in J_2$. As by 2.4.3(i) for each four distinct elements $\langle \overline{x_i} \rangle \in J_1$, $i = 1, ..., 4$, we have that $\overline{x_1}$,...,$\overline{x_4}$ are linear independent, so for each element $\langle \overline{x} \rangle \in J_2$ such that $\overline{x} = \overline{x_1 x_2}$, where $\overline{x_i} \neq \overline{b}$ for $i = 1, 2$, we have $\left\langle \overline{xb} \right\rangle \in J_2$ as well. Therefore $\overline{V_1}$ has an element $\overline{x}$ conjugate to $\overline{t}$ in $\overline{C_a}$. Let $\overline{t}^g = \overline{x}$, then by 3.1.2 $C_{\overline{C_a}}(\overline{x})/\overline{R_1}^g \cong SL_2(3) \times 2$ where $\overline{R_1}^g$ is a special group of order 3^6 with $Z(\overline{R_1}^g) = \left\langle \overline{c}^g, \overline{b}^g \right\rangle$. Since $|\overline{V}| = 3^4$, we have that a $\overline{V}$-invariant $3'$-subgroup in $C_{\overline{C_a}}(\overline{x})$ is contained in $C_{\overline{C_a}}(\overline{E})$ and so centralizes $\overline{V}$. Since $\widetilde{K_2}/\widetilde{V}$ is faithful and irreducible on $\widetilde{V}$, no $3'$-subgroup of $C_{\widetilde{X}}(\widetilde{V_1})$ is $\widetilde{V}$-invariant. Let $\widetilde{T} \in Syl_3(\widetilde{K_1})$ containing $\widetilde{V}$ and $\widetilde{y} \in N_{\widetilde{K_1}}(\widetilde{T})$ which inverts $\widetilde{t}$. Then $\widetilde{y}$ normalizes $\widetilde{V}$. We have that $\langle \widetilde{t} \rangle$ is in the orbit of length 4 under the action of $\widetilde{K_2}/\widetilde{V}$ on $P(\widetilde{V})$, so $N_{\widetilde{K_2}}(\langle \widetilde{t} \rangle) = C_{\widetilde{K_2}}(\widetilde{t})$. This gives us that $\widetilde{y}$ centralizes $\widetilde{t}$, but $\widetilde{y}$ inverts $\widetilde{t}$, a contradiction. Therefore $\widetilde{t}$ is not conjugate to it's invers in $\widetilde{X}$. As there is no normal subgroup of index 3 in $\widetilde{K_2}$, we get that $\widetilde{X} \cong PSp_4(3)$ by 2.3.6. Since both subgroups $\widetilde{K_1}$ and $\widetilde{K_2}$ are in $\widetilde{M}$,

i) holds. By ([AT],page 26) we have that $O_2(C_{\widetilde{X}}(\widetilde{z})) = O_2(C_{\widetilde{X}}(\widetilde{z},\widetilde{h}))O_2(C_{\widetilde{X}}(\widetilde{z},\widetilde{t}))$ is a $\widetilde{W}$-invariant 2-subgroup of $\widetilde{M}$. Since $O_2(C_{\widetilde{X}}(\widetilde{t},\widetilde{z})) = O_2(C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{t})) \cong Q_8$ and $O_2(C_{\widetilde{X}}(\widetilde{h},\widetilde{z})) = O_2(C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{h})) \cong Q_8$, we get that $\widetilde{Y}$ is an extraspecial 2-group of order 32 of plus type and for each element $\widetilde{x} \in \widetilde{W}$ of order three we have $O_2(C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{x})) \leq \widetilde{Y}$. Therefore v) holds and by 3.2.5(i) we have that ii) and iii) hold. Further iv) follows from iii) and 3.2.3(i). □

We are going to find the structure of $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})$ and then we will show that $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y}) = \widetilde{C_{\widetilde{z}}}$. In this section we use of the notation * for the natural homomorphism $\widetilde{C_{\widetilde{z}}} \mapsto \widetilde{C_{\widetilde{z}}}/\left<\widetilde{z}\right>$.

Lemma 3.2.7 *Let $\widetilde{x} \in \widetilde{W}$ be an element of order three. Then $C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{x}) \leq N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})$.*

Proof: For $\widetilde{x} = \widetilde{t}$ and $\widetilde{x} = \widetilde{h}$ we have that $C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{x})/\left<x\right> \cong SL_2(3)$, so $C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{x}) = \left<\widetilde{W}, O_2(C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{x}))\right>$ and therefore $C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{x}) \leq N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})$. For $\widetilde{x} = \widetilde{h_2}$ and $\widetilde{x} = \widetilde{h_1}$, we get by 3.2.4 that $C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{x}) \leq \widetilde{N}$ and so we get by 3.2.6 that $C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{x}) \leq N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})$. □

Lemma 3.2.8 $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})/\widetilde{Y} \cong (S_3 \times S_3)$.

Proof: As no element of order three in $\widetilde{W}$ centralizes $\widetilde{Y}$ and by 3.2.3(ii), $\widetilde{W} \in Syl_3(\widetilde{C_{\widetilde{z}}})$, we get that $C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})$ is a $3'$-group. As by 3.2.6(iii) $\widetilde{Y}$ is the unique maximal $\widetilde{W}$-invariant $3'$-subgroup of $\widetilde{C_{\widetilde{z}}}$, we have $C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y}) = \left<u\right>$. Since by 3.2.6(ii) $\widetilde{Y}$ is an extraspecial 2-group of order 2^5, by ([GLS2],theorem 10.6) we get that $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})/\widetilde{Y}$ is isomorphic to a subgroup of $O_4^+(2)$. As $\widetilde{N}\widetilde{Y}/\widetilde{Y} \cong (S_3 \times S_3)$ and $S_3 \times S_3$ is a maximal subgroup of $O_4^+(2)$ of index 2, we have $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})/\widetilde{Y} \cong S_3 \times S_3$. □

By 3.2.8 we get that $O^2(N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})) = \widetilde{G_1}\widetilde{G_2}$ where $\widetilde{G_i} \cong SL_2(3)$, $\widetilde{G_i} \lhd O^2(N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y}))$ and $\widetilde{G_1} \cap \widetilde{G_2} = \left<\widetilde{z}\right>$. The following lemma follows from the structure of $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})$.

Lemma 3.2.9 *i) Let $\widetilde{r}^\star\widetilde{Y}^\star \in N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star/\widetilde{Y}^\star$ be an involution, then $|C_{\widetilde{Y}^\star}(\widetilde{r}^\star)| = 4$. If 3 divides the order of $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{r}^\star)$ then a Sylow 3-subgroup of $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{r}^\star)$ is conjugate to either $\widetilde{h_1}^\star$ or $\widetilde{h_2}^\star$ in $\widetilde{C_{\widetilde{z}}}^\star$.*

ii) $\widetilde{Y}^\star = O_2(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{h}^\star)) \oplus O_2(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{t}^\star))$.

iii) Let $\left<\widetilde{x}^\star\right> \in (P(O_2(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{h}^\star))) \cup P(O_2(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{t}^\star))))$, then 3 divides the order of $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{x}^\star)$ and a Sylow 3-subgroup of $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{x}^\star)$ is conjugate to $\widetilde{h}^\star$ in $\widetilde{C}^\star$.

Let $\langle \widetilde{x}^\star\rangle \in P(Y^\star)$ *and* $\widetilde{x}^\star = \widetilde{r_1}^\star\widetilde{r_2}^\star$, *where* $\left\langle \widetilde{r}_1^\star \right\rangle \in P(O_2(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{h}^\star)))$ *and* $\langle \widetilde{r_2}^\star\rangle \in P(O_2(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{t}^\star)))$, *then 3 does not divide the order of* $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{x}^\star)$ *and* $\widetilde{x}^\star$ *is a 2-central involution in* $\widetilde{C_{\widetilde{z}}}^\star$.

iv) Under the action of $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star/\widetilde{Y}^\star$ *on* $P(\widetilde{Y}^\star)$ *we have two orbits, one of them is of length 6 and the other one has length 9.*

v) Let $\widetilde{T_1}^\star$ *be a Sylow 2-subgroup of* $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star$, *then* $\widetilde{Y^\star}$ *is a characteristic subgroup of* $\widetilde{T_1}^\star$, $Z(\widetilde{T_1}^\star)$ *is of order 2 and* $\widetilde{T_1}^\star$ *is a Sylow 2-subgroup of* $\widetilde{C_{\widetilde{z}}}^\star$.

vi) Let $\widetilde{x}^\star \in N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star$ *be an involution and* $\widetilde{x}^\star \notin \widetilde{Y}^\star$ *then* $C_{\widetilde{Y}^\star}(\widetilde{x}^\star) = [\widetilde{Y}^\star, \widetilde{x}^\star]$. *So by 2.4.6 all involutions in* $\widetilde{x}^\star\widetilde{Y}^\star$ *are conjugate.*

Proof: Let $\widetilde{r}^\star \in N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star$ be an involution such that $\widetilde{r}^\star \notin \widetilde{Y}^\star$ and $\widetilde{r}^\star$ normalizes $\widetilde{W}^\star$. Then either $\widetilde{r}^\star$ inverts $\widetilde{t}^\star$ and $\widetilde{h}^\star$ or $\langle \widetilde{t}^\star\rangle^{\widetilde{r}^\star} = \left\langle \widetilde{h}^\star \right\rangle$. In the first case by 3.2.1 we get that $|C_{O_2(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{t}^\star))}(\widetilde{r}^\star))| = 2 = |C_{O_2(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{h}^\star))}(\widetilde{r}^\star))|$. Therefore in this case $|C_{\widetilde{Y}^\star}(\widetilde{r}^\star)| = 4$. If $\langle \widetilde{t}^\star\rangle^{\widetilde{r}^\star} = \left\langle \widetilde{h}^\star \right\rangle$, then for each involution $\widetilde{x}^\star$ in $O_2(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{t}^\star))$ we have $\widetilde{x}^\star(\widetilde{x}^\star)^{\widetilde{r}^\star}$ is centralized by $\widetilde{r}^\star$. Since $O_2(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{t}^\star))$ is an elementary abelian group of order 4, in this case $|C_{\widetilde{Y}^\star}(\widetilde{r}^\star)| = 4$ as well. Part v) follows from i). The proof of the other parts is easy and follows from the structure of $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star$ in 3.2.8. □

We recall that if $T \leq T_1 \leq H$ be groups, then T is *strongly closed* in T_1 with respect to H, if $T^h \cap T_1 \leq T$ for each $h \in H$. We are going to show that $\widetilde{Y}^\star$ is strongly closed in $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star$ with respect to $\widetilde{C_{\widetilde{z}}}^\star$.

Lemma 3.2.10 *Let* $\widetilde{r}^\star \in \widetilde{Y}^\star$ *be an involution and* $\widetilde{g}^\star \in \widetilde{C_{\widetilde{z}}}^\star$ *such that* $\widetilde{e}^\star = (\widetilde{r}^\star)^{\widetilde{g}^\star} \in N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star$. *If 3 divides* $|C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{e}^\star)|$, *then* $\widetilde{e}^\star \in \widetilde{Y}^\star$.

Proof: Let $\widetilde{r}^\star \in \widetilde{Y}^\star$ be an involution and $\widetilde{g}^\star \in \widetilde{C_{\widetilde{z}}}^\star$ such that $\widetilde{e}^\star = (\widetilde{r}^\star)^{\widetilde{g}^\star} \in N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star$, $\widetilde{e}^\star \notin \widetilde{Y}^\star$ and 3 divides $|C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{e}^\star)|$. By 3.2.9(i) we have that a Sylow 3-subgroup of $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{e}^\star)$ is conjugate to either $\widetilde{h_1}^\star$ or $\widetilde{h_2}^\star$ in $\widetilde{C_{\widetilde{z}}}^\star$. By 3.2.9(iii) we have that either 3 divides the order of $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{r}^\star)$ and a Sylow 3-subgroup of $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{r}^\star)$ is conjugate to $\widetilde{h}^\star$ in $\widetilde{C^\star}$ or 3 does not divide the order of $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{r}^\star)$ and $\widetilde{r}^\star$ is a 2-central involution in $\widetilde{C_{\widetilde{z}}}^\star$. Since $\widetilde{h}^\star$ is not conjugate to $\widetilde{h_1}^\star$ or $\widetilde{h_2}^\star$ in $\widetilde{C_{\widetilde{z}}}^\star$, we get by 3.2.9(iii) that $\widetilde{r}^\star$ is a 2-central involution in $\widetilde{C_{\widetilde{z}}}^\star$. By 3.2.3(i), no element of order 2 in $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star$ centralizes a Sylow 3-subgroup. So by 3.2.9(i) and 3.2.4 we get that a Sylow 3-subgroup of $C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{e}^\star)/\langle\widetilde{e}^\star\rangle$ is of order 3 and self centralizing. Let $(\widetilde{h_2}^\star)^{\widetilde{e}^\star} = \widetilde{h_2}^\star$ and $\widetilde{f}^\star \in N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star$ be an involution such that $\widetilde{e}^\star \neq \widetilde{f}^\star \notin \widetilde{Y}^\star$ and $(\widetilde{h_1}^\star)^{\widetilde{f}^\star} = \widetilde{h_1}^\star$. As $\widetilde{h_2}^\star$ is not conjugate to $\widetilde{h_1}^\star$ in $\widetilde{C_{\widetilde{z}}}^\star$, we deduce that $\widetilde{e}^\star$ is not conjugate to $\widetilde{f}^\star$ in $\widetilde{C_{\widetilde{z}}}^\star$. Since $\widetilde{e}^\star$ is conjugate to a 2-central involution of $\widetilde{C_{\widetilde{z}}}^\star$, by 3.2.9 (iii) $\widetilde{f}^\star$ is not conjugate to any involution of $\widetilde{Y}^\star$ in $\widetilde{C_{\widetilde{z}}}^\star$. By 3.1.4(i) there is a subgroup $\widehat{X}$ such that $\widehat{X}/B_2 \cong D_8 \cong N_{H_2}(U)/C_{H_2}(U)$ as $U = \langle a, b\rangle \leq E$. Let β be an element in $\widehat{X}$ such that $(\widetilde{h_2}^\star)^\beta = \widetilde{h_1}^\star$ and $(\widetilde{e}^\star)^\beta = \widetilde{f}^\star$. This gives us that $\widetilde{f}^\star$ is a 2-central involution in $\widetilde{C_{\widetilde{z}}}^\star$ and so $\widetilde{f}^\star$ is conjugate to some involutions of $\widetilde{Y}^\star$ in $\widetilde{C_{\widetilde{z}}}^\star$, which is a contradiction. Hence the lemma is proved. □

Lemma 3.2.11 *Let $\widetilde{x}^\star \in N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star$ be an involution such that $\widetilde{x}^\star \notin \widetilde{Y}^\star$ and $3||C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{x}^\star)|$. Then $|C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{x}^\star)|_2 = 16$. Further $\widehat{P} = C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{x}^\star)/\langle\widetilde{x}^\star\rangle$ contains a nilpotent normal subgroup $\widehat{P_1}$ such that $\widehat{P}/\widehat{P_1} \cong S_3$.*

Proof: By 3.2.9(i) and vi) we have that $\widetilde{X_1}^\star = \langle C_{\widetilde{Y}^\star}(\widetilde{x}^\star), \widetilde{x}^\star\rangle$ is an elementary abelian 2-group of order 8 and $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{x}^\star)/\widetilde{X_1}^\star \cong S_3$. By 3.2.9(i) we may assume that $\left\langle \widetilde{h_2}^\star \right\rangle \in Syl_3(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{x}^\star))$, since $\left\langle \widetilde{x}^\star, \widetilde{h_2}^\star \right\rangle = C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{h_2}^\star)$. So a Sylow 3-subgroup of $\widehat{P} = C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{x}^\star)/\langle\widetilde{x}^\star\rangle$ is of order 3 and it is self-centralizing. Now by Feit and Thompson's theorem [FT], we get that $\widehat{P}$ contains a nilpotent normal subgroup $\widehat{P_1}$ such that $\widehat{P}/\widehat{P_1} \cong S_3, A_5$ or $L_3(2)$. Let $\widetilde{P_1}^\star \leq C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{x}^\star)$ such that $\widetilde{x}^\star \in \widetilde{P_1}^\star$ and $\widehat{P_1} = \widetilde{P_1}^\star/\langle\widetilde{x}^\star\rangle$. Let further $\widetilde{T}^\star \in Syl_2(\widetilde{P_1}^\star)$. Then there is a subgroup $\widetilde{P_3}^\star$ in $C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{x}^\star)$ such that $\widetilde{P_3}^\star/\widetilde{T}^\star \cong S_3, A_5$ or $L_3(2)$. By 3.2.10 we have that $\langle\widetilde{x}^\star\rangle$ is not conjugate to any element of $P(\widetilde{Y}^\star)$, so $\widetilde{x}^\star$ is not a 2-central involution. Therefore $16 \leq |C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{x}^\star)|_2 \leq 32$ and $\widetilde{h_2}^\star$ acts fixed point freely on $\widetilde{T}^\star/\langle\widetilde{x}^\star\rangle$. So either $|\widetilde{T}^\star| = 8$ and $\widetilde{P_3}^\star/\widetilde{T}^\star \cong S_3$ or $\widetilde{T}^\star = \langle\widetilde{x}^\star\rangle$ and $\widetilde{P_3}^\star/\widetilde{T}^\star \cong L_3(2)$. Let $\widetilde{f}^\star \in C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{x}^\star)$ be an involution such that $\widetilde{f}^\star \notin \widetilde{Y}^\star$ and $\left\langle \widetilde{h_1}^\star \right\rangle \in Syl_3(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{f}^\star))$. Then $|\left\langle \widetilde{x}^\star, \widetilde{f}^\star \right\rangle| = 4$ and $\left\langle \widetilde{x}^\star, \widetilde{f}^\star \right\rangle \cap \widetilde{Y}^\star = 1$. Also, we have that $\widetilde{f}^\star$ is not conjugate to $\widetilde{x}^\star$ and as $3||C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{f}^\star)|$, we deduce that $\widetilde{f}^\star$ is not conjugate to any involution in $\widetilde{Y}^\star$ by 3.2.10. Therefore the case that $\widetilde{P_3}^\star/\widetilde{T}^\star \cong L_3(2)$ does not happen and hence the lemma holds. □

Lemma 3.2.12 *$\widetilde{Y}^\star$ is strongly closed in $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star$ with respect to $\widetilde{C_{\widetilde{z}}}^\star$.*

Proof: Let $\widetilde{r}^\star \in \widetilde{Y}^\star$ be an involution and let $\widetilde{g}^\star \in \widetilde{C_{\widetilde{z}}}^\star$ such that $\widetilde{e}^\star = (\widetilde{r}^\star)^{\widetilde{g}^\star} \in N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star$. If 3 divides the order of $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{e}^\star)$, then by 3.2.10 we get that $\widetilde{e}^\star \in \widetilde{Y}^\star$. So assume that 3 does not divide the order of $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{e}^\star)$.

Assume $3||C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{r}^\star)|$. As $\widetilde{e}^\star$ inverts $\widetilde{h}^\star$ and $\widetilde{t}^\star$, we conclude that $\widetilde{e}^\star$ centralizes some involution in $\widetilde{Y}^\star$ which is conjugate to $\widetilde{r}^\star$ in $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star$. Therefore we may assume that $\widetilde{e}^\star$ centralizes $\widetilde{r}^\star$. Since $\langle \widetilde{r}^\star \rangle$ is in the orbit of length 6 in $P(\widetilde{Y}^\star)$, we get that $\widetilde{r}^\star$ is not a 2-central involution. Therefore a Sylow 2-subgroup of $C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{r}^\star)$ is of order 32 and there is a Sylow 2-subgroup $\widetilde{X}^\star$ of $C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{r}^\star)$ which contains an elementary abelian 2-group $\widetilde{Y}^\star$ of order 16 and contains the involution $\widetilde{e}^\star$. We have that $\widetilde{e}^\star \in \widetilde{X}^\star$, $\widetilde{e}^\star \notin \widetilde{Y}^\star$ and $\widetilde{e}^\star$ is conjugate to involution $\widetilde{r}^\star \in \widetilde{Y}^\star$ in $\widetilde{C_{\widetilde{z}}}^\star$. Let $\widetilde{e}^\star = \widetilde{e_1}^\star \widetilde{e_2}^\star$, where for $i = 1, 2$, we have that $3||C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{e_i}^\star)|$, $\widetilde{e_i}^\star \notin \widetilde{Y}^\star$ and $[\widetilde{e}^\star, \widetilde{e_i}^\star] = 1$. By 3.2.10 we get that $\widetilde{e_i}^\star$ is not conjugate to $\widetilde{e}^\star$ in $\widetilde{C_{\widetilde{z}}}^\star$, $\widetilde{e_i}^\star$ is not in any elementary abelian subgroup of $\widetilde{C_{\widetilde{z}}}^\star$ of order 16 and $\widetilde{e_i}^\star$ is not conjugate to any involution in any elementary abelian subgroup of order 16 in $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{e}^\star)$. Therefore a Sylow 2-subgroup of $C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{e}^\star)$ is not isomorphic to a Sylow 2-subgroup of $C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{r}^\star)$. But this is a contradiction to $\widetilde{e}^\star$ being conjugate to $\widetilde{r}^\star$ in $\widetilde{C_{\widetilde{z}}}^\star$. Therefore this case does not happen.

So we have that 3 does not divide the order of $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{r}^\star)$ and therefore $\widetilde{r}^\star$ is a 2-central involution in $\widetilde{C_{\widetilde{z}}}^\star$. We have that $C_{\widetilde{Y}^\star}(\widetilde{e}^\star) = \langle \widetilde{r_1}^\star \rangle \times \langle \widetilde{r_2}^\star \rangle$ where $\langle \widetilde{r_i}^\star \rangle \in P(\widetilde{Y}^\star)$ such that $\langle \widetilde{r_i}^\star \rangle$ is not a 2-central involution in $\widetilde{C_{\widetilde{z}}}^\star$, for $i = 1, 2$, and $\langle \widetilde{r_1}^\star \widetilde{r_2}^\star \rangle$ is a 2-central involution in $\widetilde{C_{\widetilde{z}}}^\star$. Set $\widetilde{x}^\star = \widetilde{r_1}^\star \widetilde{r_2}^\star$. Then by our assumption $\widetilde{e}^\star$ is conjugate to $\widetilde{x}^\star$ in $\widetilde{C_{\widetilde{z}}}^\star$. Let $\widetilde{X}^\star$ be a Sylow 2-subgroup of $C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{e}^\star)$ which contains $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{e}^\star)$ and $\widetilde{X_1}^\star \le \widetilde{X}^\star$ be an elementary abelian subgroup of order 16 in $\widetilde{X}^\star$. Let $\widetilde{e}^\star = \widetilde{e_1}^\star \widetilde{e_2}^\star$ with $\widetilde{e_i}^\star \notin \widetilde{Y}^\star$ for $i = 1, 2$. Then we have that $3||N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star(\widetilde{e_i}^\star)|$ and so $\langle \widetilde{e_i}^\star \rangle$ is not conjugate to an element of $P(\widetilde{Y}^\star)$ in $\widetilde{C_{\widetilde{z}}}^\star$. So $\widetilde{e_i}^\star \notin \widetilde{X_1}^\star$. By 3.2.11 $|C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{e_1}^\star)|_2 = 16$ and $\widehat{P} = C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{e_1}^\star) / \langle \widetilde{e_1}^\star \rangle$ contains a nilpotent normal subgroup $\widehat{P_1}$ such that $\widehat{P}/\widehat{P_1} \cong S_3$. Since $C_{N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star}(\widetilde{e_1}^\star) / \left\langle \widetilde{e_1}^\star, C_{\widetilde{Y}^\star}(\widetilde{e_1}^\star) \right\rangle \cong S_3$, we have $\left\langle C_{\widetilde{Y}^\star}(\widetilde{e_1}^\star), \widetilde{e_1}^\star \right\rangle / \langle \widetilde{e_1}^\star \rangle \le \widehat{P_1}$ and therefore $\left\langle C_{\widetilde{Y}^\star}(\widetilde{e_1}^\star), \widetilde{e_1}^\star \right\rangle = O_2(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{e_1}^\star))$. By a similar argument we can show that $\left\langle C_{\widetilde{X_1}^\star}(\widetilde{e_1}^\star), \widetilde{e_1}^\star \right\rangle = O_2(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{e_1}^\star))$. Therefore we have that $\widetilde{e}^\star \in C_{\widetilde{X_1}^\star}(\widetilde{e_1}^\star) \le \left\langle C_{\widetilde{Y}^\star}(\widetilde{e_1}^\star), \widetilde{e_1}^\star \right\rangle$. This gives us that $\widetilde{e_2}^\star \in O_2(C_{\widetilde{C_{\widetilde{z}}}^\star}(\widetilde{e_1}^\star))$ which is a contradiction to our assumptions $\widetilde{e_2}^\star \notin \widetilde{Y}^\star$. So this case does not happen and hence $\widetilde{Y}^\star$ is strongly closed in $N_{\widetilde{C_{\widetilde{z}}}}(\widetilde{Y})^\star$ with respect to $\widetilde{C_{\widetilde{z}}}^\star$. $\square$

Lemma 3.2.13 $\widetilde{C_{\tilde{z}}} = N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})$.

Proof: Set $\widetilde{H}^\star = \left\langle (\widetilde{Y}^\star)^{\widetilde{C_{\tilde{z}}}^\star} \right\rangle$. Assume that 3 does not divide the order of $\widetilde{H}^\star$. Then as $\widetilde{H}^\star \cap N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})^\star$ is normal in $N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})^\star$, by the structure of $N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})$ in 3.2.8 we get that $\widetilde{H}^\star \cap N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})^\star = \widetilde{Y}^\star$. Therefore $\widetilde{Y}^\star \in Syl_2(\widetilde{H}^\star)$ and so $\widetilde{Y}^\star \leq Z(N_{\widetilde{H}^\star}(\widetilde{Y}^\star))$ and Burnside's p-complement theorem gives us that $\widetilde{H}^\star = O(\widetilde{H}^\star)\widetilde{Y}^\star$. Now by the Frattini argument $\widetilde{C_{\tilde{z}}}^\star = O(\widetilde{H}^\star)N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})^\star$. Since 3 does not divide the order of $\widetilde{H}^\star$, we get that $O(\widetilde{H}^\star)$ is a $\widetilde{W}^\star$- invariant $3'$-subgroup of $\widetilde{C_{\tilde{z}}}^\star$, so by 3.2.5(i) either $O(\widetilde{H}^\star) = 1$ or $O(\widetilde{H}^\star) = \widetilde{Y}^\star$ and hence $\widetilde{C_{\tilde{z}}} = N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})$.

Now assume $3||\widetilde{H}^\star|$. We recall that by 3.2.3 $\widetilde{W}$ is a Sylow 3-subgroup of $\widetilde{C_{\tilde{z}}}$. If $|\widetilde{H}^\star|_3 = 3$, then either some element conjugate to $\widetilde{h_2}^\star$ or $\widetilde{h_1}^\star$ is in $\widetilde{H}^\star$ or some element conjugate to $\widetilde{t}^\star$ is in $\widetilde{H}^\star$. Let $\widetilde{h_2}^\star$ or $\widetilde{h_1}^\star$ be in $\widetilde{H}^\star$. Then by 3.2.4 and as $\widetilde{H}^\star \cap N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})^\star$ is normal in $N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})^\star$, by the structure of $N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})$ in 3.2.8 we get that $\widetilde{H}^\star$ is a group whose Sylow 3-subgroup is of order three and it is self-centralizing. Now by [FT] and as $|\widetilde{H}^\star|_2 \geq 16$, we get that $O_2(\widetilde{H}^\star) \neq 1$. Since $\widetilde{H}^\star$ is normal in $\widetilde{C_{\tilde{z}}}^\star$, we deduce that $O_2(\widetilde{H}^\star)$ is $\widetilde{N}^\star$-invariant and therefore $O_2(\widetilde{H}^\star) = \widetilde{Y}^\star$. Hence $\widetilde{H}^\star \leq N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})^\star$ and by the Frattini argument we get that $\widetilde{C_{\tilde{z}}} = N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})$.

Let $\widetilde{t}^\star \in \widetilde{H}^\star$, then as by 3.2.3 $\widetilde{t}^\star$ is conjugate to $\widetilde{h}^\star$ in $N_{\widetilde{C_{\tilde{z}}}}(\widetilde{W})^\star$ and by 3.2.6 $N_{\widetilde{C_{\tilde{z}}}}(\widetilde{W})^\star \leq N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})^\star$, we have $\widetilde{W}^\star = \left\langle \widetilde{h}^\star, \widetilde{t}^\star \right\rangle \leq \widetilde{H}^\star$ and a Sylow 3-subgroup of $\widetilde{H}^\star$ is of order 9. Let $\widetilde{H_1}^\star$ be a minimal normal subgroup of $\widetilde{H}^\star$. As $\widetilde{W}^\star \leq \widetilde{H}^\star$, by 3.2.6(iii) we have $O_{3'}(\widetilde{H}^\star)$ is a subgroup of $\widetilde{Y}^\star$. Let $\widetilde{H_1}^\star$ be a $3'$-group. Then $\widetilde{H_1}^\star \leq O_{3'}(\widetilde{H}^\star) \leq \widetilde{Y}^\star$ and as $\widetilde{W}^\star \leq \widetilde{H}^\star$, we get that $\widetilde{H_1}^\star = \widetilde{Y}^\star$ or $\widetilde{H_1}^\star$ is an elementary abelian group of order 4. Therefore either $C_{\widetilde{H}^\star}(\widetilde{H_1}^\star) \leq N_{\widetilde{H}^\star}(\widetilde{Y}^\star)$ or $C_{\widetilde{H}^\star}(\widetilde{H_1}^\star)/\widetilde{H_1}^\star$ is a group whose Sylow 3-subgroup is of order 3 and is self-centralizing. In the first case by the Frattini argument we have that $\widetilde{C_{\tilde{z}}} = N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})$ and so the lemma holds. In the second case by [FT] we get that $C_{\widetilde{H}^\star}(\widetilde{H_1}^\star)/\widetilde{H_1}^\star$ is isomorphic to A_5 or $L_3(2)$. Since $|Out(A_5)| = |Out(L_3(2))| = 2$ ([AT],pages 2,3), $\widetilde{W}^\star \leq \widetilde{H}^\star$ and by 3.2.1 and 3.2.4 there is no subgroup isomorphic to A_5 or $L_3(2)$ in the centralizer of each element of order three of $\widetilde{W}^\star$, this case does not happen. Hence 3 divides the order of $\widetilde{H_1}^\star$. Since there is no section isomorphic to a non abelian simple group in the centralizer of each element of $\widetilde{W}^\star$ of order three, $\widetilde{H_1}^\star$ is a simple group and it is the unique minimal normal subgroup in $\widetilde{H}^\star$. As $\widetilde{H_1}^\star \cap N_{\widetilde{H}^\star}(\widetilde{Y}^\star)$ is normal in $N_{\widetilde{H}^\star}(\widetilde{Y}^\star)$, we have $\widetilde{H_1}^\star \cap \widetilde{Y}^\star \neq 1$. Now by 3.2.12, $\widetilde{H_1}^\star$ satisfies the conditions of the Goldschmidt's Theorem [Go]. By [Go] we get that $\widetilde{H_1}^\star$ is isomorphic to one of the groups $L_2(2^n)$ for $n \geq 3$, $Sz(2^{n+1})$, for $n \geq 1$, $U_3(2^n)$, for $n \geq 2$, $L_2(q)$ for $q \equiv 3, 5 \pmod 8$, J_1 (the smallest Janko group) or a group of Ree type. By 2.3.9 and 2.3.10 and as $|\widetilde{H_1}^\star|_3 = 3$ or 9 and $|\widetilde{H_1}^\star \cap \widetilde{Y}^\star|$ is of order 4 or 16, we get that the only possibility for $\widetilde{H_1}^\star$ is to be isomorphic to $L_2(4) \cong A_5$. But as $\widetilde{W}^\star \leq \widetilde{H}^\star$, $|Out(A_5)| = 2$ ([AT],page 2) and by 3.2.4 and 3.2.1 there is no subgroup isomorphic to A_5 in the centralizer of some element of order three of $\widetilde{W}^\star$, this case does not happen and hence the lemma is proved. □

Theorem 3.2.14 $\overline{M} \cong 3U_4(3)$.

Proof: We have that $\widetilde{C_{\tilde{z}}}$ is isomorphic to the centralizer of an involution in $U_4(3)$ (we remark that the structure of the centralizer of an involution in $U_4(3)$ has completly determined in [Ph]). Let $\widetilde{X}$ be a subgroup of index 2 in $\widetilde{M}$. We have that $\widetilde{M}$ contains a perfect subgroup $\widetilde{M_1} = N_{\widetilde{M}}(\widetilde{E})$ which by 2.4.3(i) $\widetilde{M_1}/\widetilde{E} \cong A_6$ and by 3.2.2(iii) $\widetilde{M_1}$ contains the normalizer of a Sylow 3-subgroup of $\widetilde{M}$. Now we have that $\widetilde{M_1} = \widetilde{M_1}' \leq \widetilde{M}' \leq \widetilde{X}$ and by the Frattini argument we get that $\widetilde{M} = \widetilde{X}$. But this is a contradiction to $\widetilde{X}$ being of index 2 in $\widetilde{M}$. Hence there is no subgroup of index 2 in $\widetilde{M}$. Now the theorem follows from 2.3.5. □

3.3 Identifying Suz

In this section we shall find the structure of C_a. We will show that $C_a \cong 3Suz$. We recall our notations:

- $Z(O_3(H_1)) = \langle t \rangle$, $t = abc$ and $C_a = C_G(a)$ where $\langle a \rangle$,$\langle b \rangle$ and $\langle c \rangle$ are in L.
- We use the bar notation for $\overline{C_a} = C_a/\langle a \rangle$ and $\overline{M} = C_{\overline{C_a}}(\bar{b})$.
- We have J_1 of length 11 and J_2 of length 110 are the orbits of $C_{H_2/E}(a)$ on $P(\overline{E})$.
- $\overline{R_1} = O_3(C_{\overline{C_a}}(\bar{t}))$. By 3.1.2 $\overline{R_1}$ is a special 3-group of order 3^6, $\langle \bar{b}, \bar{c} \rangle = Z(\overline{R_1})$ and $\bar{t} = \bar{b}\bar{c}$, where $\langle \bar{c} \rangle$ and $\langle \bar{b} \rangle$ are in the orbit J_1 and $\langle \bar{t} \rangle$ is in the orbit J_2.
- We have $\widetilde{M} = \overline{M}/\langle \bar{b} \rangle$ and $\tilde{z} \in C_{\widetilde{M}}(\tilde{t})$ is an involution such that $\tilde{z}\widetilde{R_1} \in Z(C_{\widetilde{M}}(\tilde{t})/\widetilde{R_1})$.

Further notations: By 3.1.2 we have that $C_{\overline{C_a}}(\bar{t})/\overline{R_1} \cong SL_2(3) \times Z_2$. Let

$\overline{z} \in C_{\overline{C_a}}(\overline{t})$ be an involution such that $\overline{z}$ acts trivially on $Z(\overline{R_1})$. Then $\overline{z}$ is the preimage of $\widetilde{z}$. Set

$$\overline{W} = C_{\overline{E}}(\overline{z}) \text{ and } \overline{C_{\overline{z}}} = C_{\overline{C_a}}(\overline{z}).$$

We have the following lemma.

Lemma 3.3.1 *i)* $|\overline{R_1} \cap \overline{E}| = 3^4$.

ii) $|\overline{W}| = 27$.

iii) Let $\langle \overline{x} \rangle \in (J_1 \cap P(\overline{W}))$, *then* $C_{\overline{C_{\overline{z}}}}(\overline{x}) / \langle \overline{x} \rangle$ *is an extension of an extraspecial 2-group of order 32 by* $(S_3 \times S_3)$.

Proof: Part i) follows from the 3.1.1(i). Since $\overline{z}$ is the preimage of $\widetilde{z}$, ii) follows from 3.2.2(ii). Part iii) follows from 3.2.14 and ([AT],page 52). □

Since $\langle a \rangle \in L$, by 2.4.2(i) we have that $\overline{C_a \cap H_2}/\overline{E} \cong M_{11}$. Now the following lemma follows from ([AT],page 18).

Lemma 3.3.2 $C_{\overline{C_a \cap H_2}}(\overline{z}) = \overline{WX}$ *where* $\overline{X} \cong GL_2(3)$.

By 2.4.5, under the action of $C_{\overline{C_a \cap H_2}}(\overline{z}) / \langle \overline{W}, \overline{z} \rangle$ on $P(\overline{W})$ we have three orbits $N_i, i = 1, 2, 3$ such that;

$$|N_1| = 3, |N_2| = 4 \text{ and } |N_3| = 6.$$

By 3.2.14 and ([AT],page 52), we have that $|N_{\overline{M} \cap \overline{C_{\overline{z}}}}(\overline{W}) / \langle \overline{W}, \overline{z} \rangle| = 4$, so $\langle \overline{b} \rangle \in N_1$ or $\langle \overline{b} \rangle \in N_3$. By 3.1.2(ii) we get that $C_{N_{\overline{C_a}}(\overline{W})}(\overline{t}, \overline{z}) / \langle \overline{W}, \overline{z} \rangle$ is of order 2, so $\langle \overline{t} \rangle \in N_3$. By 3.2.14 we get that $\overline{t}$ and $\overline{b}$ are not conjugate in $\overline{C_a}$. Therefore $\langle \overline{b} \rangle \in N_1$. By 2.4.3(i) for each four distinct elements $\langle \overline{x_i} \rangle \in J_1$, $i = 1, ..., 4$, we have that $\overline{x_1}$,...,$\overline{x_4}$ are linear independent. Since $\langle \overline{b} \rangle \in J_1$, and by the representations of the elements in the orbit N_2 in 2.4.5 (for $\langle \overline{x} \rangle \in N_2$ we have $\overline{x} = \overline{x_1 x_2 x_3}$, where $\langle \overline{x_i} \rangle \in N_1$ for $i = 1, 2, 3$) we get that N_1 is the orbit of $N_{\overline{C_{\overline{z}}}}(\overline{W}) / \overline{W}$ on $P(\overline{W})$ containing $\langle \overline{b} \rangle$ and $N_2 \subseteq J_2$. Hence $|N_{\overline{C_{\overline{z}}}}(\overline{W}) / \langle \overline{W}, \overline{z} \rangle| = 24$.

Lemma 3.3.3 *Let* $\overline{F_1} \in Syl_3(C_{\overline{C_a \cap H_2}}(\overline{z}))$, *then*

i) $C_{\overline{C_a \cap H_2}}(\overline{z}) = N_{\overline{C_{\overline{z}}}}(\overline{W})$.

ii) $\overline{W} = J(\overline{F_1})$.

iii) $\overline{F_1} \in Syl_3(\overline{C_{\overline{z}}})$.

Proof: We have that $|N_{\overline{C_{\overline{z}}}}(\overline{W})/\left\langle \overline{W},\overline{z}\right\rangle| = 24$ and $|C_{\overline{C_a\cap H_2}}(\overline{z})/\left\langle \overline{W},\overline{z}\right\rangle| = 24$. Since $C_{\overline{C_a\cap H_2}}(\overline{z})/\left\langle \overline{W},\overline{z}\right\rangle$ is a subgroup of $N_{\overline{C_{\overline{z}}}}(\overline{W})/\left\langle \overline{W},\overline{z}\right\rangle$, we get that $C_{\overline{C_a\cap H_2}}(\overline{z}) = N_{\overline{C_{\overline{z}}}}(\overline{W})$. We have $C_{\overline{C_a\cap H_2}}(\overline{z})/\left\langle \overline{W},\overline{z}\right\rangle = \overline{X}$ where $\overline{X} \cong S_4$. Let $\overline{x} \in \overline{X}$ be an element of order three. As $\langle \overline{x}, \overline{x}^{\overline{y}}\rangle \cong Alt(4)$ for some element $\overline{y} \in \overline{X}$ and $\overline{W}$ is a faithful, irreducible $\overline{X}$-module, we get $|C_{\overline{W}}(\overline{x})| \leq 3$. Therefore $\overline{W} = J(\overline{F_1})$ and ii) holds. Part iii) follows from i) and ii). □

We note that $N_1 \subseteq J_1$, $N_2 \cup N_3 \subseteq J_2$ and $\left\langle \overline{t}\right\rangle \in N_3$. Let $\langle \overline{x}\rangle \in J_2 \cap P(\overline{W})$ and let $\overline{z}$ act trivially on $Z(O_3(C_{\overline{C_a}}(\overline{x})))$. By 3.1.2 and as there is an involution α in $C_{\overline{C_a\cap H_2}}(\overline{x})$ which does not act trivially on $Z(O_3(C_{\overline{C_a}}(\overline{x})))$, we get that $C_{C_{\overline{C_a}}(\overline{x})}(Z(O_3(C_{\overline{C_a}}(\overline{x}))))/O_3(C_{\overline{C_a}}(\overline{x})) \cong SL_2(3)$. Therefore $C_{\overline{C_{\overline{z}}}}(\overline{x})/Z(O_3(C_{\overline{C_a}}(\overline{x}))) \cong SL_2(3)\times 2$. Hence $|N_{C_{\overline{C_{\overline{z}}}}(\overline{x})}(\overline{W})/\left\langle \overline{W},\overline{z}\right\rangle| = 2$ and this gives us that $\langle \overline{x}\rangle \notin N_2$. Therefore for $\langle \overline{x}\rangle \in N_2$ we have $\overline{z}$ does not act trivially on $Z(O_3(C_{\overline{C_a}}(\overline{x})))$.

Lemma 3.3.4 *Let $\langle \overline{x}\rangle$ be in the orbit N_2. Then*

i) $O_3(C_{\overline{C_{\overline{z}}}}(\overline{x}))$ is an extraspecial 3-group of order 27.

ii) $C_{\overline{C_{\overline{z}}}}(\overline{x})/O_3(C_{\overline{C_{\overline{z}}}}(\overline{x})) \cong SL_2(3) \times Z_2$.

iii) $O_2(C_{\overline{C_{\overline{z}}}}(\overline{x})) = \langle \overline{z}\rangle$.

iv) $\overline{x}$ is not conjugate to its inverse in $\overline{C_{\overline{z}}}$.

Proof: Let $\overline{t}^{\overline{g}} = \overline{x}$ where $\overline{g} \in \overline{C_a}$. Then by 3.1.2 $C_{\overline{C_{\overline{z}}}}(\overline{x})/C_{\overline{R_1}^{\overline{g}}}(\overline{z}) \cong SL_2(3) \times Z_2$. As $C_{\overline{R_1}^{\overline{g}}}(\overline{z})$ is normal in $C_{\overline{C_{\overline{z}}}}(\overline{x})$, $\overline{z}$ does not act trivially on $Z(\overline{R_1}^{\overline{g}})$ and $\overline{W}$ is a subgroup of $C_{\overline{C_{\overline{z}}}}(\overline{x})$, we get that $C_{\overline{R_1}^{\overline{g}}}(\overline{z})$ is an extraspecial 3-group of order 3^3 or 3^5. By 3.3.3(iii) we get that the order of a Sylow 3-subgroup of $\overline{C_{\overline{z}}}$ is 3^4. Therefore $C_{\overline{R_1}^{\overline{g}}}(\overline{z})$ is an extraspecial 3-group of order 3^3 and hence i) and ii) hold. As $C_{C_{\overline{C_{\overline{z}}}}(\overline{x})}(C_{\overline{R_1}^{\overline{g}}}(\overline{z})) = \langle \overline{z}\rangle$, iii) holds. Let $\overline{y} \in \overline{C_{\overline{z}}}$ and $\overline{x}^{\overline{y}} = \overline{x}^{-1}$, then $\overline{y} \in N_{\overline{C_{\overline{z}}}}(\langle \overline{x}\rangle)$. Now by the structure of $C_{\overline{C_{\overline{z}}}}(\overline{x})$ in i) and ii) we get that $N_{C_{\overline{C_{\overline{z}}}}(\overline{x})}(\overline{W}) < N_{\left\langle C_{\overline{C_{\overline{z}}}}(\overline{x}),\overline{y}\right\rangle}(\overline{W})$. By i) and ii) we have that $[C_{\overline{C_{\overline{z}}}}(\overline{x}) : N_{C_{\overline{C_{\overline{z}}}}(\overline{x})}(\overline{W})] = 4$. As $|N_2| = 4$, we have $N_{N_{\overline{C_{\overline{z}}}}(\overline{x})}(\overline{W}) \leq C_{\overline{C_{\overline{z}}}}(\overline{x})$ and hence $N_{\left\langle C_{\overline{C_{\overline{z}}}}(\overline{x}),\overline{y}\right\rangle}(\overline{W}) \leq N_{C_{\overline{C_{\overline{z}}}}(\overline{x})}(\overline{W})$. But this is a contradiction to $N_{\left\langle C_{\overline{C_{\overline{z}}}}(\overline{x}),\overline{y}\right\rangle}(\overline{W}) > N_{C_{\overline{C_{\overline{z}}}}(\overline{x})}(\overline{W})$. So iv) holds. □

Lemma 3.3.5 *Let $\overline{K}$ be a $\overline{W}$-invariant $3'$-subgroup of $\overline{C_{\overline{z}}}$. Then*

i) $\overline{K}$ is a 2-group and $|\overline{K}| \leq 2^7$.

ii) For each element $\overline{x} \in \overline{W}^{\sharp}$ we have $C_{\overline{K}}(\overline{x}) = \langle \overline{z}\rangle$ or $C_{\overline{K}}(\overline{x})$ is an extraspecial 2-group.

Proof: As $\overline{K}$ is $\overline{W}$-invariant, by coprime action we have

$$\overline{K} = \left\langle C_{\overline{K}}(\overline{x}), \overline{x} \in \overline{W}^{\sharp} \right\rangle.$$

Let $\overline{x} = \overline{b}$, then by 3.3.1(iii), we have $C_{\overline{C_{\overline{z}}}}(\overline{x})/\langle \overline{x} \rangle$ is an extension of an extraspecial 2-group of order 2^5 by $(S_3 \times S_3)$. Since $(O_2(C_{\overline{C_{\overline{z}}}}(\overline{x})) \cap K)/\langle \overline{z} \rangle$ is $\overline{W}$-invariant, either $\overline{K} \cap C_{\overline{C_{\overline{z}}}}(\overline{x}) = \langle \overline{z} \rangle$ or $\overline{K} \cap C_{\overline{C_{\overline{z}}}}(\overline{x}) = O_2(C_{\overline{C_{\overline{z}}}}(\overline{x}))$ or $\overline{K} \cap C_{\overline{C_{\overline{z}}}}(\overline{x}) \cong Q_8$. Let $\overline{x} = \overline{t}$. We have that $\overline{z}$ acts trivially on $Z(\overline{R_1})$, $C_{\overline{C_a}}(\overline{x}, \overline{z})/Z(\overline{R_1}) \cong SL_2(3) \times 2$ and $C_{\overline{C_a}}(\overline{x}, \overline{z})$ contains an involution α such that $\overline{b}^{\alpha} = \overline{c}$. So $\overline{K} \cap C_{\overline{C_{\overline{z}}}}(\overline{x}) = \langle \overline{z} \rangle$ or $\overline{K} \cap C_{\overline{C_{\overline{z}}}}(\overline{x}) = O_2(C_{\overline{C_{\overline{z}}}}(\overline{x})) \cong Q_8$. Let $\langle \overline{x} \rangle \in N_2$, then by 3.3.4 we have $\overline{K} \cap C_{\overline{C_{\overline{z}}}}(\overline{x}) = \langle \overline{z} \rangle$.

Therefore either for each element $\overline{x} \in \overline{W}^{\sharp}$ we have $C_{\overline{K}}(\overline{x}) = \langle \overline{z} \rangle$ and then $\overline{K} = \langle \overline{z} \rangle$ or $C_{\overline{K}}(\overline{x})$ is an extraspecial 2-group. So $\overline{K}$ is a 2-group. In the latter case by Wielandt's order formula ([BH],XI.12.6) we get that $\left|\overline{K}/\langle \overline{z} \rangle\right|^{13-1} \leq (2^4)^{3\times3}(2^2)^{6\times3}$. Hence $\left|\overline{K}\right| \leq 2^7$ and the lemma is proved. $\square$

Let $\overline{u} \in \overline{W}$ be an element of order three such that $\langle \overline{u} \rangle \in N_1$ and $\overline{u} \notin \langle \overline{b}, \overline{c} \rangle$. Set $\overline{K} = O_2(C_{\overline{C_{\overline{z}}}}(\overline{t}))O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}))$, we are going to show that $\overline{K}$ is an extraspecial 2-group of order 2^7.

Lemma 3.3.6 *i)* $O_2(C_{\overline{C_{\overline{z}}}}(\overline{u})) = O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}, \overline{b}))O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}, \overline{c}))$

ii) $\overline{K} = O_2(C_{\overline{C_{\overline{z}}}}(\overline{b}))O_2(C_{\overline{C_{\overline{z}}}}(\overline{c}))$ *and* $\overline{K}$ *is an extraspecial 2-group of order* 2^7.

Proof: By 3.3.1(iii) $O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}))$ is an extraspecial 2-group of order 2^5. We have that $O_2(C_{\overline{C_{\overline{z}}}}(\overline{t}))$ is an extraspecial 2-group of order 2^3 and $C_{\overline{C_{\overline{z}}}}(\overline{u})/\langle \overline{u}, O_2(C_{\overline{C_{\overline{z}}}}(\overline{u})) \rangle \cong (S_3 \times S_3)$. Since $\overline{u} \notin \langle \overline{b}, \overline{c} \rangle$, we have $\langle \overline{b}, \overline{c}, \overline{u} \rangle = \overline{W}$. We have $O_2(C_{\overline{C_{\overline{z}}}}(\overline{t})) \leq (O_2(C_{\overline{C_{\overline{z}}}}(\overline{b})) \cap O_2(C_{\overline{C_{\overline{z}}}}(\overline{c})))$ (we remark that $O_2(C_{\widetilde{M}}(\widetilde{t}, \widetilde{z}))$ was a subgroup of $\widetilde{Y} = O_2(C_{\widetilde{M}}(\widetilde{z}))$ and $\overline{t}$ is the preimage of $\widetilde{t}$.) and so $O_2(C_{\overline{C_{\overline{z}}}}(\overline{t})) \cap O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}))$ is a subgroup of $C_{\overline{C_{\overline{z}}}}(\overline{W}) = \langle \overline{z} \rangle$. Hence $O_2(C_{\overline{C_{\overline{z}}}}(\overline{t})) \cap O_2(C_{\overline{C_{\overline{z}}}}(\overline{u})) = \langle \overline{z} \rangle$. As $\langle \overline{b} \rangle$ and $\langle \overline{c} \rangle$ are two 3-central elements in $C_{\overline{C_a}}(\overline{u})$, so by 3.2.14 and ([AT],page 52)we get that $O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}, \overline{b})) \cong O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}, \overline{c})) \cong Q_8$ and $O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}, \overline{b}))$ and $O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}, \overline{c}))$ both are subgroups of $O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}))$. We have $O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}, \overline{b})) \cap O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}, \overline{c})) \leq O_2(C_{\overline{C_{\overline{z}}}}(\overline{t})) \cap O_2(C_{\overline{C_{\overline{z}}}}(\overline{u})) = \langle \overline{z} \rangle$ and hence $|O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}))| = |O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}, \overline{b}))O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}, \overline{c}))| = 2^5$. Therefore $O_2(C_{\overline{C_{\overline{z}}}}(\overline{u})) = O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}, \overline{b}))O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}, \overline{c}))$ and now i) holds.

We have $O_2(C_{\overline{C_{\overline{z}}}}(\overline{t})) \leq (O_2(C_{\overline{C_{\overline{z}}}}(\overline{b})) \cap O_2(C_{\overline{C_{\overline{z}}}}(\overline{c})))$ and $O_2(C_{\overline{C_{\overline{z}}}}(\overline{t})) \cap O_2(C_{\overline{C_{\overline{z}}}}(\overline{u})) = \langle \overline{z} \rangle$, so by i) we get that $O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}))$ normalizes $O_2(C_{\overline{C_{\overline{z}}}}(\overline{t}))$ and then $O_2(C_{\overline{C_{\overline{z}}}}(\overline{t}))O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}))$

is an extraspecial 2-group of order 2^7. Also since $O_2(C_{\overline{C_{\overline{z}}}}(\overline{t})) \cap O_2(C_{\overline{C_{\overline{z}}}}(\overline{u})) = \langle \overline{z} \rangle$, we get that $O_2(C_{\overline{C_{\overline{z}}}}(\overline{b})) = O_2(C_{\overline{C_{\overline{z}}}}(\overline{b}, \overline{u}))O_2(C_{\overline{C_{\overline{z}}}}(\overline{t}))$ and $O_2(C_{\overline{C_{\overline{z}}}}(\overline{c})) = O_2(C_{\overline{C_{\overline{z}}}}(\overline{c}, \overline{u}))O_2(C_{\overline{C_{\overline{z}}}}(\overline{t}))$. Now by i) we have that $O_2(C_{\overline{C_{\overline{z}}}}(\overline{b}))O_2(C_{\overline{C_{\overline{z}}}}(\overline{c})) = O_2(C_{\overline{C_{\overline{z}}}}(\overline{t}))O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}))$ and ii) holds. $\square$

We note that $|N_1| = 3$ so $N_1 = \{\langle \overline{b} \rangle, \langle \overline{c} \rangle, \langle \overline{u} \rangle\}$. By 3.3.6 we get that for each $\langle \overline{x} \rangle \in N_1$, $O_2(C_{\overline{C_{\overline{z}}}}(\overline{x}))$ is a subgroup of $\overline{K}$. By the structures of the centralizers of the elements of order three in $\overline{W}$ in 3.1.2, 3.3.1 and 3.3.4, we get that any $\overline{W}$-invariant $3'$-subgroup in $\overline{C_{\overline{z}}}$ say $\overline{X}$ is a 2-group (see 3.3.5(i)) and it is generated by $\{\overline{X} \cap O_2(C_{\overline{C_{\overline{z}}}}(\overline{x})), \overline{x} \in \overline{W}^{\sharp}\}$. We remark that for $\langle \overline{x} \rangle \in N_3$ we have $O_2(C_{\overline{C_{\overline{z}}}}(\overline{x}))$ centralizes $Z(O_3(C_{\overline{C_{\overline{z}}}}(\overline{x})))$, $Z(O_3(C_{\overline{C_{\overline{z}}}}(\overline{x})))$ contains some elements $\overline{y}$ and $\overline{r}$ such that $\langle \overline{y} \rangle$ and $\langle \overline{r} \rangle$ are from the orbit N_1 with $\overline{x} = \overline{yr}$ and $O_2(C_{\overline{C_{\overline{z}}}}(\overline{x})) \leq O_2(C_{\overline{C_{\overline{z}}}}(\overline{y})) \cap O_2(C_{\overline{C_{\overline{z}}}}(\overline{r}))$. Also for $\langle \overline{x} \rangle \in N_3$ we have $C_{\overline{C_{\overline{z}}}}(\overline{x})/O_2(C_{\overline{C_{\overline{z}}}}(\overline{x}))$ is an elementary abelian group of order 27.

Lemma 3.3.7 *i) $\overline{K}$ is the unique maximal $\overline{W}$-invariant $3'$-subgroup of $\overline{C_{\overline{z}}}$.*

ii) $N_{\overline{C_{\overline{z}}}}(\overline{W}) \leq N_{\overline{C_{\overline{z}}}}(\overline{K})$.

iii) Let $\overline{x} \in \overline{W}$ be an element of order three, then $O_2(C_{\overline{C_{\overline{z}}}}(\overline{x})) \leq \overline{K}$.

iv) Let $\overline{W_1}$ be a subgroup of $\overline{W}$ of order 9 such that $|P(\overline{W_1}) \cap N_3| = 3$ and $P(\overline{W_1}) \cap N_1 = \varnothing$, then $\overline{K} = \left\langle O_2(C_{\overline{C_{\overline{z}}}}(\overline{x})) | \overline{x} \in \overline{W_1}^{\sharp} \right\rangle$.

Proof: By 3.3.5(i), $\overline{K}$ is a maximal $\overline{W}$-invariant $3'$-subgroup of $\overline{C_{\overline{z}}}$ and a $\overline{W}$-invariant $3'$-subgroup of $\overline{C_{\overline{z}}}$ is a 2-group. Let $\langle \overline{x} \rangle \in N_2$ then by 3.3.4 $O_2(C_{\overline{C_{\overline{z}}}}(\overline{x})) = \langle \overline{z} \rangle$ and so $O_2(C_{\overline{C_{\overline{z}}}}(\overline{x}))$ is a subgroup of $\overline{K}$. Let $\langle \overline{x} \rangle \in N_3$. Then $O_2(C_{\overline{C_{\overline{z}}}}(\overline{x}))$ centralizes $Z(O_3(C_{\overline{C_{\overline{z}}}}(\overline{x})))$. We have $Z(O_3(C_{\overline{C_{\overline{z}}}}(\overline{x})))$ contains some elements $\overline{y}$ and $\overline{r}$ such that $\langle \overline{y} \rangle$ and $\langle \overline{r} \rangle$ are from the orbit N_1 and $\overline{x} = \overline{yr}$ and $O_2(C_{\overline{C_{\overline{z}}}}(\overline{x})) \leq O_2(C_{\overline{C_{\overline{z}}}}(\overline{y})) \cap O_2(C_{\overline{C_{\overline{z}}}}(\overline{r}))$. By 3.3.6 we have $O_2(C_{\overline{C_{\overline{z}}}}(\overline{d})) \leq \overline{K}$ for each element $\langle \overline{d} \rangle \in N_1$. Therefor i) and iii) hold and ii) follows from i).

Let $\overline{W_1}$ be a subgroup of $\overline{W}$ of order 9 such that $|P(\overline{W_1}) \cap N_3| = 3$ and $P(\overline{W_1}) \cap N_1 = \varnothing$. We have $|N_3| = 6$ and so $N_3 = \{\langle \overline{t} \rangle, \langle \overline{ub} \rangle, \langle \overline{u}^{-1}\overline{b} \rangle, \langle \overline{uc} \rangle, \langle \overline{u}^{-1}\overline{c} \rangle, \langle \overline{c}^{-1}\overline{b} \rangle\}$. This gives us that $|P(\overline{W_1}) \cap N_2| = 1$. Since $\overline{W} = \langle \overline{u}, \overline{b}, \overline{c} \rangle$ and $C_{\overline{C_{\overline{z}}}}(\overline{W}) = \langle \overline{z} \rangle$ (see 3.3.3), we get that $O_2(C_{\overline{C_{\overline{z}}}}(\overline{y})) \cap O_2(C_{\overline{C_{\overline{z}}}}(\overline{x})) = \langle \overline{z} \rangle$ for $\langle \overline{x} \rangle \neq \langle \overline{y} \rangle$ where $\langle \overline{x} \rangle$ and $\langle \overline{y} \rangle$ are from $P(\overline{W_1}) \cap N_3$. Now since by 3.1.2 $O_2(C_{\overline{C_{\overline{z}}}}(\overline{x})) \cong Q_8$ for each $\langle \overline{x} \rangle \in N_3$, $|\overline{K}| = 2^7$ and by iii) we get that $\overline{K}$ is generated by $\{O_2(C_{\overline{C_{\overline{z}}}}(\overline{x})), \overline{x} \in \overline{W_1}^{\sharp}\}$ and the lemma is proved. $\square$

Lemma 3.3.8 *Let $\overline{x} \in \overline{W}$ be an element of order three, then $C_{\overline{C_{\overline{z}}}}(\overline{x}) \leq N_{\overline{C_{\overline{z}}}}(\overline{K})$.*

Proof: Let $\langle\overline{x}\rangle \in N_1 \cup N_3$, then by 3.3.1(ii) and 3.1.1 we have $C_{\overline{C_{\overline{z}}}}(\overline{x}) \leq O_2(C_{\overline{C_{\overline{z}}}}(\overline{x}))N_{\overline{C_{\overline{z}}}}(\overline{W})$. Let $\langle\overline{x}\rangle \in N_2$, then there is a subgroup $\overline{X} \leq C_{\overline{C_{\overline{z}}}}(\overline{x})$ such that $\overline{X} \cong Q_8$, $\langle\overline{X}, \overline{y}\rangle \cong SL_2(3)$ with $\overline{y} \in \overline{W}$ is an element of order three and $C_{\overline{C_{\overline{z}}}}(\overline{x}) = \left\langle \overline{X}, N_{C_{\overline{C_{\overline{z}}}}(\overline{x})}(\overline{W}) \right\rangle$. Let $\overline{s} \in \overline{X}$ be an element of order 4 and $\overline{W_1} = \overline{W} \cap O_3(C_{\overline{C_{\overline{z}}}}(\overline{x}))$. By the representation of an element in the orbit N_2 (we remark that $|N_2| = 4$ in 2.4.5 we may assume that $\overline{x} = \overline{u}\overline{b}\overline{c}$. Let $\overline{f} = \overline{s}^2 \in Z(\overline{X})$, then $\overline{f}$ is an involution which acts fixed point freely on $O_3(C_{\overline{C_{\overline{z}}}}(\overline{x}))/\langle\overline{x}\rangle$, $\overline{f}$ normalizes $\overline{W}$ and so $\overline{f}$ does not invert $\overline{b}$ or $\overline{u}$ or $\overline{c}$. As $\overline{f}$ acts fixed point freely on $O_3(C_{\overline{C_{\overline{z}}}}(\overline{x}))/\langle\overline{x}\rangle$, neither $\overline{b}$ or $\overline{c}$ nor $\overline{u}$ is in $O_3(C_{\overline{C_{\overline{z}}}}(\overline{x}))$. We have $|N_2| = 4$ and so $N_2 = \{\langle\overline{c}\overline{u}\overline{b}\rangle, \langle\overline{u}^{-1}\overline{b}\overline{c}\rangle, \left\langle\overline{b}^{-1}\overline{c}\overline{u}\right\rangle, \langle\overline{c}^{-1}\overline{b}\overline{u}\rangle\}$. Since neither $\overline{b}$ or $\overline{c}$ nor $\overline{u}$ is in $\overline{W_1}$, we get that $|P(\overline{W_1}) \cap N_2| = 1$, in fact $P(\overline{W_1}) \cap N_2 = \langle\overline{x}\rangle$. Therefore $\overline{W_1}$ is of order 9 and $|P(\overline{W_1}) \cap N_3| = 3$. Let $\overline{r} \in O_3(C_{\overline{C_{\overline{z}}}}(\overline{x}))$, then $[\overline{s}, \overline{r}] = \overline{d} \in O_3(C_{\overline{C_{\overline{z}}}}(\overline{x}))$. Let $\overline{g} \in O_3(C_{\overline{C_{\overline{z}}}}(\overline{x}))$ such that $\overline{s}\overline{d} = \overline{g}\overline{s}$. Since by 3.3.7(ii) $N_{\overline{C_{\overline{z}}}}(\overline{W}) \leq N_{\overline{C_{\overline{z}}}}(\overline{K})$ and $O_3(C_{\overline{C_{\overline{z}}}}(\overline{x})) \leq N_{\overline{C_{\overline{z}}}}(\overline{W})$, we get that $\overline{K}^{\overline{s}^{\overline{r}}} = \overline{K}^{\overline{s}\overline{d}} = \overline{K}^{\overline{g}\overline{s}} = \overline{K}^{\overline{s}}$. Therefore $\overline{K}^{\overline{s}}$ is a $\overline{W_1}$-invariant $3'$-subgroup of $\overline{C_{\overline{z}}}$. By 3.3.7(iii) for each element $\overline{w}$ of order three in $\overline{W}$ we have $O_2(C_{\overline{C_{\overline{z}}}}(\overline{w}))$ is a subgroup of $\overline{K}$, so by 3.3.7(iv) we get that $\overline{K}^{\overline{s}} = \overline{K}$. Hence $\overline{s} \in N_{\overline{C_{\overline{z}}}}(\overline{K})$ and therefore $\overline{X} \leq N_{\overline{C_{\overline{z}}}}(\overline{K})$. As by 3.3.7(ii) we have $N_{\overline{C_{\overline{z}}}}(\overline{W}) \leq N_{\overline{C_{\overline{z}}}}(\overline{K})$, the lemma is proved. □

We are going to show that $N_{\overline{C_{\overline{z}}}}(\overline{K}) = \overline{C_{\overline{z}}}$.

Lemma 3.3.9 $N_{\overline{C_{\overline{z}}}}(\overline{K})/\overline{K} \cong PSp_4(3)$.

Proof: By 3.3.7(ii) and i) we get that there is an elementary abelian subgroup $\widehat{W}$ of order 27 in $N_{\overline{C_{\overline{z}}}}(\overline{K})/\overline{K}$ such that there is no $\widehat{W}$-invariant $3'$-subgroup in $N_{\overline{C_{\overline{z}}}}(\overline{K})/\overline{K}$. Let $\langle\overline{x}\rangle$ be an element from the orbit N_2. Then by 3.3.4 we get that $C_{\overline{C_{\overline{z}}}}(\overline{x})\overline{K}/\overline{K}$ satisfies the conditions of theorem 2.3.7 and 2.3.6(if we set $\overline{x}\overline{K}/\overline{K} = d$ in 2.3.6, then d is not conjugate to its inverse in $\overline{C_{\overline{z}}}$ by 3.3.4(iv)). Therefore by 2.3.6 we have $N_{\overline{C_{\overline{z}}}}(\overline{K})/\overline{K} \cong PSp_4(3)$. We note that by the structure of $N_{\overline{C_{\overline{z}}}}(\overline{W})$ in 3.3.3(i) and 3.3.2 we get that there is no normal subgroup of index 3 in $N_{\overline{C_{\overline{z}}}}(\overline{W})$ so $N_{\overline{C_{\overline{z}}}}(\overline{K})/\overline{K}$ has no normal subgroup of index 3. □

We recall that for a prime p, a subgroup T of a group H is called *strongly p-embedded* in H if p divides $|T|$ and p does not divide $|T \cap T^g|$ for all $g \in H$ such that $g \notin T$.

Lemma 3.3.10 *$N_{\overline{C_z}}(\overline{K})$ is strongly 3-embedded in $\overline{C_z}$.*

Proof: Let $\overline{x} \in N_{\overline{C_z}}(\overline{K})$ be an element of order three, then by ([AT],page 26), x is conjugate to an element of $\overline{W}$ in $N_{\overline{C_z}}(\overline{K})$. So we may assume that $\overline{x} \in \overline{W}$ and then by 3.3.8 we have that $C_{\overline{C_z}}(\overline{x}) \leq N_{\overline{C_z}}(\overline{K})$. Now assume that 3 divides $\left|N_{\overline{C_z}}(\overline{K}) \cap N_{\overline{C_z}}(\overline{K})^{\overline{g}}\right|$ for some $\overline{g} \in \overline{C_z}$ and let $\overline{X_1} \in Syl_3(N_{\overline{C_z}}(\overline{K}) \cap N_{\overline{C_z}}(\overline{K})^{\overline{g}})$. Let $\overline{X_1} \triangleleft \overline{E_0}$ and $\overline{E_0}$ be a 3-subgroup of $N_{\overline{C_z}}(\overline{K})$. Then for some element $\overline{x} \in \overline{X_1}$ of order three we have $\overline{E_0} \leq C_{\overline{C_z}}(\overline{x})$. So by 3.3.8, $\overline{E_0} \leq N_{\overline{C_z}}(\overline{K})^{\overline{g}}$. Therefore $\overline{X_1} \in Syl_3(N_{\overline{C_z}}(\overline{K})) \cap Syl_3(N_{\overline{C_z}}(\overline{K})^{\overline{g}})$. We may assume that $\overline{W} \leq \overline{X_1}$. Then $N_{\overline{C_z}}(\overline{X_1}) \leq N_{\overline{C_z}}(\overline{W}) \leq N_{\overline{C_z}}(\overline{K}) \cap N_{\overline{C_z}}(\overline{K})^{\overline{g}}$ follows from 3.3.7(ii). Now Sylow's theorem gives that $N_{\overline{C_z}}(\overline{K}) = N_{\overline{C_z}}(\overline{K})^{\overline{g}}$, so $\overline{g} \in N_{\overline{C_z}}(N_{\overline{C_z}}(\overline{K}))$. As $\overline{K}$ is a characteristic subgroup of $N_{\overline{C_z}}(\overline{K})$, we have $\overline{g} \in N_{\overline{C_z}}(\overline{K})$ and hence $N_{\overline{C_z}}(\overline{K})$ is strongly 3-embedded in $\overline{C_z}$. $\square$

We use of the notation * for the natural homomorphism $\overline{C_z} \mapsto \overline{C_z}/\langle \overline{z} \rangle$.

Lemma 3.3.11 *Let $\overline{r}^\star \in \overline{K}^\star$ be an involution and $\overline{g}^\star \in \overline{C_z}^\star$ such that $(\overline{r}^\star)^{\overline{g}^\star} \in N_{\overline{C_z}}(\overline{K})^\star$ and $(\overline{r}^\star)^{\overline{g}^\star} \notin \overline{K}^\star$. Then 3 does not divide the order of $C_{N_{\overline{C_z}}(\overline{K})^\star}((\overline{r}^\star)^{\overline{g}^\star})$.*

Proof: Let $\overline{r}^\star \in \overline{K}^\star$ be an involution and $\overline{g}^\star \in \overline{C_z}^\star$ such that $(\overline{r}^\star)^{\overline{g}^\star} \in N_{\overline{C_z}}(\overline{K})^\star$, $(\overline{r}^\star)^{\overline{g}^\star} \notin \overline{K}^\star$ and 3 divides the order of $C_{N_{\overline{C_z}}(\overline{K})^\star}((\overline{r}^\star)^{\overline{g}^\star})$. By 2.4.7(ii) we get that 3 divides $|C_{N_{\overline{C_z}}(\overline{K})^\star}(\overline{r}^\star)|$. Let $\overline{P}^\star \in Syl_3(C_{N_{\overline{C_z}}(\overline{K})^\star}(\overline{r}^\star))$ and $\overline{P_1}^\star \in Syl_3(C_{N_{\overline{C_z}}(\overline{K})^\star}(\overline{r}^{\overline{g}^\star}))$. Then as by 3.3.10 $N_{\overline{C_z}}(\overline{K})^\star$ is strongly 3-embedded in $\overline{C_z}^\star$, we have $\overline{P}^\star \in Syl_3(C_{\overline{C_z}^\star}(\overline{r}^\star))$ and $\overline{P_1}^\star \in Syl_3(C_{\overline{C_z}^\star}((\overline{r}^\star)^{\overline{g}^\star}))$. Since $\overline{g}^\star \in \overline{C_z}^\star$, we have $(\overline{P}^\star)^{\overline{g}^\star} \in Syl_3(C_{(N_{\overline{C_z}}(\overline{K})^\star)^{\overline{g}^\star}}((\overline{r}^\star)^{\overline{g}^\star}))$ and as by 3.3.10 $(N_{\overline{C_z}}(\overline{K})^\star)^{\overline{g}^\star}$ is strongly 3-embedded in $\overline{C_z}^\star$, we get that $(\overline{P}^\star)^{\overline{g}^\star} \in Syl_3(C_{\overline{C_z}^\star}((\overline{r}^\star)^{\overline{g}^\star}))$. Hence $(\overline{P}^\star)^{\overline{g}^\star \overline{g_1}^\star} = \overline{P_1}^\star$ for some $\overline{g_1}^\star \in C_{\overline{C_z}^\star}((\overline{r}^\star)^{\overline{g}^\star})$. Now we have $\overline{P_1}^\star \leq N_{\overline{C_z}}(\overline{K})^\star \cap (N_{\overline{C_z}}(\overline{K})^\star)^{\overline{g}^\star \overline{g_1}^\star}$ and as by 3.3.10 $N_{\overline{C_z}}(\overline{K})^\star$ is strongly 3-embedded in $\overline{C_z}^\star$, we get that $\overline{g}^\star \overline{g_1}^\star \in N_{\overline{C_z}}(\overline{K})^\star$. Therefore $(\overline{r}^\star)^{\overline{g}^\star \overline{g_1}^\star} \in \overline{K}^\star$. As $\overline{g_1}^\star \in C_{\overline{C_z}^\star}((\overline{r}^\star)^{\overline{g}^\star})$, we have $(\overline{r}^\star)^{\overline{g}^\star} = (\overline{r}^\star)^{\overline{g}^\star \overline{g_1}^\star} \in \overline{K}^\star$. But this is a contradiction to our assumption that $(\overline{r}^\star)^{\overline{g}^\star} \notin \overline{K}^\star$. Hence the lemma is proved. $\square$

Lemma 3.3.12 $\overline{K}^\star$ *is strongly closed in* $N_{\overline{C_{\overline{z}}}}(\overline{K})^\star$ *with respect to* $\overline{C_{\overline{z}}}^\star$.

Proof: Let $\overline{r}^\star \in \overline{K}^\star$ be an involution and $\overline{g}^\star \in \overline{C_{\overline{z}}}^\star$ such that $(\overline{r}^\star)^{\overline{g}^\star} \in N_{\overline{C_{\overline{z}}}}(\overline{K})^\star$ and $(\overline{r}^\star)^{\overline{g}^\star} \notin \overline{K}^\star$. Then by 3.3.11, 2.4.6 and 2.4.7(iii) and ([AT],page 26) we get that the only possibility is that $(\overline{r}^\star)^{\overline{g}^\star}\overline{K}^\star$ is a non 2-central involution and then under the action of $C_{N_{\overline{C_{\overline{z}}}}(\overline{K})^\star}((\overline{r}^\star)^{\overline{g}^\star})$ on the involutions in $(\overline{r}^\star)^{\overline{g}^\star}\overline{K}^\star$ we have two orbits of lengths 4 and 12. Since $(\overline{r}^\star)^{\overline{g}^\star}\overline{K}^\star$ is a non 2-central involution, by ([AT],page 26) and 3.3.11 we get that $|C_{N_{\overline{C_{\overline{z}}}}(\overline{K})^\star}((\overline{r}^\star)^{\overline{g}^\star})| = 2^{4+5}$. Let $\overline{S}^\star$ be a Sylow 2-subgroup of $N_{\overline{C_{\overline{z}}}}(\overline{K})^\star$, $\overline{D}^\star = (\overline{K}^\star)^{\overline{g}^\star}$ and $\overline{T}^\star$ be a Sylow 2-subgroup of $N_{\overline{C_{\overline{z}}}}(\overline{D})^\star$, then $(\overline{r}^\star)^{\overline{g}^\star} \in \overline{D}^\star$. We may assume that $C_{\overline{S}^\star}((\overline{r}^\star)^{\overline{g}^\star})$ is a subgroup of $\overline{T}^\star$. By 2.4.7(i)and ii)) we have that $(\overline{r}^\star)^{\overline{g}^\star} = \overline{x}^\star\overline{k}^\star$ where $\overline{x}^\star \notin \overline{K}^\star$, $1 \neq \overline{k}^\star \in C_{\overline{K}^\star}((\overline{r}^\star)^{\overline{g}^\star})$ and 3 divides the order of the centralizers of $\overline{x}^\star$ and $\overline{k}^\star$ in $N_{\overline{C_{\overline{z}}}}(\overline{K})^\star$. Let $C_{\overline{K}^\star}((\overline{r}^\star)^{\overline{g}^\star})$ be contained in $\overline{D}^\star$. Then $\left\langle(\overline{r}^\star)^{\overline{g}^\star}\right\rangle C_{\overline{K}^\star}((\overline{r}^\star)^{\overline{g}^\star})$ is contained in $\overline{D}^\star$ and so $\overline{x}^\star \in \overline{D}^\star$. This gives us that $\overline{x}^\star$ is conjugate to an involution in $\overline{K}^\star$ which is a contradiction to 3.3.11. Hence $C_{\overline{K}^\star}((\overline{r}^\star)^{\overline{g}^\star})$ is not contained in $\overline{D}^\star$. By ([AT],page 26) we get that the 2-rank of $PSp_4(3)$ is 4. Since there is no elementary abelian group of order 8 in $PSp_4(3)$ all of whose non trivial elements are non 2-central (see 2.4.7(v)), we get that $C_{\overline{K}^\star}((\overline{r}^\star)^{\overline{g}})\overline{D}^\star/\overline{D}^\star$ is of order 2 or 4. Since $|C_{N_{\overline{C_{\overline{z}}}}(\overline{K})^\star}((\overline{r}^\star)^{\overline{g}^\star})| = 2^{4+5}$, $|\overline{D}^\star| = 2^6$ and by 3.3.11 and 2.4.7(i) we have that $C_{\overline{S}^\star}((\overline{r}^\star)^{\overline{g}^\star}) \cap \overline{D}^\star = (\overline{K}^\star \cap \overline{D}^\star)\left\langle(\overline{r}^\star)^{\overline{g}^\star}\right\rangle$, we have $|C_{\overline{S}^\star}((\overline{r}^\star)^{\overline{g}^\star})\overline{D}^\star| \geq 2^{4+5+6-4} = 2^{11}$.

Now set $\overline{A}^\star = C_{\overline{S}^\star}((\overline{r}^\star)^{\overline{g}^\star})\overline{D}^\star$, then $\overline{A}^\star$ is of index at most 2 in $\overline{T}^\star$. Let $\overline{V}^\star \leq C_{\overline{K}^\star}((\overline{r}^\star)^{\overline{g}^\star})$ such that $|\overline{V}^\star| \leq 4$, $\overline{V}^\star \cap \overline{D}^\star = 1$ and $\overline{V}^\star\overline{D}^\star/\overline{D}^\star = C_{\overline{K}^\star}((\overline{r}^\star)^{\overline{g}^\star})\overline{D}^\star/\overline{D}^\star$. We note that by 3.3.11, 2.4.6 and 2.4.7(iii) and ([AT],page 26) we get that $\overline{f}^\star\overline{D}^\star$ is a non 2-central involution in $N_{\overline{C_{\overline{z}}}}(\overline{D})^\star/\overline{D}^\star$ for each involution $\overline{f}^\star \in \overline{V}^\star$(we recall that $\overline{V}^\star \leq \overline{K}^\star$ and $\overline{D}^\star = (\overline{K}^\star)^{\overline{g}^\star}$). Now let $\overline{A}^\star = \overline{T}^\star$, then as $\overline{V}^\star\overline{D}^\star/\overline{D}^\star$ is a normal subgroup of $\overline{A}^\star/\overline{D}^\star$, we get that $Z(\overline{A}^\star/\overline{D}^\star) \cap \overline{V}^\star\overline{D}^\star/\overline{D}^\star \neq 1$. But this gives us that there is an involution in $\overline{V}^\star$ say $\overline{f}^\star$ such that $\overline{f}^\star\overline{D}^\star$ is a 2-central involution in $N_{\overline{C_{\overline{z}}}}(\overline{D})^\star/\overline{D}^\star$ which is a contradiction. Therefore $|\overline{A}^\star| = 2^{11}$ and $\overline{V}^\star$ is of order 2. This gives us that $\left\langle\overline{K}^\star \cap \overline{D}^\star, (\overline{r}^\star)^{\overline{g}^\star}\right\rangle$ is of order 16 and by 2.4.7(iii) we get that $C_{\overline{D}^\star}(\overline{f}^\star) = \left\langle\overline{K}^\star \cap \overline{D}^\star, (\overline{r}^\star)^{\overline{g}^\star}\right\rangle$. As $\overline{f}^\star\overline{D}^\star$ is a non 2-central involution in $N_{\overline{C_{\overline{z}}}}(\overline{D})^\star/\overline{D}^\star$, we have that $\overline{A}^\star/\overline{D}^\star$ is a subgroup of order 2^5 of $C_{N_{\overline{C_{\overline{z}}}}(\overline{D})^\star/\overline{D}^\star}(\overline{f}^\star\overline{D}^\star)$. By the structure of the centralizer of a non 2-central involuton in $PSp_4(3)$ ([AT],page 26) we get that $\overline{A}^\star/\overline{D}^\star$ contains an elementary abelian group of order 16 and $Z(\overline{A}^\star/\overline{D}^\star) = \left\langle\overline{f}^\star\overline{D}^\star\right\rangle$.

Thus by 2.4.7(v) we get the following contradiction.

$$(\overline{r}^\star)^{\overline{g}^\star} \in C_{\overline{D}^\star}(\overline{A}^\star) \le \left[C_{\overline{D}^\star}(\overline{f}^\star), \overline{A}^\star\right] = \left[C_{\overline{D}^\star}(\overline{f}^\star), C_{\overline{S}^\star}((\overline{r}^\star)^{\overline{g}^\star})\overline{D}^\star\right]$$
$$= \left[(\overline{K}^\star \cap \overline{D}^\star)\left\langle(\overline{r}^\star)^{\overline{g}^\star}\right\rangle, C_{\overline{S}^\star}((\overline{r}^\star)^{\overline{g}^\star})\right] = \left[\overline{K}^\star \cap \overline{D}^\star, C_{\overline{S}^\star}((\overline{r}^\star)^{\overline{g}^\star})\right] \le \overline{K}^\star \cap \overline{D}^\star.$$

This contradiction shows that $\overline{K}^\star$ is strongly closed in $N_{\overline{C_{\overline{z}}}}(\overline{K})^\star$ with respect to $\overline{C_{\overline{z}}}^\star$ and hence the lemma is proved. □

Lemma 3.3.13 $\overline{C_{\overline{z}}} = N_{\overline{C_{\overline{z}}}}(\overline{K})$.

Proof: By 3.3.12, $\overline{K}^\star$ is strongly closed in $N_{\overline{C_{\overline{z}}}}(\overline{K})^\star$ with respect to $\overline{C_{\overline{z}}}^\star$. So by Goldschmidt's theorem [Go] we have $\overline{H}^\star = \left\langle(\overline{K}^\star)^{\overline{C_{\overline{z}}}^\star}\right\rangle$ contains no section isomorphic to $\Omega_6^-(2)$. Therefore $N_{\overline{C_{\overline{z}}}}(\overline{K})^\star \cap \overline{H}^\star = \overline{K}^\star$. As $N_{\overline{C_{\overline{z}}}}(\overline{K})^\star = N_{\overline{C_{\overline{z}}}^\star}(\overline{K}^\star)$, we have $\overline{K}^\star \in Syl_2(\overline{H}^\star)$. Hence $\overline{K}^\star \le Z(N_{\overline{H}^\star}(\overline{K}^\star))$ and Burnside's p-complement theorem gives us that $\overline{H}^\star = O(\overline{H}^\star)\overline{K}^\star$. Now by the Frattini argument $\overline{C_{\overline{z}}}^\star = O(\overline{C_{\overline{z}}}^\star)N_{\overline{C_{\overline{z}}}}(\overline{K})^\star$. Since by 3.3.10, $N_{\overline{C_{\overline{z}}}}(\overline{K})^\star$ contains a Sylow 3-subgroup of $\overline{C_{\overline{z}}}^\star$, we conclude that $O(\overline{C_{\overline{z}}}^\star)$ is a $\overline{W}^\star$- invariant $3'$-subgroup of $\overline{C_{\overline{z}}}^\star$. By 3.3.7(i), $O(\overline{C_{\overline{z}}}^\star) = 1$ and hence $\overline{C_{\overline{z}}} = N_{\overline{C_{\overline{z}}}}(\overline{K})$. □

Theorem 3.3.14 $C_a \cong 3Suz$.

Proof: By 3.2.14 $C_{\overline{C_a}}(\overline{u}) \cong 3U_4(3)$ and by ([AT],page 52), $U_4(3)$ has only one class of involutions and so $\overline{z}$ is not weakly closed in $O_2(C_{\overline{C_{\overline{z}}}}(\overline{u}))$ with respect to $C_{\overline{C_a}}(\overline{u})$. Therefore $\overline{z}$ is not weakly closed in $\overline{K}$ with respect to $\overline{C_{\overline{z}}}$. Now $\overline{C_a} \cong Suz$ follows from 3.3.13 and 2.3.4 and the theorem holds. □

3.4 2-central involution

In this section we try to find an ivolution z in the group G such that $C_G(z)$ is an extension of an extraspecial 2-group of order 2^9 by $\Omega_8^+(2)$. We recall our last notations:

- $R = O_3(H_1)$, $E = O_3(H_2)$, $Z(R) = \langle t\rangle$;
- $t = abc$, $C_a = C_G(a) \cong 3Suz$, $U = \langle a, b\rangle$ and $C_U = C_G(U)$ where $\langle a\rangle$,$\langle b\rangle$ and $\langle c\rangle$ are in L.

• $\overline{z}$ is a 2-central involution in $\overline{C_a} = C_a/\langle a\rangle$ such that $C_{\overline{E}}(\overline{z}) = \left\langle \overline{c}, \overline{b}, \overline{u}\right\rangle$ is of order 27.

• We have J_1 of length 11 and J_2 of length 110 are the orbits of $C_{H_2/E}(a)$ on $P(\overline{E})$.

• We have $\overline{M} = C_{\overline{C_a}}(\overline{b})$ and $\widetilde{M} = \overline{M}/\left\langle \overline{b}\right\rangle$.

• $\widetilde{z} \in C_{\widetilde{M}}(\widetilde{t})$ is an involution such that $\widetilde{z}\widetilde{R_1} \in Z(C_{\widetilde{M}}(\widetilde{t})/\widetilde{R_1})$. By 3.2.13 $\widetilde{Y} = O_2(C_{\widetilde{M}}(\widetilde{z}))$ is an extraspecial group of order 32.

Further notations: Let $s = ba$ and z be the preimage of $\overline{z}$ of order 2, then

$$C_s = C_G(s) \text{ and } C = C_{C_s}(z).$$

• $W = C_E(z)$ is the preimage of $\overline{W}$, and $Y = O_2(C_{C_U}(z))$ is a Sylow 2-subgroup of the pre-preimage of $\widetilde{Y}$.

At first, we determine the structure of C_s. We shall show that $C_s/U \cong U_4(3) : 2$. The following lemma follows from 3.2.14.

Lemma 3.4.1 *i) $O_3(C_U) = U$ and $C_U/U \cong U_4(3)$.*

ii) Y is an extraspecial group of order 32. Further $Y = O_2(C_G(U, c, z))O_2(C_G(U, u, z))$.

Proof: i) follows from 3.2.14. We have $\widetilde{Y} = \left\langle O_2(C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{h})), O_2(C_{\widetilde{C_{\widetilde{z}}}}(\widetilde{t}))\right\rangle = O_2(C_{\widetilde{M}}(\widetilde{z}))$. Hence $Y = O_2(C_G(U, c, z))O_2(C_G(U, u, z))$. By 3.2.6 we get that Y is an extraspecial group of order 32 and the lemma is proved.□

Lemma 3.4.2 *i) $|W| = 3^4$.*

ii) $W \in Syl_3(C)$.

iii) $N_C(W)/W \cong D_8.2$.

iv) $\langle c\rangle$ is not normalized by $N_C(W)$.

v) Under the action of $N_{C\cap C_U}(W)/W$ on $P(W)$, the orbit containing $\langle c\rangle$ is of length 2.

Proof: As z is the preimage of $\overline{z}$, by 3.3.1(ii), we get that W is of order 3^4 and by 3.2.3(ii) we get that $W \in Syl_3(C \cap C_U)$. We have that $\langle a, b, c\rangle \leq W$ and as $t = abc$, we get that $C_{C_U}(c) = C_{C_U}(t)$. By 3.1.2 we have that $C_{C_U}(t)/O_3(C_{C_U}(t)) \cong SL_2(3)$ where $O_3(C_{C_U}(t))$ is a special group of order 3^7 with center $\langle a, b, c\rangle =$

$W \cap O_3(C_{C_U}(t))$. Further z acts fixed point freely on $O_3(C_{C_U}(t))/Z(O_3(C_{C_U}(t))$. Therefore $N_{C_{C_U \cap C(c)}}(W)/W$ is of order 2. On the other hand by 3.2.3(i) we have that $N_{C_U \cap C}(W)/W \cong D_8$. Hence $\langle c \rangle$ is not normalized by $N_{C \cap C_U}(W)$. So there is an element $u \neq c$ in W conjugate to c in $N_C(W)$. By 2.4.2(i) for each five distinct elements $\langle x_i \rangle$, $i = 1, ..., 5$, from the orbit L we have that $x_1, ..., x_5$ are linear independent. As $|W| = 81 = 3^4$, we have $L \cap P(W) = \{\langle a \rangle, \langle u \rangle, \langle b \rangle, \langle c \rangle\}$, and further as $s = ab$, we get that $W \in Syl_3(C)$ and $\{\langle c \rangle, \langle u \rangle\}$ is the orbit of $N_C(W)/W$ on $P(W)$ which contains $\langle c \rangle$. Hence $N_{C_U}(W)$ is of index at most 2 in $N_C(W)$. By 3.2.6(v) there is an involution $\alpha \in H_2 \cap C$ such that $\alpha \notin C_U$. Since $H_2 \cap C$ is a subgroup of $N_C(W)$ and $\alpha \in H_2 \cap C$, we get that $N_{C_U}(W)$ is of index 2 in $N_C(W)$. By 3.2.3(i) $N_{C_U}(W)/W \cong D_8$, so $N_C(W)/W \cong D_8.2$ and the lemma is proved. □

We have that $\langle a, b, c \rangle \leq W$ and by 3.4.2(iv) $\langle c \rangle$ is not normalized by $N_{C \cap C_U}(W)$. So there is an element $u \neq c$ in W conjugate to c in $N_{C \cap C_U}(W)$. By 2.4.2(i) for each five distinct elements $\langle x_i \rangle$, $i = 1, ..., 5$, from the orbit L we have that $x_1, ..., x_5$ are linear independent, hence

$$L \cap P(W) = \{\langle a \rangle, \langle u \rangle, \langle b \rangle, \langle c \rangle\}.$$

Let $L_1 = L \cap P(W)$, $L_2 = \{\langle a^{-1}bcu \rangle, \langle ab^{-1}cu \rangle, \langle ac^{-1}bu \rangle, \langle au^{-1}cb \rangle\}$, $L_3 = \{\langle abcu \rangle, \langle a^{-1}b^{-1}cu \rangle, \langle a^{-1}c^{-1}bu \rangle, \langle a^{-1}u^{-1}cb \rangle\}$, $L_4 = \{\langle x_1x_2x_3 \rangle$ where $\langle x_1 \rangle$, $\langle x_2 \rangle$ and $\langle x_3 \rangle$ are three distinct elements of $L_1\}$ and $L_5 = \{\langle x_1x_2 \rangle$ where $\langle x_1 \rangle$ and $\langle x_2 \rangle$ are two distinct elements of $L_1\}$. Then $|L_1| = |L_2| = |L_3| = 4$, $|L_4| = 16$, $|L_5| = 12$ and $P(W) = \bigcup_{i=1}^{5} L_i$. Now let F be a subgroup of W of order 27. Then we can see that $|P(F) \cap L_4| \neq 0$ and if $\langle x_1x_2x_3 \rangle \in L_4$ is in $P(F)$, then $\left\langle x_1^{-1}x_2 \right\rangle$ or $\langle x_1x_2 \rangle$ is in $P(F)$. In fact we have proved the following lemma.

Lemma 3.4.3 *i) Let $F \leq W$ be of index 3. Then F contains elements from the orbits I and J as well. F contains elements $x = x_1x_2x_3$ and y where $y = x_1x_2^{-1}$ or $y = x_1x_2$ and $\langle x_i \rangle \in \{\langle c \rangle, \langle b \rangle, \langle u \rangle, \langle a \rangle\}$, for $i = 1, 2, 3$.*
ii) $L \cap P(W) = \{\langle a \rangle, \langle u \rangle, \langle b \rangle, \langle c \rangle\}$.

Lemma 3.4.4 *i) Y is the unique maximal W-invariant $3'$-subgroup in C.*
ii) $N_C(W) \leq N_C(Y)$.

Proof: Let $F \leq W$ be a subgroup of order 27, then by 3.4.3(i) and 3.1.3 we get that, $C_C(F)$ is a subgroup of a group X such that $O_3(X)$ is a special 3-group of order 3^7, $Z(O_3(X))$ is of order 27, $|P(Z(O_3(X)))\cap L| = 3$, $Z(O_3(X)) = W \cap O_3(X)$, $X/O_3(X) \cong SL_2(3) \times 2$ and $zO_3(X) \in Z(X/O_3(X))$. Let $g \in C_C(a)$ then as $s = ab$, we have $g \in C_C(b)$. Hence $g \in (C \cap C_U)$ and therefore $C_C(a) = C_C(b)$. Since $C_C(a) = C_C(b)$, $|P(Z(O_3(X))) \cap L| = 3$ and $|P(W)) \cap L| = 4$, we have $U \leq Z(O_3(X))$. Since $|P(Z(O_3(X))) \cap L| = 3$ and $U \leq Z(O_3(X))$, we deduce that $Z(O_3(X)) = \langle a, b, c\rangle$ or $Z(O_3(X)) = \langle a, b, u\rangle$. Since $O_2(C_C(F))$ is W-invariant and $Z(O_3(X)) \leq W$, we get that only in the case of $F = Z(O_3(X))$, we have $O_2(C_C(F)) \neq \langle z\rangle$. Therefore if $O_2(C_C(F)) \neq \langle z\rangle$, then $F = \langle U, c\rangle$ or $F = \langle U, u\rangle$. Now 3.4.1(ii) gives us that $O_2(C_C(F)) \leq Y$. This and coprime action give us that the order of a maximal W-invariant $3'$-subgroup in C is 32 and is contained in Y. So we have proved i) and ii) follows from i). □

We note that $Y = O_2(C_G(U, c, z))O_2(C_G(U, u, z))$ and c and u are conjugate in $N_{C\cap C_U}(W)$. By 3.4.1 we have $C_U/U \cong U_4(3)$. Let $Y_1 \leq Y$ be normal in $N_{C_U}(Y)$, then either $Y_1 = Y$ or $Y_1 = \langle z\rangle$.

Lemma 3.4.5 *i)* $N_{C_s}(Y)/\langle C_{C_s}(Y), Y\rangle \cong (S_3 \times S_3)$.
ii) $C_C(Y)Y/\langle Y, s\rangle \cong S_3$.

Proof: Since $Z(Y) = \langle z\rangle$, we have $N_{C_s}(Y) = N_C(Y)$. Let $\widehat{X} = C_C(Y)/\langle z, s\rangle$. Since $U \leq C_C(Y)$, we have $3||\widehat{X}|$. We have that $W \leq C_{C_U}(z)$ and $C_{C_U}(Y) \cap W = U$(see 3.2.1(i) and 3.2.4(ii),(i)). So by 3.4.2(ii) we have $U \in Syl_3(C_C(Y))$. Hence $|\widehat{X}|_3 = 3$ and $\langle\widehat{a}\rangle = U\langle z\rangle / \langle z, s\rangle \in Syl_3(\widehat{X})$. Let $\widehat{x} \in C_{\widehat{X}}(\widehat{a})$ be a $3'$-element then $x \in C \cap C_U($ x is a preimage of $\widehat{x})$. Since $C_{C_U\cap C}(Y) = \langle U, z\rangle$, we have $C_{\widehat{X}}(\widehat{a}) = \langle\widehat{a}\rangle$. Now by [FT] we get that $\widehat{X}/O_{3'}(\widehat{X}) \cong A_5, L_3(2), Z_3$ or S_3. Since $O_{3'}(\widehat{X})$ is W-invariant, by 3.4.4(i) we get that $O_{3'}(\widehat{X}) = 1$. Let $\widehat{X} \cong A_5$ or $L_3(2)$. Since $\widehat{X}$ is W-invariant, W normalizes X where X is the preimage of $\widehat{X}$. Therefore W normalizes $X' \cong SL_2(5)$, $SL_2(7)$, A_5 or $L_3(2)$. By ([AT], pages 2 and 3) we get that $|Out(A_5)| = |Out(L_3(2))| = 2$. Therefore a subgroup of order 27 in W say F centralizes X'. By 3.4.3(i) and 3.1.1 we get that, $C_C(F)$ is a subgroup of a group X_1 such that $O_3(X_1)$ is a special group of order 3^7 and $X_1/O_3(X_1) \cong SL_2(3)\times 2$. So there is no section isomorphic to A_5 or $L_3(2)$ in $C_C(F)$.

Hence this case does not happen and $\widehat{X} \cong Z_3$ or S_3. Since by 3.2.6(v) there is an involution $\alpha \in C \cap H_2$, $z \neq \alpha$ which centralizes Y, we have $\widehat{X} \cong S_3$. Since Y is an extraspecial 2-group of order 2^5 (see 3.2.6(ii)), by ([GLS2],theorem 10.6) we get that $N_C(Y)/C_C(Y)Y$ is isomorphic to a subgroup of $O_4^+(2) \cong S_3 \times S_3$. By 3.2.3(i) we have that $N_{C_U \cap C}(W)/\langle U, z\rangle \cong S_3 \times S_3$. Since $\alpha \in C_{C\cap H_2}(Y)$ and $C \cap H_2 \leq N_C(W)$ by 3.4.2(iii), we get that $N_C(W)Y/C_C(Y)Y \cong S_3 \times S_3$. As $N_C(W)Y/C_C(Y)Y \cong S_3 \times S_3$ and $O_4^+(2) \cong (S_3 \times S_3).2$, we have that $N_C(Y)/C_C(Y)Y \cong (S_3 \times S_3)$ and hence the lemma holds. □

Lemma 3.4.6 *i) There is an involution α in $N_G(U) \cap C_s \cap H_2$ such that α is not conjugate to an involution of C_U in C_s.*

ii) $C_G(\alpha, s)/\langle \alpha, s\rangle \cong U_4(2)$ and $C_G(\alpha, s, z) \leq N_{C_s}(Y)$.

ii) Let $T \in Syl_2(N_{C_s}(Y))$, then $T \in Syl_2(C_s)$.

Proof: By 3.1.4(i), $N_G(U)/C_U \cong D_8$. By 3.2.6(v), there is an involution $\alpha \neq z$ in $X = N_G(U) \cap C_s \cap H_2$ conjugate to z in H_2 which centralizes Y and $a^\alpha = b$. Since by 3.4.1 $C_U/U \cong U_4(3)$ and α centralizes Y, by ([AT],page 52) we get that $C_{C_U/U}(\alpha) \cong 2 \times U_4(2)$. By ([AT],page 26) we get that $|U_4(2)|_3 = 3^4$ and hence $|C_{C_s}(\alpha)|_3 \geq 3^5$. By 3.4.2(i),(ii) we have that for an involution $f \in C_U$, $|C_{C_s}(f)|_3 = 3^4$. Hence α is not conjugate to an involution of C_U in C_s and i) holds.

We have $C_G(\alpha, U)/\langle s, \alpha\rangle \cong U_4(2)$ and by 3.2.6(v) $\langle s, u, c\rangle \leq C_{C_U}(\alpha)$. Since $t = sc$, we get that $t \in C_{C_U}(\alpha)$. Since α is conjugate to z in H_2, by 3.4.2(i), $C_E(\alpha)$ is of order 81. Set $V = C_E(\alpha)$, as $E \leq C_U$, we have $V \leq C_G(\alpha, U)$. We use the notation $\widehat{}$ for the natural homomorphism onto $C_G(\alpha, s)/\langle s, \alpha\rangle$. We have that $\widehat{V}$ is an elementary abelian group of order 27, $\widehat{C_G(\alpha, U)} \cong U_4(2)$ and $\widehat{V} \leq \widehat{C_G(\alpha, U)}$. So by ([AT], page 26) we get that $N_{\widehat{C_G(\alpha,U)}}(\widehat{V})/\widehat{V} \cong S_4$ and $\widehat{V}$ is a faithful $N_{\widehat{C_G(\alpha,U)}}(\widehat{V})/\widehat{V}$-module. By 2.4.5, under the action of $N_{\widehat{C_G(\alpha,U)}}(\widehat{V})/\widehat{V}$ on $P(\widehat{V})$ we have three orbits I_i, $i = 1, 2, 3$, such that $|I_1| = 3$, $|I_2| = 4$ and $|I_3| = 6$. By 3.1.3 we get that $C_G(t, s)/O_3(C_G(t, s)) \cong SL_2(3) \times 2$, $O_3(C_G(t, s)$ is a special group of order 3^7, $Z(O_3(C_G(t, s))) = \langle U, c\rangle$, $C_G(Z(O_3(C_G(t, s))))/O_3(C_G(t, s)) \cong SL_2(3)$ and $C_G(Z(O_3(C_G(t, s))))/O_3(C_G(t, s))$ is faithful on $O_3(C_G(t, s))/Z(O_3(C_G(t, s)))$. Since $|C_U(\alpha)| = 3$, $\langle s, c\rangle \leq V$, $|V| = 81$, we get that $\alpha \notin C_G(Z(O_3(C_G(t, s))))$ and $\widehat{C_G(t, s, \alpha)}/O_3(\widehat{C_G(t, s, \alpha)}) \cong SL_2(3)$. Further $O_3(\widehat{C_G(t, s, \alpha)})$ is an extraspecial group of order 27. This gives us that $|\widehat{C_G(\alpha, s)}|_3 = 81$ and $\widehat{C_G(\alpha, U, t, s)} =$

$\widehat{C_G(\alpha, t, s)}$.

Set $\widehat{D} = \widehat{C_G(s,t,\alpha)}$ and $\widehat{F} = N_{\widehat{C_G(\alpha,s)}}(\widehat{V})$. Let $x \in N_{C_G(s,\alpha)}(V)$. Then as $s = ab$ and by 2.4.2(i) for each four distinct elements $\langle x_i \rangle$, $i = 1, ..., 4$, from the orbit L we have that $x_1, ..., x_4$ are linear independent, we deduce that $x^2 \in C_U$. Hence $N_{C_G(U,\alpha)}(V)$ is of index at most 2 in $N_{C_G(s,\alpha)}(V)$ and as $\alpha \notin C_U$, we get that $\widehat{F} = N_{\widehat{C_G(\alpha,U)}}(\widehat{V})$. Therefore $\widehat{F}/\widehat{V} \cong S_4$. We note that by 3.3.14 we get that $C_G(c) \cong 3Suz$ and there is no section isomorphic to $C_G(c)$ in H_1, so c is not conjugate to t in G. Let $y \in C_s$ and suppose that $t^y = t^{-1}$, then as $s = ab$ and $t = abc$, we get that $c^y = abc^{-1}$. But this and 2.4.2(iii) give us that c is conjugate to t, which is a contradiction. So t is not inverted in C_s. In fact $C_G(t, s) = C_G(s, c)$. From the structure of $\widehat{D}$ we get that $|N_{\widehat{D}}(\widehat{V})| = 3^4 2$, so $\langle \widehat{t} \rangle$ is in the orbit of lenght 4 under that action of $\widehat{F}/\widehat{V}$ on $P(\widehat{V})$ and hence $\langle \widehat{t} \rangle \in I_2$. Let $\widehat{V_1} \leq \widehat{V}$ be of index 3. If $\widehat{t} \in \widehat{V_1}$, then from the structure of $\widehat{D}$ we get that there is no $\widehat{V}$-invariant $3'$-subgroup in $C_{\widehat{C_G(\alpha,s)}}(\widehat{V_1})$. We recall that $\langle \widehat{t} \rangle \in I_2$.

Now assume that $I_2 \cap P(\widehat{V_1}) = \varnothing$. We note that $s = ab$, $t = sc$ and $\widehat{t}$ is in the orbit of length 4. So there are at least 4 elements conjugate to c in V. Since by 2.4.2(i) for each four distinct elements $\langle x_i \rangle$, $i = 1, ..., 4$, from the orbit L we have that $x_1, ..., x_4$ are linear independent and V is of order 81, we have $|P(V) \cap L| = 4$ and as $P(\widehat{V_1})$ does not contain an element conjugate to $\widehat{t}$, we have $|P(V_1) \cap L| = 0$. Let $\langle g \rangle$ and $\langle r \rangle$ be two elements of $P(V) \cap L$ such that $P(V) = \{\langle g \rangle, \langle r \rangle, \langle c \rangle, \langle u \rangle\}$. We remark that $|P(U) \cap L \cap P(V)| = 0$. By 2.4.2(i) for each five distinct elements $\langle x_i \rangle$, $i = 1, ..., 5$, from the orbit L we have that $x_1, ..., x_5$ are linear independent, so $s = ab = c^{\pm1}u^{\pm1}g^{\pm1}r^{\pm1}$. As V is of order 81, the same argument shows that $V = \langle c, u, g, r \rangle$. Now as $|P(V_1) \cap L| = 0$, $s = ab = c^{\pm1}u^{\pm1}g^{\pm1}r^{\pm1}$ and V_1 is of order 27, we can see that V_1 contains two elements $x = x_1^{-1}x_2$ and $y = x_1x_2x_3$ where $\langle x_i \rangle \in (P(V) \cap L)$, for $i = 1, 2, 3$. Hence by 3.1.3 we get that $C_G(V_1)$ is a subgroup of X where $X/O_3(X) \cong SL_2(3) \times 2$, $O_3(X)$ is a special group of order 3^7, $Z(O_3(X)) = \langle x_1, x_2, x_3 \rangle$, $C_G(Z(O_3(X)))/O_3(X) = C/O_3(X) \cong SL_2(3)$ and $C_G(Z(O_3(X)))/O_3(X)$ is faithful on $O_3(X)/Z(O_3(X))$. Since α is trivial on $Z(O_3(X))$, we have $\alpha \in C_C(Z(O_3(X)))$. Hence there is $X_1 \leq X$ with $C_{C_s}(V_1, \alpha) \leq X_1$ and $X_1/Z(O_3(X)) \cong SL_2(3) \times 2$. Since $V \leq X_1$, V is of order 81 and $Z(O_3(X)) \leq V$, there is no $\widehat{V}$-invariant $3'$-subgroup in $C_{\widehat{C_G(\alpha,s)}}(\widehat{V_1})$. So in any case there is no $\widehat{V}$-invariant $3'$-subgroup in $C_{\widehat{C_G(\alpha,s)}}(\widehat{V_1})$. By coprime action we get that there is no $\widehat{V}$-invariant $3'$-subgroup in $\widehat{C_G(\alpha,s)}$. We have $|\widehat{C_G(\alpha,s)}|_3 = 81$ and $\widehat{C_G(\alpha,U)} \cong U_4(2)$. Since $\widehat{C_G(\alpha,U)} \le \widehat{C_G(s,\alpha)}$, $|U_4(2)|_3 = 81$ (see ([AT, page 26])) and there is no subgroup of index 3 in $\widehat{C_G(\alpha,U)}$, we get that there is no subgroup of index 3 in $\widehat{C_G(\alpha,s)}$. Now we can see that $\widehat{C_G(\alpha,s)}$ satisfies the conditions of 2.3.6 (in 2.3.6 set: $\widehat{t} = d$, $H = \widehat{C_G(\alpha,s)}$, $\widehat{V} = E_2$ and $\widehat{D} = X$, we note that as $\widehat{D} = C_G(\widehat{t,U,\alpha},s)$ and $\widehat{C_G(\alpha,U)} \cong U_4(2)$, we get that $\widehat{D}$ is isomorphic to the centralizer of a non-trivial 3-central element in $U_4(2)$). Now by 2.3.6 we get that $\widehat{C_G(\alpha,s)} \cong U_4(2)$. This gives us that $\widehat{C_G(\alpha,U)} = \widehat{C_G(\alpha,s)}$. By 3.2.13 we have that $C_G(U,z) = N_{C_U}(Y)$. As α centralizes Y and $\widehat{C_G(\alpha,U)} = \widehat{C_G(\alpha,s)}$, we conclude that $C_G(s,z,\alpha) \le N_{C_s}(Y)$ and ii) holds.

Let $T \in Syl_2(N_C(Y))$, $T_1 \in Syl_2(N_{C_U}(Y))$ with $T_1 \le T$ and $\alpha \in T$. Then by lemmas 3.2.9(v) and 3.2.8 we have Y is a characteristic subgroup in T_1 and $|T_1| = 2^7$. By 3.4.5(i), (ii) we get that T is of order 2^8 and so $T = T_1\langle\alpha\rangle$. We remark that from the structure of T_1 in 3.2.9 and as $|C_T(\alpha)| = 2^7$ (by ii)), $T = T_1\langle\alpha\rangle$ is of order 2^8 and $C_T(Y) = \langle\alpha,z\rangle$ is normal in T, we get that $Z_2(T) \cong Z_2 \times Z_4$ and $\langle\alpha,z\rangle \le Z_2(T)$. Therefore $\langle\alpha,z\rangle = \Omega_1(Z_2(T))$ is a characteristic subgroup of T and $\alpha^y = \alpha z$ for some element $y \in T$. Now let $x \in N_C(T)$ be a 2-element, then x normalizes $\langle\alpha,z\rangle$. If x centralizes α, then by ii) we get that $x \in N_{C_s}(Y)$ and so $x \in T$. If x does not centralize α we get that $\alpha^x = \alpha z$ and then xy centralizes α. Now ii) gives us that $xy \in T$ and as $y \in T$, we get that $x \in T$. Hence $T \in Syl_2(C)$ and the lemma holds. □

Lemma 3.4.7 *There is a subgroup H of index 2 in C_s containing C_U.*

Proof: Let $F \in Syl_2(C_U)$ and $T \in Syl_2(C_s)$ containing F. We recall that $U = \langle a,b\rangle$, $s = ab$ and so $C_U \le C_s$. By 3.4.5(i),(ii) and 3.4.6(iii) we have $|T| = 2^8$ and by 3.2.9(v) and 3.2.8 we get that $|F| = 2^7$. So F is a maximal subgroup of T. By 3.4.6(i) there is an involution α in T such that α is not conjugate to any involution of F in C_s. Therefore Thompson's transfer lemma ([BH], XII.8.2) gives us that C_s has a subgroup H of index 2. Now we have $H \cap C_U$ is a normal subgroup of C_U of index at most 2, as $C_U/U \cong U_4(3)(3.4.1)$, we have $C_U \le H$ and the lemma is proved. □

Further notations: By 3.4.7 there is a subgroup H of index 2 in C_s containing C_U, we fix the notation H for such a subgroup of index 2 in C_s. Set

$$H_z = C_H(z) \text{ and } D = N_{C_U}(Y).$$

Lemma 3.4.8 *i)* $D = N_H(Y) = C_{C_U}(z)$.

ii) $N_H(W) \leq D$.

iii) $N_H(W) = N_{C_U}(W)$.

iv) D *contains a Sylow 2-subgroup of* H_z.

v) $|R \cap E| = 3^3$.

Proof: By 3.2.13 we conclude that $D = C_G(U, z)$ and by 3.2.8 $D/\langle U, Y\rangle \cong S_3 \times S_3$. Since C_U is a subgroup of H, we have $D \leq N_H(Y)$ and by 3.4.5(i),ii)) we get that D is of index 2 in $N_C(Y)$. As H is of index 2, by 3.4.5(i),(ii),(iii) we get that $D = N_H(Y)$ and i) holds. Since $\langle Y, W\rangle \leq C_U \leq H$, by 3.4.4(i) we get that Y is the unique maximal W-invariant $3'$-subgroup in H. Therefore $N_H(W) \leq N_H(Y)$ and by i) $N_H(W) \leq D$ so ii) holds. Since $D \leq C_U \leq H$, by 3.2.6(iv) $N_{C_U}(W) \leq D$ and by ii) $N_H(W) \leq D$, we get that $N_H(W) = N_{C_U}(W)$. Since D is of index 2 in $N_C(Y)$ and H is of index 2 in C_s, by 3.4.5(iii) we get that D contains a Sylow 2-subgroup of H_z. As $RE \leq O_3(H_1 \cap H_2)$ and the order of a maximal elementary abelian subgroup of R is 3^3, we have $|R \cap E| = 3^3$ and the lemma is proved. □

Remark: 1) Let $N \cong U_4(2)$, then by ([AT],page 26) we get that N has two classes of involutions. Let $x \in N$ be a 2-central involution, then $C_N(x)/O_2(C_N(x))$ is an extension of an elementary abelian group of order 9 by a group of order 2 and $O_2(C_N(x))$ is an extraspecial group of order 32 (this gives us that the center of a Sylow 2-subgroup of N is of order 2). Let V be a Sylow 3-subgroup of $C_N(x)$ then $C_{O_2(C_N(x))}(V) = \langle x\rangle$. Let y be a non 2-central involution in N, then $C_N(y)/O_2(C_N(y)) \cong S_3$ and $O_2(C_N(y))$ is an elementary abelian group of order 16.

2) Let $y = x_1x_2x_3x_4$ or $y = x_1x_2x_3$ where $\langle x_i\rangle \neq \langle x_j\rangle$ for $i \neq j$ and $\langle x_i\rangle$ and $\langle x_j\rangle$ are in $P(W) \cap L$ for $j, i = 1, ..., 4$. Then as by 2.4.3(i) for each four distinct elements $\langle \overline{x_i}\rangle$, $i = 1, ..., 4$, from the orbit J_1 we have that $\overline{x_1}, ..., \overline{x_4}$ are linear independent, we get that y is conjugate to t in C_a. Let $y = t^g$, $F = R^g$ and $N = H_1^g$ where $g \in C_a$. By 3.1.2 we get that $C_G(a, y)/O_3(C_G(a, y)) \cong SL_2(3) \times 2$ with $O_3(C_G(a, y))$ is special group of order 3^7. This gives us that $|C_G(a, y)|_3 = 3^{7+1} = 3^8$. Since $a \notin Z(F)$, we have $|C_F(a)| \leq 3^4$ and by ([AT],page 26) $|U_4(2)|_3 = 81 = 3^4$. Since by the general assumption $N/F \cong 2U_4(2) : 2$ and $|C_G(a, y)|_3 = 3^8$, we get that if $a \notin F$, then aF is a 3-central element in N/F.

3) Let T be a Sylow 3-subgroup of C_s containing E. Then by 2.4.2(iv), E is a characteristic subgroup of T and therefore $T \in Syl_3(C_s \cap H_2)$. Let $x \in C_{H_2}(s)$, then $s^x = a^x b^x = s = ab$, since $\langle a \rangle$ and $\langle b \rangle$ are in the orbit L and by 2.4.2(i) for each five distinct elements $\langle x_i \rangle$, $i = 1, ..., 5$, from the orbit L we have that $x_1, ..., x_5$ are linear independent, we get that $x^2 \in C_U$. Therefore $T \leq C_U$.

We are going to show that $C_U = H$ and then 3.4.1 will give us the structure of H. In the following lemma we show that $D/\langle s \rangle$ is strongly 3-embedded in $H_z/\langle s \rangle$, this will help us to prove that $D = H_z$. Finally we will show that C_U is strongly 2-embedded in H and then by Bender's theorem [Be] we get that $C_U = H$.

Lemma 3.4.9 *$D/\langle s \rangle$ is strongly 3-embedded in $H_z/\langle s \rangle$.*

Proof: Let $\langle s \rangle \neq \langle x \rangle \leq D$ be of order three. Since by 3.4.2(ii), $W \in Syl_3(C)$ and $W \leq D$, we may assume that $x \in W$. If $x \in U$, then $C_G(s, x) = C_U$ and so $C_G(s, x, z) \leq D$. So we assume that $x \notin U$, then by 3.2.3(iii), x is conjugate to an element $\beta \in \{yr | y \in U, r \in \{c, cu, c^{-1}u\}\}$. Since $s = ab$, we see that $\langle s, x \rangle$ contains an element $x_1x_2x_3x_4$ or $x_1x_2x_3$ where $\langle x_i \rangle \neq \langle x_j \rangle$ for $i \neq j$ and $\langle x_i \rangle$ and $\langle x_j \rangle$ are in $P(W) \cap L$, $j, i = 1, ..., 4$. Since by 2.4.2(i) for each five distinct elements $\langle x_i \rangle$, $i = 1, ..., 5$, from the orbit L we have that $x_1, ..., x_5$ are linear independent, we get that $\langle s, x \rangle$ contains an element conjugate to t in H_2. Let $y \in \langle s, x \rangle$ be conjugate to t, $y = t^g$, $F = R^g$ and $N = H_1^g$ with $g \in H_2$. Assume first that $s \notin F$. Assume further thst $a \notin F$ and $b \notin F$. By the general assumption $C_N(s, z, x)/C_F(s, z, x)$ is isomorphic to a subgroup of $2U_4(2)$. By 3.1.2 we get that aF and bF are two 3-central elements in N/F ($|C_{N/F}(aF)|_3 = |C_{N/F}(bF)|_3 = |U_4(2)|_3 = 3^4$). As $FE \leq O_3(N \cap H_2)$ and the order of a maximal elementary abelian subgroup of F is 3^3, we have $|F \cap E| = 3^3$ and hence EF/F is an elementary abelian group of order 27 containing aF and bF. Since $s = ab$, by 2.4.7(vii) we get that sF is a non 3-central element in N/F and $C_N(s, x)/O_3(C_N(s, x))$ is a 2-group. Since by 3.4.2(ii) $W \in Syl_3(C)$, we get that $C_N(s, z, x) \leq N_C(W)$ (we remark that $W \leq O_3(C_N(s, x))$ and as by 3.4.2(ii) $W \in Syl_3(C)$, we conclude that $W = C_{O_3(C_N(s,x))}(z)$). This gives us that $C_{H_z}(x) \leq N_H(W)$. Since by 3.4.8(ii) $N_H(W) \leq D$, we get that $C_{H_z}(x) \leq D$. Now suppose that $s \notin F$ and one of the elements a or b are in F, as $F = O_3(N)$ and $W \in Syl_3(C)$, by 3.4.2(ii) we get that $C_F(x, z) \leq W$. Since $s = ab$, $C_F(x, z) \leq W$

and by 2.4.2(i) for each five distinct elements $\langle x_i \rangle$, $i = 1, ..., 5$, from the orbit L we have that $x_1, ..., x_5$ are linear independent, we get that $C_N(s, x, z) \leq C_{C_U}(x, z) \leq D$.

Suppose that $s \in F$. As by the general assumption $C_N(F) = \langle y \rangle$, we get that involutions $rF \in Z(N/F)$ act fixed point freely on $F/Z(F)$. Therefore as $\langle y, s \rangle \leq C$, we have $zF \notin Z(N/F)$. As $C_N(y)/F \cong 2U_4(2)$ and $zF \notin Z(N/F)$, by 2.4.7(viii) we get that $C_N(s, x, z)/C_F(s, x, z)$ is an extension of a 2-group by an elementary abelian group of order 9. Since W is of order 3^4, $W \in Syl_3(C)$ (3.4.2(i),(ii)) and 9 divides the order of $C_{N/F}(z)$, we get that $C_F(z) = \langle x, s \rangle$. Therefore $O_2(C_N(s, x, z))$ is W-invariant and hence by 3.4.4(i), we have $O_2(C_N(s, x, z)) \leq Y$. In fact $C_N(s, x, z) \leq \langle Y, N_C(W) \rangle$, then by 3.4.4(ii) we have $C_C(x) \leq N_C(Y)$ and then $C_{H_z}(x) \leq N_H(Y)$. Now by 3.4.9(i) we conclude that $C_{H_z}(x) \leq D$. Hence we have proved that for each subgroup $\langle s \rangle \neq \langle x \rangle \leq W$ of order three we have $C_{H_z}(x) \leq D$.

Assume that $h \in H_z$ and 3 divides $|D/\langle s \rangle \cap D^h/\langle s \rangle|$. Let $X_1 \in Syl_3(D \cap D^h)$. As $W \in Syl_3(C)$ and W is an elementary abelian group, we may asume that $X_1 \leq W$. Let X_1 be a normal subgroup of a 3-subgroup X_2 of D^h, then $Z(X_1) \cap Z(X_2) \neq 1$. As for each $x \in W$ we have $C_{H_z}(s, x) \leq D$, we get that $X_2 \leq D$. Therefore $X_1 \in Syl_3(D)$ and we may assume that $W = X_1$ (by 3.4.2(ii)) $W \in Syl_3(C)$ and as $W \leq D \leq H_z$, we get that $W \in Syl_3(H_z)$). Hence we may assume that $h \in N_{H_z}(W)$ and by 3.4.8(ii) we get that $h \in D$. Therefore D is strongly 3-embedded in H_z and the lemma is proved. $\square$

We recall that H is a subgroup of index 2 in C_s and $C_U \leq H$, so by 3.4.8(i) $D \leq H_z$.

Theorem 3.4.10 $C_s/U \cong U_4(3) : 2$.

Proof: We begin by showing that $D = H_z$. Denote by * be the natural homomorphism $H_z \to H_z/\langle s, z \rangle$. Then by 3.4.9 $D^\star$ is strongly 3-embedded in $H_z^\star$. Let $r^\star \in Y^\star$ be an involution and $g^\star \in H_z^\star$ such that $(r^\star)^{g^\star} \in D^\star$. Let $T^\star$ be a Sylow 2-subgroup of $D^\star$. Then as $U \leq D$, we get that 3 divides the order of $C_{D^\star}(T^\star)$. Therefore 3 divides $|C_{D^\star}(r^\star)|$ and 3 divides $|C_{D^\star}((r^\star)^{g^\star})|$ as well. Let $P^\star \in Syl_3(C_{D^\star}(r^\star))$ and $P_1^\star \in Syl_3(C_{D^\star}((r^\star)^{g^\star}))$. Then as by 3.4.9 $D^\star$ is strongly 3-embedded in $H_z^\star$, we have $P^\star \in Syl_3(C_{H_z^\star}(r^\star))$ and $P_1^\star \in Syl_3(C_{H_z^\star}((r^\star)^{g^\star}))$. Since $g^\star \in H_z^\star$, we have $(P^\star)^{g^\star} \in Syl_3(C_{(D^\star)^{g^\star}}((r^\star)^{g^\star}))$ and as by 3.4.9 $(D^\star)^{g^\star}$ is strongly 3-embedded in $H_z^\star$, we have $(P^\star)^{g^\star} \in Syl_3(C_{H_z^\star}((r^\star)^{g^\star}))$. Hence $(P^\star)^{g^\star g_1^\star} = P_1^\star$ for some $g_1^\star \in C_{H_z^\star}((r^\star)^{g^\star})$. Now we have $P_1^\star \leq D^\star \cap (D^\star)^{g^\star g_1^\star}$ and as by 3.4.9 $D^\star$ is strongly 3-embedded in $H_z^\star$, we get that $g^\star g_1^\star \in D^\star$. Therefore $(r^\star)^{g^\star g_1^\star} \in Y^\star$. As $g_1^\star \in C_{H_z^\star}((r^\star)^{g^\star})$, we have $(r^\star)^{g^\star} = (r^\star)^{g^\star g_1^\star} \in Y^\star$. Hence $Y^\star$ is strongly closed in $D^\star$ with respect to $H_z^\star$.

Let $X^\star = \langle (Y^\star)^{H_z^\star} \rangle$ and $Z^\star = O(X^\star)$. Assume that 3 divides the order of $Z^\star$. As $O_{3'}(Z^\star)$ is of odd order and $O_{3'}(Z^\star)$ is $W^\star$-invariant, we get that $O_{3'}(Z^\star) = 1$ by 3.4.4(i). Hence $F(Z^\star) = Z^\star \cap D^\star = U^\star$. Since $F(Z^\star)$ is normal in $H_z^\star$, $F(Z^\star) = U^\star$ and $D^\star$ is strongly 3-embedded in $H_z^\star$, by 3.4.9, we get that $H_z^\star \leq D^\star$ and hence $H_z^\star = D^\star$.

Now assume that 3 does not divide the order of $Z^\star$. Then as $Z^\star$ is of odd order and $Z^\star$ is $W^\star$-invariant, we get that $Z^\star = 1$ by 3.4.4(i). Let $B_1 = \{L_2(2^n),\ n \geq 3,\ Sz(2^{n+1}),\ n \geq 1,\ U_3(2^n),\ n \geq 2\}$ and $B_2 = \{L_2(q),\ q \equiv 3, 5 \pmod 8),\ J_1$, group of Ree type $\}$. Let $O_2(X^\star) = 1$. Then since $Y^\star$ is strongly 2-closed in $D^\star$ with respect to $H_z^\star$, $|Y^\star| = 16$ and $Z^\star = 1$, we get with Goldschmidt's theorem [Go] that either $X^\star \in B_1 \cup B_2$ or $X^\star$ is the central product of groups $P_1^\star$ and $P_2^\star$ where $P_i^\star \in B_1 \cup B_2$, $i = 1, 2$. We note that, if $Q^\star$ is a quasisimple normal subgroup of $X^\star$, then, as $O_2(X^\star) = 1 = Z^\star$, we get that $Z(Q^\star) = 1$. First we assume that $X^\star$ is the central product of groups $P_i^\star$ and $P_i^\star \in B_1 \cup B_2$, $i = 1, 2$. Then since $|Y^\star| = 16$, we have $|Y \cap P_i^\star| = 4$, $i = 1, 2$, and by 2.3.9 and 2.3.10 we get that 3 divides the order of $P_i^\star$, $i = 1, 2$. Since $W^\star \in Syl_3(H_z^\star)$ by 3.4.2(ii), $W^\star \leq D^\star$, by 3.4.9 $D^\star$ is strongly 3-embedded in $H_z^\star$ and there is no section isomorphic to a non abelian simple group in $D^\star$ (see 3.2.13), we get that there is no section isomorphic to a non abelian simple group in the centralizers of the elements of order three in $H_z^\star$. But this gives us that 3 does not divide the order of $P_i^\star$, $i = 1, 2$ which is a contradiction. Therefore $X^\star \in B_1 \cup B_2$. By 3.4.8(i),iv)) and 3.2.13 we get that $D^\star = N_{H_z^\star}(Y^\star)$ contains a Sylow 2-subgroup of $H_z^\star$. So by 2.3.9 and 2.3.10, we get that $X^\star = L_2(16)$ and $Y^\star \in Syl_2(X^\star)$. This gives us that 15 divide the order of $N_{H_z^\star}(Y^\star)$ (see 2.3.9(ii)). By 3.4.8(i) $D^\star = N_{H_z^\star}(Y^\star)$ and by 3.2.13, 5 does not divide the order of $D^\star$. Hence this case does not happen. Now we assume that $O_2(X^\star) \neq 1$. Then as $O_2(X^\star)$ is $W^\star$-invariant, we get by 3.4.4(i) that $O_2(X^\star) \leq Y^\star$. Since $O_2(X^\star)$ is normal in $H_z^\star$, we get that $O_2(X^\star) = Y^\star$ and so $H_z^\star \leq N_{H_z^\star}(Y^\star)$. We note that $D^\star/Y^\star \cong S_3 \times S_3$ acts irreducibly on $Y^\star$. By 3.4.8(i) $D^\star = N_{H_z^\star}(Y^\star)$, so $H_z^\star \leq D^\star$ and hence $H_z^\star = D^\star$.

In fact as $C_U/U \cong U_4(3)$ by 3.4.1 and by ([AT],page 52) $U_4(3)$ has just one class of involutions, we have proved that for each involution $f \in C_U$, $C_H(f)$ is a subgroup of C_U. We will show that C_U is strongly 2-embedded in H. Let $h \in H$ such that $|C_U/\langle s\rangle \cap C_U^h/\langle s\rangle|$ is even and let $X_1 \in Syl_2(C_U \cap C_U^h)$. Let X_2 be a 2-subgroup of C_U^h such that $X_1 \lhd X_2$. Then $Z(X_1) \cap Z(X_2) \neq 1$. As for each involution $x \in C_U$, $C_H(x) \leq C_U$ we get that $X_2 \leq C_U$. Therefore $X_1 \in Syl_2(C_U)$ and we may assume that h normalizes a Sylow 2-subgroup T of C_U. By conjugations in C_U if necessary we may assume that $Y \leq T$. Then by 3.2.9(v) we have that $h \in N_H(Y)$. Therefore $h \in D \leq C_U$ and we get that C_U is strongly 2-embedded in H.

Assume $C_U < H$. By 3.4.1 $C_U/U \cong U_4(3)$ so $H \neq O(H)H_z$. Since C_U is strongly 2-embedded in H and $H \neq O(H)H_z$, by Bender's theorem [Be] we get that $H/O(H)$ has a normal subgroup $L/O(H)$ of odd index such that $L/O(L)$ is isomorphic to $Sz(q)$, $L_2(q)$ or $U_3(q)$, $q > 2$, a power of 2. We have $C_U \leq C_a$ and by 3.3.14 $C_a/\langle a\rangle \cong Suz$. As there is no subgroup isomorphic to $3 \times U_4(3)$ in Suz (see([AT],page 131)), we get that C_U is a non split extension of U by $U_4(3)$ (see 3.4.1). Since L is of odd index in H, we have $C_U \leq L$. By ([Gor], theorem 16.4) we get that the centralizer of each involution in $L/O(H)$ is 2-closed, but $D = C_{C_U}(z)$ is not 2-closed (see 3.2.13). Therefore $C_U = H$ and then by 3.4.1 we have $H/U \cong U_4(3)$ which proves the theorem. □

For the remainder of this chapter we denote by A the centralizer of z in G.

Lemma 3.4.11 *i) $C_A(a)/\langle a\rangle$ is an extension of an extraspecial 2-group of order 2^7 by $\Omega_6^-(2)$.*

ii) $O_3(C) = U$ and C/U is an extension of an extraspecial 2-group of order 2^5 by $(S_3 \times S_3).2$.

iii) Let X be a W-invariant $3'$-subgroup of C, then $X \leq Y$. In particular $O_2(C) = Y$ and $C = N_{C_s}(Y)$.

Proof: Since z is the preimage of $\overline{z}$, i) follows from 3.3.13 and ii) follows from 3.2.13 and 3.4.10. We have Y is extraspecial with center $\langle z\rangle$, so $N_{C_s}(Y) \leq C$. Now by ii) and 3.4.5 we get that $C = N_{C_s}(Y)$. Then iii) follows from 3.4.4(i) and the lemma holds. □

We note that each involution in M_{11} is a 2-central involution in M_{12} and the preimage of a 2-central involution in M_{12} is an involution in $2M_{12}$ ([GLS3] table

5.3b). In fact, each involution in $2M_{12}$ is 2-central.

Let $N \cong M_{12}$ and $r \in N$ be a 2-central involution, then by ([AT],page 32) we get that $C_N(r)/O_2(C_N(r)) \cong S_3$, $O_2(C_N(r))$ is extraspecial of order 32 and $C_N(r)/O_2(C_N(r))$ acts faithfully on $O_2(C_N(r))/Z(O_2(C_N(r)))$. By the general assumption we have $H_2/E \cong 2M_{12}$ and E/U is an elementary abelian group of order 81. By 3.4.1 $C_U/U \cong U_4(3)$, so by ([AT],page 52) we get that $N_{C_U}(E)/E \cong A_6$ and hence $(H_2 \cap C_U)/E \cong A_6$. Since $z \in H_2 \cap C_U$ and $(H_2 \cap C_U)/E \cong A_6$, we get that $zE \notin Z(H_2/E)$ and by ([AT],page 32) $C_{H_2}(z)/\langle W, z\rangle$ is an extension of a 2-group of order 2^5 by S_3. By 3.3.3(i) and 3.3.2 we get that $|N_{A\cap C_a}(W)/W| = 2^4 \cdot 3$. By 2.4.2(i) we get that $(C_a \cap H_2)/E \cong M_{11}$ and $N_{H_2}(\langle a\rangle)/E \cong M_{11} \times 2$. Since $C_{H_2}(z) \leq N_A(W)$, $z \in C_a \cap H_2$, $N_{H_2}(\langle a\rangle)/E \cong M_{11} \times 2$ and the centralizer of an involution in M_{11} is isomorphic to $GL_2(3)$ ([AT],page 18), we get that $N_{A\cap H_2}(W, \langle a\rangle)/W$ is of order $2^5 \cdot 3$. As $N_{A\cap C_a}(W)$ is of index at most 2 in $N_A(W, \langle a\rangle)$, $|N_{A\cap C_a}(W)/W| = 2^4 \cdot 3$ and $N_{A\cap H_2}(W, \langle a\rangle)/W$ is of order $2^5 \cdot 3$, we get that $N_A(W, \langle a\rangle)/W$ is of order $2^5 \cdot 3$. Since $|L \cap P(W)| = 4$ (see 3.4.3(ii)), $C_{H_2}(z) \leq N_A(W)$ and $|C_{H_2}(z)| = 4|N_A(W, \langle a\rangle)|$, we get that $C_{H_2}(z) = N_A(W)$. Under the action of $N_A(W)/W$ on $P(W)$ we see that $L \cap P(W)$ is the orbit containing $\langle a\rangle$. We collect this in the following lemma.

Lemma 3.4.12 *i)* $N_A(W) = C_{H_2}(z)$ *and* $|N_A(W)/W| = 2^7 \cdot 3$.

ii) Under the action of $N_A(W)/W$ *on* $P(W)$ *we have that* $L \cap P(W)$ *is the orbit containing* $\langle a\rangle$*.*

Lemma 3.4.13 *Let* $F \in Syl_3(C_A(a))$ *such that* $W \leq F$*. Then*

i) $W = J(F)$*.*

ii) $F \in Syl_3(A)$*.*

Proof: By 3.3.2 and 3.3.3(i) $N_{C_a\cap A}(W)/\langle W, z\rangle = \widehat{X}$ with $\widehat{X} \cong S_4$ and $W/\langle a\rangle$ a faithful, irreducible $\widehat{X}$-module. Let $\widehat{x} \in \widehat{X}$ be an element of order three. We have that $\langle \widehat{x}, \widehat{x}^{\widehat{y}}\rangle \cong A_4$ for some element $\widehat{y} \in \widehat{X}$. By 3.4.2(i) we have that $|W| = 81 = 3^4$. Hence $|C_W(\widehat{x})| \leq 9$. Therefore $W = J(F)$. Let F_1 be a 3-subgroup of A such that $F \lhd F_1$, then $W \lhd F_1$. Now by 3.4.12(i) we get that $|F_1| \leq 3^5$. Hence $F = F_1$ and the lemma is proved. □

We remark that for an element $x \in W$ of order three, we have that $N_A(W, \langle x\rangle) = N_{H_2}(\langle x\rangle, W) \cap A$ as we have $N_A(W) = C_{H_2}(z)$ by 3.4.12(i). By the general assumption $C_{H_1}(R) = \langle t \rangle$. Hence for an involution $rR \in Z(H_1/R)$ we have that r acts fixed point freely on $R/Z(R)$.

Lemma 3.4.14 *Under the action of $N_A(W)/\langle W, z\rangle$ on $P(W)$ we have 5 orbits L_1, L_2, L_3, L_4 and L_5 such that $L_1 = L \cap P(W)$ and*

i) $|L_1| = |L_2| = |L_3| = 4$, $|L_4| = 16$, $|L_5| = 12$.

ii) $L_2 = \{\langle a^{-1}bcu\rangle, \langle ab^{-1}cu\rangle, \langle ac^{-1}bu\rangle, \langle au^{-1}cb\rangle\}$.

iii) $L_3 = \{\langle abcu\rangle, \langle a^{-1}b^{-1}cu\rangle, \langle a^{-1}c^{-1}bu\rangle, \langle a^{-1}u^{-1}cb\rangle\}$.

iv) $L_4 = \{\langle x\rangle$, such that $x = x_1x_2x_3$ and $\langle x_1\rangle$, $\langle x_2\rangle$ and $\langle x_3\rangle$ are three distinct elements of $L_1\}$.

v) $L_5 = \{\langle x\rangle$, such that $x = x_1x_2$ and $\langle x_1\rangle$ and $\langle x_2\rangle$ are two distinct elements of $L_1\}$.

Proof: By 2.4.2(ii) we have $N_{H_2}(\langle s\rangle)/E \cong M_{10} : 2 \cong A_6.2^2$. We have E/U is an elementary abelian group of order 81 and by 3.4.1 $C_U/U \cong U_4(3)$. So by ([AT],page 52) we get that $N_{C_U}(E)/E \cong A_6$ and hence $(H_2 \cap C_U)/E \cong A_6$. Since $z \in H_2 \cap C_U$ and $N_{H_2}(\langle s\rangle)/E \cong A_6.2^2$, we get with ([AT],page 4) that $|(N_{H_2}(\langle s\rangle) \cap A)/W| = 2^5$. As by 3.4.12(i) we have $N_A(W) = C_{H_2}(z)$, we get that $|N_A(\langle s\rangle, W)/W| = 2^5$. Since by 3.4.12(i) $|N_A(W)/W| = 2^7 \cdot 3$, we conclude that under the action of $N_A(W)/W$ on $P(W)$ the orbit containing $\langle s\rangle$ is of length 12. Let $L_1 = L \cap P(W)$, then by 3.4.12(ii) L_1 is the orbit containing $\langle a\rangle$ and $|L_1| = 4$. Set $L_5 = \{\langle x\rangle$, such that $x = x_1x_2$ and $\langle x_1\rangle$ and $\langle x_2\rangle$ are two distinct elements of $L_1\}$. Then $|L_5| = 2\binom{4}{2} = 12$. By 3.3.2 and 3.3.3(i) we have $N_{A\cap C_a}(W)/W \cong GL_2(3)$ and $\overline{W}$ is a faithful, irreducible $N_{A\cap C_a}(W)/\langle W, z\rangle$-module. Now by 2.4.5 we get that $N_{A\cap C_a}(W)/W$ acts 3-transitively on $L_1 \setminus \{\langle a\rangle\}$. Let $B_1 = \{\langle u\rangle, \langle c\rangle, \langle b\rangle\} = L_1 \setminus \{\langle a\rangle\}$, $B_2 = \{\langle cu^{-1}\rangle, \langle cu\rangle, \langle bu^{-1}\rangle, \langle bu\rangle, \langle bc^{-1}\rangle, \langle bc\rangle\}$ and $B_3 = \{\langle b^{-1}cu\rangle, \langle c^{-1}ub\rangle, \langle u^{-1}cb\rangle\}$. Then by 2.4.5 $N_{A\cap C_a}(W)/W$ acts transitively on B_i, $i = 1, 2, 3$. Since $N_A(W)/W$ is transitive on B_2 and L_1, we get that all elements of L_5 are conjugate in $N_A(W)$. Since $\langle s\rangle \in L_5$ and the orbit containing $\langle s\rangle$ is of length 12, we get that L_5 is the orbit containing $\langle s\rangle$. By 3.1.2 we get that $W \in Syl_3(C_A(t, a))$. If z acts fixed point freely on $R/Z(R)$, then $zR \in Z(H_1/R)$. As aR is a 3-central element in $H_1/R \cong 2U_4(2) : 2$ and $|U_4(2)|_3 = 3^4$ ([AT],page 26), we get that $|C_A(t, a)|_3 = 3^5$ which is a contradiction to $W \in Syl_3(C_A(t, a))$ and $|W| = 3^4$(see 3.4.2(i)). Hence z does not act fixed point freely on $R/Z(R)$ and so $zR \notin Z(H_1/R)$. By 3.4.12(i) we have $N_A(W, \langle t \rangle) = C_{H_1 \cap H_2}(z)$. By the general assumption $H_1 \cap H_2/E$ is an extenstion of an elementary abelian group of order 9 by $GL_2(3) \times 2$ and $O_3(H_1 \cap H_2/E)$ is the natutal module for $H_1 \cap H_2/O_3(H_1 \cap H_2)$. Since $C_{H_1 \cap H_2}(\tau)/O_3(H_1 \cap H_2) \cong GL_2(3)$, $zR \notin Z(H_1/R)$ and z centralizes t, we get that $|N_{H_1 \cap A}(W)/W| = 2^3 \cdot 3$. By 3.4.12(i) $|N_A(W)/W| = 2^7 \cdot 3$. This gives us that the orbit containing $\langle t \rangle$ is of length 16. Set $L_4 = \{\langle x \rangle$, such that $x = x_1x_2x_3$ and $\langle x_1 \rangle$, $\langle x_2 \rangle$ and $\langle x_3 \rangle$ are three distinct elements of $L_1\}$. Then $|L_4| = 4\binom{4}{3} = 16$. We have $N_A(W)/W$ is transitive on L_1 and B_2, so all elements of L_4 are conjugate in $N_A(W)$. Since $\langle t \rangle \in L_4$ and the orbit containing $\langle t \rangle$ is of length 16, we get that L_4 is the orbit containing $\langle t \rangle$. Now set $L_2 = \{\langle a^{-1}bcu \rangle, \langle ab^{-1}cu \rangle, \langle ac^{-1}bu \rangle, \langle au^{-1}cb \rangle\}$ and $L_3 = \{\langle abcu \rangle, \langle a^{-1}b^{-1}cu \rangle, \langle a^{-1}c^{-1}bu \rangle, \langle a^{-1}u^{-1}cb \rangle\}$. Then $|L_3| = |L_2| = |L_1| = 4$. We note that by 3.3.4(iv) bcu and $c^{-1}u^{-1}b^{-1}$ are not conjugate in $N_{A \cap C_a}(W)$ and by 3.2.3(iii) cu and $c^{-1}u$ are not conjugate in $N_{A \cap C_U}(W)$. By 3.2.8 and 3.2.3(i) $N_{\widetilde{C_{\tilde{z}}}}(\widetilde{Y})/\widetilde{Y} = N_{\widetilde{C_{\tilde{z}}}}(\widetilde{W})\widetilde{Y}/\widetilde{Y}$, so by 3.2.9(i) there is an involution in $N_{A \cap C_U}(W)$ which acts fixed point freely on W/U. Hence cu is conjugate to $c^{-1}u^{-1}$ in $N_{A \cap C_U}(W)$. As $N_A(W)/W$ is transitive on L_1 and B_3, all elements of L_2 are conjugate in $N_A(W)$ and all elements of L_3 are conjugate as well. Since $|P(W)| = 40$, we get that $|P(W)| - (|L_1| + |L_4| + |L_5|) = 8$ and so either L_2 and L_3 are two orbits or $L_2 \cup L_3$ is an orbit. Since by 2.4.2(i) for each five distinct elements $\langle x_i \rangle$, $i = 1, ..., 5$, from the orbit L we have that $x_1, ..., x_5$ are linear independent, we get that the elements in L_2 and L_3 are conjugate to $\langle t \rangle$ in H_2. Assume that $L_2 \cup L_3$ is an orbit of $N_A(W)/W$. Then for $\langle x \rangle \in L_2 \cup L_3$ we have $|N_A(W, \langle x \rangle)|_2 = 2^4$, as by 3.4.12(i), $|N_A(W)/W| = 2^7 \cdot 3$. We have $x = t^g$ for some $g \in H_2$, and then by 3.4.12(i) $N_A(\langle x \rangle, W) = C_{H_1^g \cap H_2}(z)$. By the general assumption we have $(H_1^g \cap H_2)/O_3(H_1^g \cap H_2) \cong GL_2(3) \times 2$. Therefore for each involution $f \in H_1^g \cap H_2$ we have $|C_{H_1^g \cap H_2}(f)|_2 = 2^5$ or 2^3. Hence $L_2 \cup L_3$ is not an orbit and hence L_2 and L_3 are two orbits of length 4 and the lemma is proved. □

Further notation: For the remainder of this chapter we adopt the notations L_1, L_2, L_3, L_4 and L_5 from 3.4.14.

Remarks:1) By 2.4.2(i) for each five distinct elements $\langle x_i \rangle$, $i = 1, ..., 5$ from the orbit L we have that $x_1, ..., x_5$ are linear independent, so $(L_2 \cup L_3 \cup L_4) \subseteq J$. By 2.4.2(ii) we get that $L_5 = I \cap P(W)$. We recall that $\langle t \rangle \in L_4$, $\langle s \rangle \in L_5$ and $\langle a \rangle \in L_1$.

2) Let $N \cong U_4(2)$ and $X \leq N$ be an elementary abelian group of order 27. Then by ([AT], page 26) $N_N(X)/X \cong S_4$, $N_N(X)$ is a maximal subgroup of N and N has just one class of subgroups isomorphic to X. By coprime action and the structure of the centralizes of the elements of order three in N ([AT],page 26) we get that there is no X-invariant $3'$-subgroup in N.

Lemma 3.4.15 *i) Let $\langle x \rangle \in L_2 \cup L_3$. Then $C_A(x)/\langle x, z \rangle \cong U_4(2)$. In particular $O_2(C_A(x)) = \langle z \rangle$.*

ii) $zR \notin Z(H_1/R)$.

Proof: Let $\langle x \rangle \in L_2 \cup L_3 \cup L_4$. Then as by 2.4.2(i) for each five distinct elements $\langle x_i \rangle$, $i = 1, ..., 5$ from the orbit L we have that $x_1, ..., x_5$ are linear independent, we get that x is conjugate to t in H_2. Let $x = t^g$, $N = H_1^g$ and $F = R^g$ with $g \in H_2$. By 3.4.12(i) we have $N_A(W, \langle x \rangle) = C_{N \cap H_2}(z)$. By the general assumption $N \cap H_2/E$ is an extenstion of an elementary abelian group of order 9 by $GL_2(3) \times 2$. If $zF \notin Z(N/F)$, as z centralizes x, we get that $|N_{N \cap A}(W)/W| = 2^3 \cdot 3$. This gives us that the orbit containing $\langle x \rangle$ is of length 16 as by 3.4.12(i) $|N_A(W)/W| = 2^7 \cdot 3$. Now by 3.4.14(i) we get that $\langle x \rangle \in L_4$. Therefore, if $\langle x \rangle \in L_2 \cup L_3$ we have that $zF \in Z(N/F)$ and $zR \notin Z(H_1/R)$. Suppose that $\langle x \rangle \in L_2 \cup L_3$. Then as $zF \in Z(N/F)$ and $C_N(F) = Z(F)$, we get that z acts fixed point freely on $F/Z(F)$ and hence $C_A(x)/\langle x, z \rangle \cong U_4(2)$. Now the lemma is proved. □

Lemma 3.4.16 *Let $\langle x \rangle$ be in the orbit L_4. Then*

i) $O_3(C_A(x))$ is an extraspecial 3-group of order 27.

ii) $C_A(x)/O_3(C_A(x)))$ is an extension of $Q_8 \times Q_8$ by an elementary abelian group of order 9.

iii) $O_2(C_A(x)) \cong Q_8$.

iv) Let X be a W-invariant $3'$-subgroup of $C_A(x)$, then $X \leq O_2(C_A(x))$ and either $X = \langle z \rangle$ or $X = O_2(C_A(x))$.

Proof: We have $\langle t \rangle \in L_4$ and so we just prove the lemma for $x = t$. Since $|L_4| = 16$, we get that $N_{A \cap H_1}(W)$ has index 16 in $N_A(W)$ and $zR \notin Z(C_G(t)/R)$ by 3.4.15(ii). Therefore zR is a non 2-central involution in $C_{H_1}(t)/R \cong 2U_4(2)$ and by 2.4.7(viii) we get that $C_A(t)/C_R(z)$ is an extension of $Q_8 \times Q_8$ by an elementary abelian group of order 9. As $C_{H_1}(R) = Z(R)$, any involution $rR \in Z(H_1/R)$ acts fixed point freely on $R/Z(R)$. We have $zR \notin Z(H_1/R)$ by 3.4.15(ii). If z acts fixed point freely on $R/Z(R)$, then rz centralizes R, which is a contradiction. By coprime action we have $R = C_R(z)\,[z, R]$. As $C_R(z)$ and $[z, R]$ are $C_A(t)/C_R(z)$-invariant, $O_2(C_A(t)/C_R(z)) \cong Q_8 \times Q_8$, $|R/Z(R)| = 81$ and by the general assumption $C_{H_1}(R) = Z(R)$, we get that $C_R(z) \cong [z, R]$ is an extraspecial group of order 27. Since $C_{R/Z(R)}(z)$ is of order 9, $|O_2(C_A(t)/C_R(z))|_2 = 2^6$ and $C_{R/Z(R)}(z)$ is $C_A(t)/C_R(z)$-invariant, we have $O_2(C_A(t)) > \langle z\rangle$. We have $O_2(C_A(t))$ centralizes $C_R(z)$ and so $|O_2(C_A(t))C_R(z)/C_R(z) \cap Z(O_2(C_A(t)/C_R(z)))| = 2$ (we remark that $rC_R(z) \in Z(O_2(C_A(t)/C_R(z))$ and r acts fixed point freely on $R/Z(R)$). As $O_2(C_A(t))$ is W-invariant, by 2.4.7(viii) either $O_2(C_A(t)) \cong Q_8$ or $O_2(C_A(t)) \cong Q_8 \times Q_8$. If $|O_2(C_A(t))| = 2^6$, then as $[R/Z(R), z]$ is of order 9 and $[R/Z(R), z]$ is $O_2(C_A(t))$-invariant, we get that 2 divides the order of $C_{O_2(C_A(t))}([R, z])$, but then 2 divides the order of $C_{H_1}(R)$ which is a contradiction to the general assumption. So $O_2(C_A(t)) \cong Q_8$. Now let X be a W-invariant $3'$-subgroup of $C_A(t)$, then $X/C_R(z) \leq O_2(C_A(t)/C_R(z))$. As $C_R(z)$ is of order 27, $|W| = 81$ (by 3.4.2(i)) and $|C_A(t)|_3 = 3^5$, we have $|W \cap C_R(z)| = 9$. Since X is W-invariant, $C_R(z)$ is an extraspecial group of order 27 and $|W \cap C_R(z)| = 9$, we have $X \leq O_2(C_A(t))$. As $O_2(C_A(x)) \cong Q_8$ and X is W-invariant we get that either $X = \langle z\rangle$ or $X = O_2(C_A(x)) \cong Q_8$ and the lemma is proved. □

Lemma 3.4.17 *Let $\widehat{K}$ be a W-invariant $3'$-subgroup of A. Then*

i) $\widehat{K}$ is a 2-group and $|\widehat{K}| \leq 2^9$.

ii) $C_A(x) \cap \widehat{K} \leq O_2(C_A(x))$ for each element $x \in W$ of order three.

iii) $\widehat{K} = \left\langle \widehat{K} \cap O_2(C_A(x)) | x \in W^\sharp \right\rangle$.

Proof: As $\widehat{K}$ is W-invariant, by coprime action we have

$$\widehat{K} = \langle C_{\widehat{K}}(x), x \in W^\sharp \rangle.$$

Suppose that $x = a$ then by 3.4.11(i) we have $C_A(x)/\langle x\rangle$ is an extension of an extraspecial 2-group of order 2^7 by $U_4(2)$. As $\widehat{K} \cap C_G(x)$ is W-invariant, by 5.13 and 5.7 we get that $\widehat{K} \cap C_G(x) \leq O_2(C_A(x))$ and by 5.5(ii) and coprime action we get that $\widehat{K} \cap C_A(x) = \langle z\rangle$ or $\widehat{K} \cap C_A(x)$ is an extraspecial 2-group and $|\widehat{K} \cap C_A(x)| \leq 2^5$. Assume that $x = t$, then $\langle x\rangle \in L_4$ and by 3.4.16(iv),(iii) we get that either $\widehat{K} \cap C_A(x) = \langle z\rangle$ or $\widehat{K} \cap C_A(x) = O_2(C_A(x)) \cong Q_8$.

Suppose that $x = s$, then $\langle x\rangle \in L_5$ and by 3.4.11(iii) we get that $\widehat{K} \cap C_A(x) \leq Y$. Since $\widehat{K} \cap C_A(x) \leq Y$ is W-invariant, by 3.2.5(ii) and coprime action we get that $O_2(C_A(x)) \cap K \cong Q_8$ or $\widehat{K} \cap C_A(x) = \langle z\rangle$ or $\widehat{K} \cap C_A(x) = O_2(C_A(x))$. Assume that $\langle x\rangle \in L_2 \cup L_3$, then by 3.4.15 (i) $\widehat{K} \cap C_G(x) = \langle z\rangle$.

Hence either for each element $x \in W^\sharp$ we have $C_{\widehat{K}}(x) = \langle z\rangle$ and then $\widehat{K} = \langle z\rangle$ or $C_{\widehat{K}}(x)$ is an extraspecial 2-group. So by Wielandt's order formula ([BH],XI.12.6) we get that $\widehat{K}$ is a 2-group and $2^{|W|}|\widehat{K}|^{|P(W)|-1} \leq |O_2(C_A(a))|^{|a||L_1|}|O_2(C_A(s))|^{|s||L_5|}|O_2(C_A(t))|^{|t||L_4|}|O_2(C_A(t$ $|O_2(C_A(tu))|^{|tu||L_3|}$. Now by lemmas 3.4.14(i), 3.4.11(i),(ii), 3.4.16(iii), 3.4.2(i) and 3.4.15(i) and as $|a| = |t| = |tu| = |tau| = |s| = 3$, we conclude that $|\widehat{K}| \leq 2^9$ and hence the lemma is proved. $\square$

We recall that $U = \langle a, b\rangle$ is a subgroup of W of order 9, $u \notin U$ and $Y = O_2(C_A(U)) = O_2(C_A(U, c))O_2(C_A(U, u))$. Set

$$K_z = O_2(C_A(u))Y.$$

Lemma 3.4.18 *i) $O_2(C_A(a)) = O_2(C_A(u, a))Y$ and $O_2(C_A(b)) = YO_2(C_A(u, b))$.*

ii) $K_z = O_2(C_A(a))O_2(C_A(b))$ and K_z is an extraspecial 2-group of order 2^9.

iii) $O_2(C_A(c)) = O_2(C_A(c, a))O_2(C_A(c, b))$ and $O_2(C_A(u)) = O_2(C_A(u, a))O_2(C_A(u, b))$.

Proof: As a, u, c and b are conjugate in $N_A(W)$ by6.12(ii), we get that i) and iii) follow from 3.3.6(ii). We recall that u and c are conjugate in $A \cap C_U$ and $N_A(W)$ acts 2-transitively on L_1. By 3.4.11(i) $O_2(C_A(a))$ is an extraspecial 2-group of order 2^7, so by 3.4.12(ii) we get that $O_2(C_A(x))$ is an extraspecial 2-group of order 2^7 for all $\langle x\rangle \in L \cap P(W)$. By i) we have that $Y \leq O_2(C_A(b)) \cap O_2(C_A(a))$. By i) and iii) and as $O_2(C_A(u))$ is an extraspecial 2-group, we get that K_z is an extraspecial 2-group. We have $O_2(C_A(u)) \cap Y \leq O_2(C_{C_U}(u))$ and as ub is conjugate to bc in C_a by 2.4.3(ii) we get with 3.1.2 that $O_2(C_{C_a}(u, b)) \cong Q_8$. Therefore $|O_2(C_A(u)) \cap Y| \leq 8$. Since both $O_2(C_A(u))$ and Y are W-invariant and K_z is a 2-group, we get that K_z is a W-invariant $3'$-subgroup of A. Hence by 3.4.17(i) we have that $|K_z| \leq 2^9$. As by 3.2.6(ii) $|O_2(C_A(u)) \cap Y| \leq 8$, $|Y| = 2^5$ and $|O_2(C_A(u))| = 2^7$, we conclude that K_z is of order 2^9 and the lemma is proved. $\square$

Lemma 3.4.19 *i) K_z is the unique maximal W-invariant $3'$-subgroup of A.*

ii) $N_A(W) \leq N_A(K_z)$.

iii) For $x \in W^\sharp$ we have that $O_2(C_A(x)) \leq K_z$.

Proof: By lemmas 3.4.17(i) and 3.4.18(ii), K_z is a maximal W-invariant $3'$-subgroup of A and any W-invariant $3'$-subgroup of A is a 2-group. Suppose that $\langle x \rangle \in L_1$, then by 3.4.18(i),iii)) we have that $O_2(C_A(x))$ is a subgroup of K_z. Assume that $\langle x \rangle \in L_2 \cap L_3$. Then by 3.4.15(i) $O_2(C_A(x)) = \langle z \rangle$ is a subgroup of K_z. Suppose that $x = s$, then $\langle x \rangle \in L_5$ and by 3.4.11(iii) $O_2(C_A(x)) = Y$ and hence $O_2(C_A(x)) \leq K_z$. Assume that $\langle x \rangle \in L_4$, then by 3.4.16(iii) $O_2(C_A(x)) \cong Q_8$ and hence $O_2(C_A(x))$ centralizes a subgroup of order 27 of W, as $|Aut(Q_8)|_3 = 3$. Therefore by 3.4.3(i) there is an element $\langle y \rangle \in L_5$ such that $O_2(C_A(x)) \leq C_A(y)$. As $\langle s \rangle \in L_5$ we may assume that $y = s$, then by 3.4.11(iii) and as $O_2(C_A(x))$ is W-invariant, we get that $O_2(C_A(x)) \leq Y$ and hence $O_2(C_A(x)) \leq K_z$. Now iii) holds, ii) follows from i) and i) follows from iii) and 3.4.17(ii). □

Lemma 3.4.20 *Let $V = \langle a, bcu, b^{-1}u \rangle$ and X be a V-invariant $3'$-subgroup of A. Then $X \leq K_z$.*

Proof: Let $V = \langle a, bcu, b^{-1}u \rangle$, then $P(V) \cap L_1 = \{\langle a \rangle\}$, $P(V) \cap L_2 = \{\langle a^{-1}bcu \rangle\}$, $P(V) \cap L_3 = \{\langle abcu \rangle\}$, $P(V) \cap L_4 = \{\langle bcu \rangle, \langle acu^{-1} \rangle, \langle abc^{-1} \rangle, \langle auc^{-1} \rangle, \langle abu^{-1} \rangle, \langle acb^{-1} \rangle, \langle aub^{-1} \rangle\}$ and $P(V) \cap L_5 = \{\langle b^{-1}u \rangle, \langle c^{-1}b \rangle, \langle cu^{-1} \rangle\}$. Let $\alpha \in N_A(W)$ be of order 2 and $[\alpha, V] = 1$, then $[\alpha, a] = 1$. Since α centralizes each element of $P(V) \cap L_5$ and L_1 is α-invariant, we have $[\alpha, W] = 1$ and hence $\alpha = z$. Now let X be a V-invariant $3'$-subgroup in A, then $X \cap N_A(W) = \langle z \rangle$. Let $F \leq V$ be of order 9, then either $P(V) \subseteq L_4$ or $|P(F) \cap L_5| \geq 1$. Let $|P(F) \cap L_5| \geq 1$ and $\langle y \rangle \in (P(F) \cap L_5)$, then $y = xv$ where $\langle x \rangle$ and $\langle v \rangle$ are from $L_1 \setminus \{\langle a \rangle\}$. By 3.4.11(ii) we have that $C_A(y)/\langle x, v \rangle$ is an extension of an extraspecial group of order 32 by $(S_3 \times S_3)2$. We have $V \cap \langle x, v \rangle = \langle y \rangle$, $C_A(y)/O_2(C_A(y))$ is irreducible on $O_2(C_A(y))/Z(O_2(C_A(y))$ and $X \cap N_A(W) = \langle z \rangle$. As $C_A(y)/O_3(C_A(y))$ is isomorphic to the centralizer of a 2-central involution in $C_s/U \cong U_4(3) : 2$, we get that $X \cap C_A(y) \leq O_2(C_A(y))$. Now let $P(F) \subseteq L_4$ and $1 \neq y \in F$, then by 3.4.16 we have $O_3(C_A(y))$ is an extraspecial 3-group of order 27, $C_A(y)/O_3(C_A(y)))$ is an extension of $Q_8 \times Q_8$ by an elementary abelian group of order 9 and $O_2(C_A(y)) \cong Q_8$. We note that as $X \cap N_A(W) = \langle z \rangle$, we have $(X \cap C_A(y))O_{2,3}(C_A(y))/O_{2,3}(C_A(y)) \cap Z(C_A(y)/O_{2,3}(C_A(y))) = 1$ and this gives us that $X \cap C_A(y) \leq O_2(C_A(y))$. Therefore $C_X(F) \leq O_2(C_A(x))$ for some element $x \in V$ of order three. Now by coprime action and 3.4.19(iii) we get that $X \leq K_z$ and the lemma is proved. □

Lemma 3.4.21 $C_A(a) \leq N_A(K_z)$.

Proof: As by 3.4.19(ii) $N_A(W)$ is a subgroup of $N_A(K_z)$, we have $N_{C_a \cap A}(W) \leq N_A(K_z)$. Set $Q = O_2(C_A(a))$. By 3.3.4 we get that for $x = bcu$ we have that $C_A(x,a)Q/\langle a, Q\rangle$ is a split extension of an extraspecial 3-group of order 27 by $SL_2(3)$. As $\overline{z}$ acts nontrivially on $Z(O_3(C_{\overline{C_a}}(\overline{x})))$, we get by 3.3.2 and 3.3.3(i) that $C_A(x,a)$ is not contained in $N_{C_a \cap A}(W)$. Set $X = C_A(x,a)$ and let $P \leq X$ be a 3-group such that $PQ/Q = O_3(XQ/Q)$. We have $W \leq X$ and so by 3.4.13(i),(ii) we get that $P \leq N_A(W)$. Now 3.4.19(ii) gives us that $P \leq N_A(K_z)$. Set $V = P \cap W$, then we have that $Z(P) = \langle a, x\rangle$ and V is of order 27. Let $y \in X$ be an element of order 4 such that $\widehat{y} = \langle y\rangle Q/Q \in XQ/Q$ is of order 4. Then $\widehat{y}^2$ acts fixed point freely on $P/Z(P)$ and $y^2 \in N_A(W)$. We have $x = bcu$ and $\widehat{y}^2$ centralizes x, so $\widehat{y}^2$ does not invert b, c and u. Hence $P(V) \cap L_1 = \{\langle a\rangle\}$. Now by the representations of the elements in the orbits L_i in 3.4.14 for $i = 1, ..., 5$ we get that $P(V) \cap L_2 = \{\langle a^{-1}bcu\rangle\}$, $P(V) \cap L_3 = \{\langle abcu\rangle\}$, $P(V) \cap L_4 = \{\langle bcu\rangle, \langle acu^{-1}\rangle, \langle abc^{-1}\rangle, \langle auc^{-1}\rangle, \langle abu^{-1}\rangle, \langle acb^{-1}\rangle, \langle aub^{-1}\rangle\}$ and $P(V) \cap L_5 = \{\langle b^{-1}u\rangle, \langle c^{-1}b\rangle, \langle cu^{-1}\rangle\}$. So $V = \langle a, bcu, b^{-1}c\rangle$ and $|P(V) \cap L_1| = |P(V) \cap L_2| = |P(V) \cap L_3| = 1$, $|P(V) \cap L_5| = 3$ and $|P(V) \cap L_4| = 7$. Now let $r \in P$, then $d = [y, r] \in PQ$. Let $yd = gy$ for some $g \in PQ$. By 3.4.19(ii),iii) we get that PQ is a subgroup of $N_A(K_z)$. Therefore we have that

$$((K_z)^y)^r = (K_z)^{y^r} = (K_z)^{yd} = (K_z)^{gy} = (K_z)^y.$$

Hence $(K_z)^y$ is V-invariant and therefore by 3.4.20 we get that $(K_z)^y = K_z$. This gives us that $y \in N_A(K_z)$ and so $C_A(a, x)$ normalizes K_z. Since by 3.4.11(i), $C_A(a)Q/Q \cong 3 \times U_4(2)$ and by 3.3.2 and 3.3.3(i), $N_{A \cap C_a}(W)Q/Q$ is an extension of an elementary abelian group of order 81 by S_4, by ([AT],page 26) $N_{C_a \cap A}(W)Q/Q$ is a maximal subgroup of $C_A(a)Q/Q$. Therefore $C_A(a) \leq N_A(K_z)$ and the lemma holds. □

Lemma 3.4.22 $N_A(K_z)/K_z \cong \Omega_8^+(2)$.

Proof: We note that for $N \cong U_4(2)$ and F an elementary abelian group in N of order 27, that by ([AT],page 26) each element of order three in N is conjugate to an element of F. So by 3.4.11(i) and 3.4.13(ii) we get that each element of order three of A is conjugate to an element of W in A. Now by 3.4.11(i),(ii), 3.4.15(i) and 3.4.16 we get that no element of order three in W centralizes K_z. Further by 3.4.19(i) K_z is the unique maximal W-invariant $3'$-subgroup of A. Therefore $C_A(K_z) = \langle z \rangle$. Since by 3.4.18(ii), K_z is an extraspecial 2-group of order 2^9, by ([GLS2],theorem 10.6) $N_A(K_z)/K_z$ is isomorphic to a subgroup of $Aut(K_z)/Inn(K_z) \cong O_8^+(2)$. By ([AT], page 85) the normalizer of an elementary abelian group of order 81 in $O_8^+(2)$ is isomorphic to $S_3 wr S_4$. Hence from the structure of $N_A(W)$ in 3.4.12(i) we conclude that $N_A(K_z)/K_z$ is not isomorphic to $O_8^+(2)$. By 3.4.21 we have that $C_A(a) \leq N_A(K_z)$. As $\Omega_8^+(2)$ is the unique simple subgroup of index 2 in $O_8^+(2)$, from the structure of $C_A(a)$ in 3.4.11(i) we get that $C_A(a)K_z/K_z$ is isomorphic to a subgroup of $\Omega_8^+(2)$. Let $\Omega_8^+(2) \cong \widehat{N} \leq Aut(K_z)/Inn(K_z)$. We have that $C_A(a)K_z/K_z$ is isomorphic to a subgroup of $\widehat{N}$. By 3.4.19(ii), $N_A(W) \leq N_A(K_z)$. Suppose that $N_A(W)K_z/K_z$ is isomorphic to a subgroup of $\widehat{N}$. Then $N_A(W)K_z/K_z$ is isomorphic to a maximal subgroup of $\Omega_8^+(2)$ ([AT],page 85) and as $C_A(a)$ is not a subgroup of $N_A(W)$ by 3.4.11(i) and 3.4.12(i), we get that $N_A(K_z)/K_z \cong \widehat{N} \cong \Omega_8^+(2)$.

So it is enough for us to show that $N_A(W)K_z/K_z$ is isomorphic to a subgroup of $\widehat{N}$. Since $\widehat{N}$ is of index 2 in $Aut(K)/Inn(K)$, either $N_A(W)K_z/K_z$ is isomorphic to a subgroup of $\widehat{N}$ or $N_A(W)K_z/K_z$ has a subgroup $F^\star$ of index 2 isomorphic to a subgroup of $\widehat{N}$. We assume that $N_A(W)K_z/K_z$ is not isomorphic to a subgroup of $\widehat{N}$ and hence $N_A(W)K_z/K_z$ has a subgroup $F^\star$ of index 2 isomorphic to a subgroup of $\widehat{N}$. We note that as $F^\star$ is of index 2 in $N_A(W)K_z/K_z$, by 3.4.12(i) and as $C_{K_z}(W) = \langle z \rangle$, we get that $|F^\star| = 3^5 \cdot 2^5$. By 3.3.2 and 3.3.3(i) we have $N_{A\cap C_a}(W)/\langle W, z\rangle \cong S_4$ and as $C_A(a)K_z/K_z$ is isomorphic to a subgroup of $\widehat{N}$, we deduce that $N_{C_a\cap A}(W)K_z/K_z$ is isomorphic to a subgroup of $F^\star$. Let $\widehat{X} \leq \widehat{N}$ be an elementary abelian group of order 81, then by ([AT],page 85) we get that $N_{\widehat{N}}(\widehat{X})/\widehat{X}$ is an extension of an elementary abelian group of order 8 by S_4. We note that $Z(O_2(N_{\widehat{N}}(\widehat{X})/\widehat{X}))$ is of order 2 and $N_{\widehat{N}}(\widehat{X})/O_{3,2}(N_{\widehat{N}}(\widehat{X}))$ acts faithfully on $O_2(N_{\widehat{N}}(\widehat{X})/\widehat{X})/Z(O_2(N_{\widehat{N}}(\widehat{X})/\widehat{X})))$. From the structure of $N_{\widehat{N}}(\widehat{X})/\widehat{X}$ we get that there is no subgroup of index 2 in $N_{\widehat{N}}(\widehat{X})/\widehat{X}$ containing a section isomorphic to S_4. But $F^\star$ is isomorphic to a subgroup of index 2 of $N_{\widehat{N}}(\widehat{X})$, $N_{A\cap C_a}(W)/\langle W, z\rangle \cong S_4$ and $N_{A\cap C_a}(W)K_z/K_z$ is a subgroup of $F^\star$. This shows that $N_A(W)K_z/K_z$ is isomorphic to a subgroup of $\widehat{N}$ and the lemma is proved. □

Lemma 3.4.23 *Let $x \in W$ be of order three, then $C_A(x) \leq N_A(K_z)$.*

Proof: By 3.4.19(ii) we have $N_A(W) \leq N_A(K_z)$. So it is enough to prove the lemma for just one element from each orbit L_i, $i = 1, 2, 3, 4, 5$. Let $x = a$, then $\langle x \rangle \in L_1$ and by 3.4.21 we have that $C_A(x) \leq N_A(K_z)$. Let $x = s$, then $\langle x \rangle \in L_5$ and by 3.4.11(ii),(iii) we have $C_A(s)/Y \cong (S_3 \times S_3).2$, so $C_A(x) \leq \langle Y, N_A(W) \rangle$. Now by 3.4.19(ii), we have $C_A(x) \leq N_A(K_z)$. Let $x = t$ then $\langle x \rangle \in L_4$. We have $W \leq N_A(K_z)$ and by 3.4.2(i) W is an elementary abelian group of order 81, so WK_z/K_z is an elementary abelian group of order 81. Since $N_A(K_z)/K_z \cong \Omega_8^+(2)$ by 3.4.22 and $|L_4| = 16$, by 2.4.8(v) we get that $|C_{N_A(K_z)/K_z}(xK_z)| = 2^3 \cdot 3^5$. On the other hand by 3.4.16 we get that $|C_A(x)/O_2(C_A(x))| = 2^3 \cdot 3^5$. Since by 3.4.19(iii) $O_2(C_A(x)) \leq K_z$ and $|C_A(x)/O_2(C_A(x)))| = |C_{N_A(K_z)/K_z}(xK_z)|$, we have $|C_{N_A(K_z)}(x)| = |C_A(x)|$ and hence $C_A(x) \leq N_A(K_z)$. Let $\langle x \rangle \in L_2 \cup L_3$, then by 3.4.15(i) $C_A(x)/\langle x, z \rangle \cong U_4(2)$. Since $|L_2| = |L_3| = 4$ and $N_A(K_z)/K_z \cong \Omega_8^+(2)$, by 2.4.8(v) we get that $C_{N_A(K_z)/K_z}(xK_z) \cong 3 \times U_4(2)$ and hence $|C_{N_A(K_z)}(x)| = |C_A(x)|$. This gives us that $C_A(x) \leq N_A(K_z)$ and the lemma is proved. □

Set $B = N_A(K_z)$. In what follows we use the notation * for the natural homomorphism $A \mapsto A/\langle z \rangle$. We are going to show that B is strongly 3-embedded in A and we will use this to show that $A = B$.

Lemma 3.4.24 *B is strongly 3-embedded in A.*

Proof: Let $x \in B$ be of order three, then by 3.4.22 and 2.4.8(v), x is conjugate to an element of W in B. So we may assume that $x \in W$ and then by 3.4.23 we have that $C_A(x) \leq B$. So we have $C_A(x) \leq B$ for all 3-element $x \in B$. Now assume that 3 divides $|B \cap B^g|$ for some $g \in A$ and let $X_1 \in Syl_3(B \cap B^g)$. Let E_0 be a 3-subgroup of B^g with $X_1 \lhd E_0$. Then for some element $x \in X_1$ of order three we have $E_0 \leq C_A(x)$. So $E_0 \leq B$. Therefore $X_1 \in Syl_3(B) \cap Syl_3(B^g)$. We may assume that $W \leq X_1$, then $W \leq N_A(K_z^g)$. Now 3.4.19(i) gives us that $K_z^g = K_z$, hence $g \in B$ and the lemma is proved. □

Lemma 3.4.25 *$K_z^\star$ is strongly closed in $B^\star$ with respect to $A^\star$.*

Proof: Let $r^\star \in K_z^\star$ be an involution and $g^\star \in A^\star$ such that $(r^\star)^{g^\star} \in B^\star$ and $(r^\star)^{g^\star} \notin K_z^\star$. By 2.4.8(ii) we get that 3 divides $|C_{B^\star}(r^\star)|$. By 2.4.8(i) $B^\star/K_z^\star$ has 5 classes, $2A$,$2B$,$2C$,$2D$ and $2E$ of involutions. Let $(r^\star)^{g^\star}K_z^\star$ be in class $2A$, then 27 divides the order of $C_{B^\star}((r^\star)^{g^\star})$. Now by 2.4.8(iv) and 2.4.6 we get that 3 divides the order of each involution in $(r^\star)^{g^\star}K_z^\star$. Let $(r^\star)^{g^\star}K_z^\star$ be in one of the classes $2C$, $2B$,$2D$ or $2E$, then by 2.4.8(iii),(i),(iv) and 2.4.6 we get that 3 divides the order of each involution in $(r^\star)^{g^\star}K_z^\star$. Therefore 3 divides the order of $C_{B^\star}((r^\star)^{g^\star})$. Let $P^\star \in Syl_3(C_{B^\star}(r^\star))$ and $P_1^\star \in Syl_3(C_{\overline{B}}((r^\star)^{g^\star}))$. Then as by 3.4.24 $B^\star$ is strongly 3-embedded in $A^\star$, we have $P^\star \in Syl_3(C_{A^\star}(r^\star))$ and $P_1^\star \in Syl_3(C_{A^\star}((r^\star)^{g^\star}))$. Since $g^\star \in A^\star$, we have $(P^\star)^{g^\star} \in Syl_3(C_{(B^\star)^{g^\star}}((r^\star)^{g^\star}))$ and as by 3.4.24 $(B^\star)^{g^\star}$ is strongly 3-embedded in $A^\star$, we get that $(P^\star)^{g^\star} \in Syl_3(C_{A^\star}((r^\star)^{g^\star}))$. Hence $(P^\star)^{g^\star g_1^\star} = P_1^\star$ for some $g_1^\star \in C_{A^\star}((r^\star)^{g^\star})$. Now we have $P_1^\star \leq B^\star \cap (B^\star)^{g^\star g_1^\star}$ and as by 3.4.24 $B^\star$ is strongly 3-embedded in $A^\star$, we get that $g^\star g_1^\star \in B^\star$. Therefore $(r^\star)^{g^\star g_1^\star} \in K_z^\star$. As $g_1^\star \in C_{A^\star}((r^\star)^{g^\star})$, we have $(r^\star)^{g^\star} = (r^\star)^{g^\star g_1^\star} \in K_z^\star$. But this is a contradiction to our assumption that $(r^\star)^{g^\star} \notin K_z^\star$. Hence $K_z^\star$ is strongly closed in $B^\star$ with respect to $A^\star$ and the lemma is proved. □

Lemma 3.4.26 $A = B$

Proof: By 3.4.25, $K_z^\star$ is strongly closed in $B^\star$ with respect to $A^\star$. So by Goldschmidt's theorem [Go] we have $H^\star = \langle (K_z^\star)^{A^\star} \rangle$ contains no section isomorphic to $\Omega_8^+(2)$. Therefore $B^\star \cap H^\star = K_z^\star$ and as $B^\star = N_{A^\star}(K_z^\star)$, we have $K_z^\star \in Syl_2(H^\star)$. Hence $K_z^\star \leq Z(N_{H^\star}(K_z^\star))$ and Burnside's p-complement theorem gives us that $H^\star = O(H^\star)K_z^\star$. Now by the Frattini argument $A^\star = O(A^\star)B^\star$. Since by 3.4.13, $B^\star$ contains a Sylow 3-subgroup of $A^\star$, we get that $O(A^\star)$ is a $W^\star$-invariant $3'$-subgroup of $A^\star$. Now by 3.4.19(i), $O(A^\star) = 1$ and hence $A = B$. □

Now we can prove the **Theorem 4**.

Proof: As by ([AT],page 52) $U_4(3)$ has just one class of involutions and by 3.4.1 $C_U/U \cong U_4(3)$ we get that z is not weakly closed in Y with respect to C_U. Therefore z is not weakly closed in K_z with respect to G. Now the theorem follows from 3.4.26 and 2.3.3. □

Chapter 4

Characterization of $M(24)'$

In this chapter we will prove theorem 5 and corolarry 7. Therefore in this chapter G is a group of $M(24)'$-type and we keep the notations G, H_1 and τ as in definition 2. We gave a sketch of the proof for theorem 5 in 1.3.2.

This chapter has three sections. In section 4.1 we give some preliminary lemmas which are needded in the next sections. In section 4.2 we will select a suitable non 2-central involution z in $O^2(H_1)$ and we shall show that $C_G(z) \cong 2M(22) : 2$. Then in section 4.3 we will select an elementary abelian subgroup M of order 2^{11} in $C_G(z)$ and we shall determine the structure of $N_G(M)$. This will able us to use ([Re], lemma 9) to find the structure of the centralizer of a 2-central involution in G and the main result follows from ([As3], theorem 34.1).

4.1 Preliminaries

In this section we give some lemmas which are required in the next sections.

Remark: Let $X \cong U_5(2)$. By ([AT], page 73) if $x \in X$ is an element of order three in class $3A$, then $N_X(\langle x \rangle) \cong 3 \times U_4(2)$. So x is not inverted in X. But x is inverted in $Aut(X)$ and $N_{Aut(X)}(\langle x \rangle) \cong (3 \times U_4(2)) : 2$. This and ([AT], page 73) give us that $Aut(X)$ has one class of subgroups isomorphic to $(3 \times U_4(2)) : 2$.

Lemma 4.1.1 *Let $X \cong U_5(2)$, then* *i) X has two classes $2A$ and $2B$ of involutions. For $x \in 2A$, x is a 2-central involution, $O_3(C_X(x)) = 1$ and $C_X(x)$ has shape $2^{1+6}.3^{1+2}.SL_2(3)$. For $x \in 2B$, $C_X(x)$ has shape $2^{4+4} : 3^2.2$.*

ii) There is an elementary abelian subgroup Y of order 16 in X such that $N_X(Y)/O_2(N_X(Y)) \cong 3 \times A_5$ and $O_2(N_X(Y))$ is a special group of order 2^8 with center Y. Under the action of $N_X(Y)/O_2(N_X(Y))$ on $P(Y)$ we have two orbits I_1 and I_2 such that $|I_1| = 5$, $|I_2| = 10$ and $N_X(Y)/O_2(N_X(Y))$ is 3-transitive on I_1. The elements of I_1 are 2-central and the elements of I_2 are non 2-central. Furthermore for an element $\langle x \rangle \in I_2$ we have $C_X(x) = C_{N_X(Y)}(x)$ and $O_3(N_X(Y)) = 1$.

iii) Let I_1 and I_2 be as in ii) and $\langle x \rangle \in I_2$ and $x = x_1x_2$, where $\langle x_1 \rangle$ and $\langle x_2 \rangle$ are two distinct elements from the orbit I_1. Then for $i = 1, 2$, $C_X(x, x_i)$ is of index 2 in $C_x(x)$ and contains $O_{2,3}(C_X(x))$.

iv) Let Y, I_1 and I_2 be as in ii). Then there are some elementary abelian subgroups A of order 4 in Y such that $P(A) \subseteq I_2$. If $\langle x_1x_2 \rangle \in I_2 \cap P(A)$, where $\langle x_1 \rangle$ and $\langle x_2 \rangle$ are two distinct elements from the orbit I_1, then $P(A) = \{\langle x_1x_2 \rangle, \langle x_3x_1 \rangle, \langle x_3x_2 \rangle\}$, where $x_1 \neq x_3 \neq x_2$ and $\langle x_3 \rangle \in I_1$ and $C_X(A) = C_X(x_1, x_2, x_3) = C_X(Y, A)$. Further all such subgroups of Y are conjugate in $N_{Aut(X)}(Y)$.

v) Let A be as in iv), then $O_2(C_X(A)) = O_2(N_X(Y))$, $C_X(A) \leq N_X(Y)$ and $C_X(A)/O_2(C_X(A))$ is of order three. Let $X_1 = Aut(X)$. Then $C_{X_1}(A)/O_2(C_X(A)) \cong S_3$ and $O_3(C_{X_1}(A)/O_2(C_X(A)))$ acts trivially on $Z(O_2(C_X(A)))$ and acts fixed point freely on $O_2(C_X(A))/Z(O_2(C_X(A)))$.

vi) Let Y be as in ii) and $X_1 = Aut(X)$. Then there is a subgroup $F \cong (3 \times U_4(2)) : 2$ in X_1 containing Y. Further X_1 has one class of subgroups isomorphic to F.

vii) By notations in ii), there is no subgroup B of order 8 in Y such that $P(B) \subset I_2$.

Proof: i) follows from ([AT], page 73). Let $X \cong U_5(2)$ and $X_1 = X : 2 \cong U_5(2) : 2$. By ([AT], pages 73) we get that X has an elementary abelian subgroup Y of order 16 such that $O_2(N_X(Y))$ is a special group of order 2^8 with center Y, $N_X(Y)/O_2(N_X(Y)) \cong 3 \times A_5$ and the extension splits. Further $C_X(Y)/O_2(N_X(Y))$ is of order 3, $C_X(Y)/O_2(N_X(Y))$ acts fixed point freely on $O_2(N_X(Y))/Y$ and $N_X(Y)/O_2(N_X(Y)) \cong 3 \times A_5$. This gives us that Y centralizes an element u

of order three in X and $C_X(u)$ contains a subgroup isomorphic to A_5 as well. Now by ([AT], page 73) we get that $C_X(u) \cong 3 \times U_4(2)$. Set $F = C_X(u)$, then F contains Y. Also u is conjugate to u^{-1} in X_1. So by ([AT], page 73) we get that X_1 has one class of subgroups isomorphic to $N_{X_1}(\langle u \rangle) \cong F : 2$ and $C_{X_1}(u) = F$. We note that $O_3(N_X(Y)) = 1$. Set $\overline{K} = N_X(Y)/O_{2,3}(N_Y(Y))$. By ([AT], page 73) $\overline{K}$ has two orbits I_1 and I_2 on $P(Y)$ such that $|I_1| = 5$, $|I_2| = 10$ and $\overline{K}$ is 3-transitive on I_1. The elements of I_1 are 2-central and the elements of I_2 are non 2-central in X. We have $I_1 = \{\langle x_1 \rangle, \langle x_2 \rangle, \langle x_3 \rangle, \langle x_4 \rangle, \langle x_5 \rangle$ where $x_5 = x_1x_2x_3x_4\}$ and $I_2 = \{\langle x_ix_j \rangle$ where $\langle x_i \rangle$ and $\langle x_j \rangle$ are two distinct elements from the orbit $I_1\}$. Now by the representations of the elements in the orbits I_1 and I_2 we get that there is no subgroup T of order 8 in Y such that $P(T) \subset I_2$. Let $A \leq Y$ be of order 4 such that $P(A) = \{\langle x_1x_2 \rangle, \langle x_3x_1 \rangle, \langle x_3x_2 \rangle\}$. Then by the representations of the elements in the orbits I_1 and I_2 we get that $P(A) \subseteq I_2$. From the natural action of $\overline{K}$ on I_1 we get that $C_{\overline{K}}(x_1x_2, x_2)$ and $C_{\overline{K}}(x_1x_2, x_1)$ are of index 2 in $C_{\overline{K}}(x_1x_2)$. Therefore $C_{\overline{K}}(A) = C_{\overline{K}}(x_1, x_2, x_3)$. Since by ([AT], page 73) $C_X(x_1x_2) = C_{N_X(Y)}(x_1x_2)$, we have $C_X(x_1, x_2, x_3) = C_X(Y)$. By ([AT], page 73) $N_{X_1}(Y)/O_2(C_X(Y)) \cong (3 \times A_5) : 2$. So $N_{X_1}(Y)/C_X(Y) \cong S_5$ and $C_{X_1}(x_1, x_2, x_3)/O_2(C_X(Y)) \cong S_3$. Let $B \leq Y$ be of order 4 and $P(B) \subset I_2$. Then by the representations of the elements in the orbits I_1 and I_2 we get that if $x_ix_j \in B$ where $\langle x_i \rangle$ and $\langle x_j \rangle$ are two distinct elements from the orbit I_1 then $B = \langle x_ix_j, x_ix_r \rangle$ where $x_i \neq x_r \neq x_j$ and $\langle x_r \rangle \in I_1$. Since $N_{X_1}(Y)/C_X(Y) \cong S_5$ acts 5-transitively on I_1, we get that B is conjugate to A in $N_{X_1}(Y)$. Hence the lemma is proved.□

Lemma 4.1.2 *Let $X \cong U_4(2)$. Then there is an elementary abelian subgroup Y of order 16 in X with $N_X(Y)/Y \cong A_5$. Also, there is a subgroup A in Y of order four all of whose involutions are non 2-central in X and they are conjugate in $N_X(Y)$. We have $C_{Aut(X)}(A)$ is an extenstion of Y by a group of order 2 and it is nonabelian. Further X has one class of subgroups isomorphic to $N_X(Y)$.*

Proof: The lemma follows from ([AT], page 26). Let $X \cong U_4(2)$ and $X_1 = Aut(X)$. By ([AT], page 26) X has an elementary abelian subgroup Y of order 16 with $N_X(Y)/Y \cong A_5$ and $N_{X_1}(Y)/Y \cong S_5$. Further by ([AT], page 26) we conclude that $N_X(Y)$ is a maximal subgroup in X and X has one class of subgroups isomorphic to $N_X(Y)$. By ([AT], page 26) $N_X(Y)/Y$ has two orbits I_1 and I_2

on $P(Y)$ such that $|I_1| = 5$, $|I_2| = 10$ and $N_X(Y)/Y$ is 3-transitive on I_1. The elements of I_1 are 2-central and the elements of I_2 are non 2-central in X. This gives us that $I_1 = \{\langle x_1 \rangle, \langle x_2 \rangle, \langle x_3 \rangle, \langle x_4 \rangle, \langle x_1x_2x_3x_4 \rangle\}$ and $I_2 = \{\langle x_ix_j \rangle, \langle x_ix_jx_r \rangle$ where $i \neq j \neq r$, $i = 1, ...4$, $j = 1, ...4$, $r = 1, ..., 4$ and $\langle x_i \rangle$, $\langle x_j \rangle$ and $\langle x_r \rangle$ are in $I_1\}$. By the representations of the elements in the orbits I_1 and I_2 we get that there is an elementary abelian subgroup A of order 4 in Y such that $P(A) \subseteq I_2$. Further if $\langle x_1x_2 \rangle \in I_2 \cap P(A)$, where $\langle x_1 \rangle$ and $\langle x_2 \rangle$ are two distinct elements from the orbit I_1, then $P(A) = \{\langle x_1x_2 \rangle, \langle x_ix_1 \rangle, \langle x_ix_2 \rangle\}$, where $x_1 \neq x_i \neq x_2$ and $\langle x_i \rangle \in I_1$. By ([AT], page 26) $C_X(x_1x_2) \leq C_{N_X(Y)}(x_1x_2)$ and $C_{X_1}(x_1x_2) \leq C_{N_{X_1}(Y)}(x_1x_2)$. Therefore from the natural action of $N_X(Y)/Y$ and $N_{X_1}(Y)/Y$ on I_1 we get that $C_X(A) = Y$ and $C_{X_1}(A)/Y$ is of order 2 and the lemma holds.□

Lemma 4.1.3 *Let $X \cong \Omega_7(3)$. Then X has 3 classes $2A$, $2B$ and $2C$ of involutions. Further*

i) If $x \in 2A$ then $C_X(x) \cong 2U_4(3) : 2$.

ii) If $x \in 2B$ then $C_X(x) \cong (2^2 \times U_4(2)) : 2$.

iii) If $x \in 2C$ then $C_X(x) \cong S_4 \times 2(A_4 \times A_4) : 2$.

iv) There is no elementary abelian subgroup A of order 4 in X such that $C_X(A)$ contains an elementary abelian group of order 16, $|C_X(A)|_2 = 32$ and a Sylow 2-subgroup of $C_X(A)$ is nonabelian.

Proof: (i),(ii) and (iii) follow from ([AT], page 106). Let $X \cong \Omega_7(3)$ and $A = \langle x, y \rangle \leq X$ be an elementary abelian subgroup of order 4 such that $C_X(A)$ contains an elementary abelian group of order 16, $|C_X(A)|_2 = 32$ and a Sylow 2-subgroup of $C_X(A)$ is nonabelian. Assume that $x \in 2A$, then by i) $\overline{W} = C_X(x)/\langle x \rangle \cong U_4(3) : 2$. We note that by ([AT], pages 52, 53), we get that the order of a Sylow 2-subgroup of the centralizer of each involution in $U_4(3) : 2$ is at least 32. So $|C_X(A)|_2 \geq 2^6$ and hence $x \notin 2A$. Assume that $x \in 2C$. Then by iii) $C_X(x) = F \times Y$ where $F \cong S_4$ and $Y \cong 2(A_4 \times A_4) : 2$. We note that $x \in Y$. If $A \cap F \neq 1$ then we get that $|C_X(A)|_2 \geq 2^6$. Hence $A \cap F = 1$. Let $A \leq Y$ then $F \leq C_X(A)$ and from the structure of Y we get that $|C_Y(A)|_2 \geq 2^3$. This gives us that $A \cap Y = \langle x \rangle$. Now let $y = fe$ where $1 \neq f \in F$ and $1 \neq e \in Y$. Let $T \in Syl_2(C_X(x))$ and $A \leq T$. We note that by i), ii) and iii) we get that each involution of X is 2-central in X and $Z(T)$ is an elementary abelian group of order 4. Hence there is an involution

$a \in Z(T)$ such that $a \in 2B$. By iii), $C_X(Z(T)) = C_F(Z(T)) \times Y \cong D_8 \times Y$. Hence $O^2(C_X(Z(Y))) = O^2(Y)$. On the other hand, by ii), ([AT], page 26) and since $a \in 2B$, we get that $O^2(C_X(Z(T))) = L_1 \star L_2$, where $L_1 \cong L_2 \cong SL_2(3)$ and $L_1 \star L_2$ is the central product of L_1 and L_2. Further, $L_1^w = L_2$ for some involution $w \in C_X(Z(T))$. This gives us that $C_X(x) = (K_1 \star K_2 \times K_3)\langle u, t\rangle$, where $\langle u, t\rangle$ is an elementary abelian group of order 4, $K_1 \cong K_2 \cong SL_2(3)$, $K_3 \cong A_4$, $K_1^t = K_2$, $[K_3, t] = 1$ and $K_1 \star K_2$ is the central product of K_1 and K_2. Now from the structures of F and Y we get that $|C_F(f)| \geq 4$ and $|C_Y(e)| \geq 16$. This gives us that $x \notin 2C$. Assume that $x \in 2B$. Then from the structure of $C_X(x)$ in ii) and as the order of a Sylow 2-subgroup of the centralizer of each involution in $U_4(2):2$ is at least 2^5 (see [AT], pages 26, 27), we get that $y \notin C_X(x)$, a contradiction. So $x \notin 2B$ and the lemma is proved.□

Lemma 4.1.4 *Let $X \cong M(22)$. Then*

i) There is an elementary abelian subgroup A of order 9 in X such that all of whose elements of order three are 3-central elements in X.

ii) X has an elementary abelian subgroup N of order 2^{10} such that $N_X(N)/N \cong M_{22}$. Further for $T \in Syl_2(N_X(N))$ we have $N = J(T)$.

Proof: By ([As3], lemma 39.4) we get that there is an elementary abelian subgroup Y of order 3^5 in X such that $N_X(Y)/Y \cong O_5(3)$. By ([As3], lemmas 39.3(ii) and 39.6) we get that under the action of $N_X(Y)/Y$ on $P(Y)$ the singular points are 3-central elements in X. Since there is a singular line in Y, we get that there is a subgroup A of order 9 in X all of whose elements of order three are 3-central elements in X and i) is proved.

By ([As3], lemma 25.7) we get that X has an elementary abelian subgroup N of order 2^{10} such that $N_X(N)/N \cong M_{22}$. Let $t \in N$ be a 3-transposition, then by ([As3], lemma 37.6) we get that $C_X(t) \cong 2U_6(2)$. Further by ([As3], 30.1, 30.3) we get that $N_{C_X(t)}(N)/N \cong L_3(4)$ and for $T \in Syl_2(N_{C_X(t)}(N))$ we have $N = J(T)$. Let $P \in Syl_3(N_{C_X(t)}(N))$, then by ([As3], lemma 22.2) we get that $C_N(P)$ is of order 4. Let $\overline{k} \in X/N$ be an involution. Then since M_{22} has just one class of involutions (see [AT], page 39) we may assume that $k \in N_{C_X(t)}(N)$. Since $L_3(4)$ has one class of involutions and there is an involution in $N_{C_X(t)}(N)/N \cong L_3(4)$ which inverts P, we may assume that $\overline{k}$ inverts P. This gives us that $|[\overline{k}, N]| = 2^4$ and so $|C_N(\overline{k})| = 2^6$.

Now let $S \in Syl_2(N_X(N))$ which contain T. Let $E \neq N$ be an elementary abelian subgroup in S of order 2^{10}. Then as $|C_N(\overline{k})| = 2^6$, M_{22} has one class of involutions and the 2-rank of M_{22} is 4, we conclude that $|E \cap N| = 2^6$. This gives us that there is an elementary abelain subgroup $\overline{K}$ of order 16 in $\overline{X} = X/N$ such that $C_N(\overline{K})$ is of order 2^6. We may assume that $\overline{k} \in \overline{K}$. Then $E \cap N = C_N(\overline{k})$ and so contains t. This gives us that N and E are two elementary abelian subgroups of order 2^{10} in $C_X(t)$. So $N = E$ by ([As3], 30.3) and the lemma is proved.$\square$

4.2 The centralizer of a non 2-central involution

In this section we select a non 2-central involution z in $O^2(H_1)$ and we will determine the structure of $C_G(z)$.

Notations: We have $O^2(H_1)/O_3(H_1) \cong U_5(2)$. So by 4.1.1(ii) $O^2(H_1)$ contains an elementary abelian subgroup U of order 16 such that $N_{O^2(H_1)}(U)O_3(H_1)/O_3(H_1)$ is an extension of a special group of order 2^8 with center $UO_3(H_1)/O_3(H_1)$ by $(3 \times A_5)$. By 4.1.1(ii) U contains some 2-central involutions and some non 2-central involutions of $O^2(H_1)$. In fact by 4.1.1(ii) we get that if a and d are two distinct involutions in U such that d and a are two 2-central involutions in $O^2(H_1)$ then ad is a non 2-central involution in $O^2(H_1)$. We keep these notations U, a and d in the remainder of this chapter. Set $z = ad$ and $R = O_3(H_1)$.

We are going to show that $C_G(z) \cong 2M(22) : 2$. To do this we will select a suitable involution $t \in U$ and we shall show that $C_G(z, t, \tau)/\langle z \rangle$ satisfies the conditions of theorem 1 in [Pa] and so $C_G(z, t)/\langle z, t \rangle \cong U_6(2)$ by ([Pa], theorem 1). This will help us to invoke theorem 4.2.1 in [DS] to show that $C_G(z)$ is as desired. We remark that $O^2(H_1)$ is a subgroup of $C_G(\tau)$.

Lemma 4.2.1 *i) $C_R(z)$ is an extraspecial group of order 3^7.*

ii) $C_R(d)$ is an extraspecial group of order 27.

iii) $C_R(U) = Z(R)$.

Proof: We have $O^2(H_1)/R \cong U_5(2)$. Set $\widehat{K} = N_{O^2(H_1)/R}(UR/R)$ and let $\widehat{P} = O_2(\widehat{K})$. Then by 4.1.1(ii) $\widehat{U} = Z(\widehat{P})$ and $\widehat{K}/\widehat{P} \cong 3 \times A_5$. Set $\widetilde{X} = \widehat{K}/\widehat{P}$. By 4.1.1(ii) we get that $\widetilde{X}$ has two orbits I_1 and I_2 on $P(\widehat{U})$ such that $|I_1| = 5$, $\widetilde{X}$

is 3-transitive on I_1 and the elements in the orbit I_1 are 2-central in $C_G(\tau)R/R$, $|I_2| = 10$ and the elements in I_2 are non 2-central involutions in $C_G(\tau)R/R$. Let $I_1 = \{\langle\widehat{x}_1\rangle, \langle\widehat{x}_2\rangle, ..., \langle\widehat{x}_5\rangle\}$, $\widehat{U} = \langle\widehat{x}_1, \widehat{x}_2, \widehat{x}_3, \widehat{x}_4\rangle$ and $\widehat{x}_5 = \widehat{x}_1\widehat{x}_2\widehat{x}_3\widehat{x}_4$. Then $I_2 = \{\langle\widehat{x}_i\widehat{x}_j\rangle$, for $i \neq j$ and $i, j = 1, ..., 5\}$. Set $\widehat{W} = \langle\widehat{x}_2, \widehat{x}_3, \widehat{x}_1\rangle$ and $\overline{R} = R/Z(R)$. As $\widetilde{X}$ is 3-transitive on I_1, there are 10 subgroups conjugate to $\widehat{W}$ in $\widehat{U}$. Therefore $\widetilde{X}$ has two orbits L and I on the set of hyperplanes in $\widehat{U}$ such that $|L| = 10$, $\widehat{W} \in L$ and $|I| = 5$. By 2.2.3(i) $\overline{R}$ is a 10-dimensional symplectic space. Let $\{\overline{v}_i, \overline{w}_i\}$, $i = 1, ..., 5$ be a symplectic base for $\overline{R}$. If $[\overline{v}_i, \widehat{x}_j] = 1$ for some $i = 1, ..., 5$ and $j = 1, ..., 5$, then $[\overline{w}_i, \widehat{x}_j] = 1$. So $|C_{\overline{R}}(\widehat{x}_j)| = 3^{2\alpha}$, $\alpha \geq 0$, $j = 1, ..., 5$, $|C_{\overline{R}}(\widehat{U})| = 3^{2\gamma}$, $\gamma \geq 0$ and $|C_{\overline{R}}(\widehat{W})| = 3^{2\beta}$, $\beta \geq 0$. Now by coprime action and as $|L| = 10$, we get that $3^{10} = |\overline{R}| \geq (3^{2\beta})^{10}$. This gives us that $|C_{\overline{R}}(\widehat{W})| = 1$ and at least iii) holds. Since $|I| = 5$ and $|C_{\overline{R}}(\widehat{W})| = 1$, we get that for $\widehat{W_1} \in I$ we have that $|C_R(\widehat{W_1})| = 27$. We remark that if $C_R(\widehat{x}_1) = Z(R)$ then $\widehat{x}_1\widehat{x}_2$ centralizes R which is a contradiction. Therefore $|C_R(d)| \geq 27$. Set $\overline{T} = [\overline{R}, d]$. Then $\overline{T}$ is $C_{H_1/R}(d)$-invariant. Since dR is a 2-central involution in H_1/R, by 4.1.1(i), we get that $C_{H_1/R}(d)$ has shape $2^{1+6}.3^{1+2}.SL_2(3).2$. But by ([AT],page 112), we get that there is no subgroup isomorphic to $C_{H_1/R}(d)$ in $PSp_6(3) : 2$ and $GSp_6(3)$. Therefore $|\overline{T}| \geq 3^8$. This gives us that $|C_R(d)| = 3^3$ and ii) holds. By ([AT], page 73) $C_{H_1/R}(d)$ is a maximal subgroup of H_1/R. Therefore $C_{\overline{R}}(\langle\widehat{x}_1, \widehat{x}_2\rangle) = 1$. Now by coprime action we have $3^{10} = |\overline{R}| = |C_{\overline{R}}(\widehat{x}_1)||C_{\overline{R}}(\widehat{x}_2)||C_{\overline{R}}(\widehat{x}_1\widehat{x}_2)| = 3^4|C_{\overline{R}}(\widehat{x}_1\widehat{x}_2)|$. Hence $|C_{\overline{R}}(\widehat{x}_1\widehat{x}_2)| = 3^6$ and i) holds. Now the lemma is proved.□

Lemma 4.2.2 *i)* $O_3(C_{H_1}(z)) = C_R(z)$*,* $O_2(C_{H_1}(z)/C_R(z))$ *is a special group of order* 2^8*,* $Z(O_2(C_{H_1}(z)/C_R(z))) = UC_R(z)/C_R(z)$ *and* $C_{H_1}(z)/O_{3,2}(C_{H_1}(z))$ *is an extension of an elementary abelian group of order 9 by a group of order 4. Further* $C_{H_1}(z)$ *contains a Sylow 3-subgroup of* $C_G(z)$*.*

ii) $O_{3,2}(C_G(\tau, d))$ *is an extension of an extraspecial group of order 27 by an extraspecial group of order* 2^7 *and* $C_G(d, \tau)/O_{3,2}(C_G(\tau, d))$ *is an extension of an extraspecial group of order 27 by* $SL_2(3)$*.*

iii) $O_{3'}(C_{H_1}(z)) = \langle z\rangle$*.*

Proof: We have $H_1 = N_G(\langle\tau\rangle)$ and by 4.2.1(i)(ii) $C_R(z)$ is an extraspecial group of order 3^7 and $C_R(d)$ is an extraspecial group of order 27. By 4.1.1(ii) we get that $C_R(z) = O_3(C_{H_1}(z))$ and by 4.2.1(iii) we get that $C_G(C_R(z)) = Z(R)$. So $\langle\tau\rangle$ is the center of each Sylow 3-subgroup of $C_{H_1}(z)$. Therefore $C_{H_1}(z)$ contains a Sylow 3-

subgroup of $C_G(z)$. Now since dR is a 2-central involution and zR is a non 2-central involution in $C_G(\tau)/R$, i) and ii) follow from 4.1.1(i). By i) if $\langle z\rangle < O_{3'}(C_{H_1}(z))$ then $|O_2(C_{H_1}(z)) \cap U| \geq 4$. This gives us that there is a subgroup B in U of order 4 such that $|C_R(B)| \geq 3^7$. This and 4.2.1(i)(ii) give us that all involutions in B are non 2-central in H_1. Since by 4.1.1(ii) $C_{H_1}(z)R/R$ is a maximal subgroup of $N_{H_1}(U)R/R$, we get that $C_R(z) \leq C_R(U)$, a contradiction to 4.2.1(iii). Therefore $O_{3'}(C_{H_1}(z)) = \langle z\rangle$ and the lemma is proved.□

Further notations: By 4.1.1(iv) there is a subgroup $A \leq U$ of order 4 containing z such that all involutions in A are non 2-central in $O^2(H_1)$. Further $A = \langle z, ab\rangle$ where b is conjugate to d in $N_{O^2(H_1)}(U)$. We fix the notation A for such a subgroup of U. Set $t = ab$.

Lemma 4.2.3 *i)* $C_R(A)$ *is an extraspecial group of order* 3^5 *and* $O_3(C_{H_1}(A)) = C_R(A)$.

ii) $O_2(C_{H_1}(A)/\langle C_R(A), A\rangle) \cong Q_8 \times Q_8$ *and* $C_{H_1}(A)/O_{3,2}(C_{H_1}(A)) \cong S_3$.

iii) $C_{H_1}(A)$ *contains a Sylow 3-subgroup of* $C_G(A)$.

iv) $O_2(C_{H_1}(A)) = A$.

Proof: Set $\overline{X} = C_{H_1}(A)R/R$. Then by 4.1.1(ii),(v) we get that $O_3(\overline{X}) = 1$ and $O_2(\overline{X})/\overline{A}$ is a group of order 2^6 with center of order at least 4. Further by 4.1.1(v) $\overline{X}/O_2(\overline{X}) \cong S_3$ and $O_3(\overline{X}/O_2(\overline{X}))$ acts trivially on $Z(O_2(\overline{X}))$ and acts fixed point freely on $O_2(\overline{X})/Z(O_2(\overline{X}))$. Assume that $C_R(A)$ is an extraspecial group of order 3^5. By 4.1.1(v) we get that $O_2(C_{H_1}(A)R/R)$ is a special group with center UR/R. So if $O_2(C_{H_1}(A)) \neq A$ then we get that $A < O_2(C_{H_1}(A)) \cap U$. This gives us that there is a subgroup B in U of order 8 such that $|C_R(B)| \geq 3^5$. This and 4.2.1(i)(ii) give us that all involutions in B are non 2-central in H_1, a contradiction to 4.1.1(vii). So $O_2(C_{H_1}(A)) = A$. Now as $C_R(A)$ is an extraspecial group of order 3^5, by 2.2.3(ii) we get that $C_{H_1}(A)/C_R(A)A$ is isomorphic to a subgroup of $Sp_4(3):2$. This and as $O_2(\overline{X})/\overline{A}$ is a group of order 2^6 with the center of order at least 4, $\overline{X}/O_2(\overline{X}) \cong S_3$ and acts fixed point freely on $O_2(\overline{X})/Z(O_2(\overline{X}))$ give us that $O_2(C_{H_1}(A)/\langle C_R(A), A\rangle) \cong Q_8 \times Q_8$ and the lemma is proved. So it is enough to show that $C_R(A)$ is an extraspecial group of order 3^5.

By 4.2.1(i),(ii) $C_R(z)$ is an extraspecial group of order 3^7 and $C_R(d)$ is an extraspecial group of order 27. Therefore as aR and bR are conjugate to dR in H_1/R, we get that $C_R(a)$ and $C_R(b)$ are two extraspecial group of order 27. Since $dR \neq aR$, we have $C_R(a) \cap C_R(d) = Z(R)$. Since $z = ad$, we get that $C_R(d, z)$ and $C_R(a, z)$ are two subgroup of $C_R(a, d)$. Hence $C_R(d, z) = C_R(a, z) = Z(R)$. This gives us that d and a act fixed point freely on $C_R(z)/Z(R)$. Suppose that b acts fixed point freely on $C_R(z)/Z(R)$. Then we get that t centralizes $C_R(z)$. Since $z \neq t$ and as $C_{H_1}(z)R/R$ is maximal in $N_{H_1}(U)R/R$ and tR is conjugate to zR in $N_{H_1}(U)R/R$, we get that $C_R(z)$ is $N_{H_1}(U)R/R$-invariant. This and 4.1.1(ii) give us that each non 2-central involution in U centralizes $C_R(z)$ and so U centralizes $C_R(z)$, a contradiction to 4.2.1(iii). Therefore $C_R(b) \leq C_R(z)$. Since $|C_R(b)| = 27$ and $|C_R(z)| = 3^7$, we have $|C_R(z, t)| = 3^5$. Hence $C_R(A)$ is an extraspecial group of order 3^5 and the lemma is proved.□

Lemma 4.2.4

Proof: Set $\overline{X} = C_{H_1}(A)/A$. By 4.2.3(i),(ii) $\overline{X}/O_{3,2}(\overline{X}) \cong S_3$ and $O_{3,2}(\overline{X})$ is an extension of ai) $C_G(A)/A \cong U_6(2)$ and the extension does not split.

ii) $C_G(z) \cong 2Aut(M(22))$. n extraspecial group of order 3^5 by $Q_8 \times Q_8$. By 4.2.3(iv) we have that $O_2(\overline{X}) = 1$. By 4.1.1(v) we get that there is an element $x \in C_{H_1}(A)$ such that $x^2 \in C_{H_1}(U)$ and x does not act trivially on U. By this and the structure of $\overline{X}$ we get that $Z(O_2(\overline{X}/O_3(\overline{X})))$ is of order 2. Therefore $O_3(\overline{X})/Z(O_3(\overline{X}))$ is an irreducible $\overline{X}/O_3(\overline{X})$-module. By 4.1.1(iv) each subgroup of U of order 4 such that all of whose involutions are non 2-central involutions in H_1, is conjugate to A in H_1. So τ is not weakly closed in $C_{H_1}(A)$. Therefore by 4.2.3(i),(ii),(iii) and ([Pa], theorem 1) we get that $C_G(A)/A \cong U_6(2)$. Since $O_2(C_{H_1}(A)/C_R(A))$ is a special group of order 2^8, we have $A \leq C_G(A)'$ and hence $C_G(A)$ is a quasisimple group and i) holds.

Set $\widetilde{M} = C_G(z)/\langle z \rangle$. By i) $F^\star(C_{\widetilde{M}}(\widetilde{t})) \cong 2U_6(2)$. Assume that $F^\star(\widetilde{M})$ is simple. Then by ([DS], theorem 3.1) we get that $\widetilde{M} \cong M(22)$ or $\widetilde{M} \cong M(22):2$. By 4.2.2(i) $C_{H_1}(z)$ is the centralizer of a 3-central element in $C_G(z)$. So by 4.2.2(i) and ([AT] ,page 163) we get that $\widetilde{M}$ is not isomorphic to $M(22)$. Therefore $\widetilde{M} \cong M(22):2$. By 4.2.2(i) we get that $z \in C_G(z)'$. Therefore $C_G(z) \cong 2M(22):2$ and the lemma is proved. Hence it is enough to show that $F^\star(\widetilde{M})$ is a nonabelian simple group.

Let $\widetilde{K}$ be a minimal normal subgroup of $F^\star(\widetilde{M})$. Assume that $3||\widetilde{K}|$. Then by 4.2.2(i) we get that $\widetilde{\tau} \in \widetilde{K}$. Therefore by i) $\widetilde{\tau} \in \widetilde{K} \cap F^\star(C_{\widetilde{M}}(\widetilde{t}))$. Hence $F^\star(C_{\widetilde{M}}(\widetilde{t})) \leq$

$\widetilde{K}$. This gives us that $\widetilde{K}$ is a nonabelian group and as $F^\star(C_{\widetilde{M}}(\widetilde{t})) \leq \widetilde{K}$, we get that $\widetilde{K} \cong M(22)$. Therefore $\widetilde{K} = F^\star(\widetilde{M})$ and the lemma is proved in this case. Now assume that 3 does not divide the order of $\widetilde{K}$. Then we get that $O_{3'}(\widetilde{M}) \neq 1$. Set $\overline{N} = \widetilde{M}/O_{3'}(\widetilde{M})$. Then $3||\overline{N}|$ and $O_{3'}(\overline{N}) = 1$. Hence $3||\overline{K_1}|$, where $\overline{K_1}$ is a minimal normal subgroup of $\overline{N}$. It follows from the previous case that $F^\star(\overline{N}) \cong M(22)$. By 4.1.4 there is an elementary abelian subgroup of order 9 in $F^\star(\overline{N})$ all of whose elements of order three are conjugate to $\overline{\tau}$. Now by 4.2.2(iii) and coprime action we get that $O_{3'}(\widetilde{M}) = 1$, a contradiction. Hence this case does not happen and the lemma is proved.□

4.3 Proof of theorem 5 and corollary 7

In this section we shall prove the theorem 5 and corollary 7. First we recall our last notations.

• $R = O_3(H_1)$. $U \leq O^2(H_1)$ is an elementary abelian group of order 16, a and d are two distinct involutions in U such that d and a are two 2-central involutions in $O^2(H_1)$ and $z = ad$ is a non 2-central involution in $O^2(H_1)$.

• $A = \langle z, t\rangle$ is a subgroup of U of order 4 such that all involutions in A are conjugate in $N_{H_1}(U)$ and $t = ab$, where b is conjugate to d in $N_{H_1}(U)$.

Lemma 4.3.1 *There is an elementary abelian subgroup M of order 2^{11} in $C_G(z)$ containing A and d such that $N_{C_G(z)}(M)/M \cong M_{22} : 2$ and $C_G(M, z) = M$.*

Proof: By 4.2.4(ii) we have $C_G(z) \cong 2M(22) : 2$. By ([As3], 23.8), the preimage of an involution of $M(22)$ is an involution in $2M(22)$. By ([AT], page 163) we conclude that there is an elementary abelian subgroup of order 2^{10} in $M(22)$. Therefore by ([AT], page 163) we get that there is an elementary abelian subgroup M of order 2^{11} in $C_G(z)$ such that $N_{C_G(z)}(M)/M \cong M(22) : 2$ and $C_G(M, z) = M$. By 4.2.4(i) we have $C_G(A)/A \cong U_6(2)$. By ([AT],page 115) we get that $N_{C_G(A)}(M)/M \cong L_3(4)$ and each involution of $C_G(A)$ is conjugate to an involution of M in $C_G(A)$. Therefore we may assume that $d \in M$ and the lemma is proved.□

Further notations: By 4.3.1 there is an elementary abelian subgroup M of order 2^{11} in $C_G(z)$ containing A and d such that $N_{C_G(z)}(M)/M \cong M_{22} : 2$ and $C_G(z, M) = M$. We fix the notation M for such a subgroup of $C_G(z)$.

Lemma 4.3.2 *i)* $N_G(M)/M \cong M_{24}$.

ii) Let $S \in Syl_2(N_G(M))$, *then* $M = J(S)$.

iii) $N_G(M)$ *contains a Sylow 2-subgroup of* G.

iv) $N_G(M)/M$ *has two orbits of lengths* 276 *and* 1771 *on* $P(M)$. *Further the orbit containing* $\langle z\rangle$ *has length 276.*

v) There is a subgroup $\overline{D}$ *in* $N_G(M)/M$ *isomorphic to* M_{23} *and containing* $O^2(N_{C_G(z)}(M)/M)$ *such that under the action of* $\overline{D}$ *on* $P(M)$ *we have that* z *and* t *are conjugate and the orbit containing* $\langle z\rangle$ *is of length 23. In particular* z *is not conjugate to* zt *under the action of* $\overline{D}$ *on* $P(M)$.

Proof: Since $C_G(z, M) = 1$, we have $C_G(M) = 1$. Therefore $N_G(M)/M$ is ismorphic to a subgroup of $GL_{11}(2)$. We have $N_{C_G(z)}(M)/M \cong M_{22} : 2$. Set $\overline{X} = N_{C_G(z)}(M)/M$, $\overline{F} = O^2(\overline{X})$ and $\overline{Y} = N_G(M)/M$. By 4.2.4(ii) and ([AT], page 163) we have $N_{C_G(z)}(A)/A \cong U_6(2).2$. So by 4.2.4(i) and ([AT], page 115) $C_{\overline{X}}(A) \cong L_3(4)$ and $N_{\overline{X}}(A) \cong L_3(4) : 2$. This gives us that t and zt are conjugate under the action of $\overline{X}$ and they are not conjugate under the action of $\overline{F}$. In fact under the action of $\overline{F}$ on $P(M)$ the orbit containing $\langle t\rangle$ is of length 22. Let N_2 be the orbit of $\overline{F}$ on $P(M)$ containing $\langle t\rangle$, then $\overline{F}$ acts 3-transitively on N_2. Therefore $\overline{F}$ has 7 orbits N_i, $i = 1, ..., 7$, on $P(M)$ such that $|N_2| = |N_7| = 22$, $|N_3| = |N_6| = 231$ and $|N_4| = |N_5| = 770$ and $N_1 = \{\langle z\rangle\}$. Further $N_2 = \{\left\langle t^{\overline{F}}\right\rangle\}$, $N_3 = \{\left\langle (tt_1)^{\overline{F}}\right\rangle$, where $t_1 \neq t$ and $\langle t_1\rangle \in N_2\}$, $N_4 = \{\left\langle (tt_1t_2)^{\overline{F}}\right\rangle$, where $t_i \neq t$ and $\langle t_i\rangle \in N_2, i = 1, 2\}$ and $N_{9-j} = \{\left\langle (zx)^{\overline{F}}\right\rangle$, where $\langle x\rangle \in N_j\}$, $j = 2, 3, 4$. We note that $\overline{F}$ acts 3-transitively on N_2 and N_7. We note that t and zt are conjugate under the action of $\overline{X}$. Hence as M_{22} is a $\{2, 3, 5, 7, 11\}$-group, either $\overline{X}$ has five orbits L_i, $i = 1, ..., 5$, on $P(M)$ such that $L_1 = \{\langle z\rangle\}$, $|L_2| = 44$, $|L_3| = |L_4| = 231$ and $|L_5| = 1540$ or $\overline{X}$ has six orbits L_i, $i = 1, ..., 6$, on $P(M)$ such that $L_1 = \{\langle z\rangle\}$, $|L_2| = 44$, $|L_3| = |L_4| = 231$ and $|L_5| = |L_6| = 770$. Let $S \in Syl_2(N_{C_G(z)}(M))$, then by ([As3], lemma 31.1) we get that $M = J(S)$. So as t is conjugate to z in H_1, we get by 2.2.2 that $\overline{Y} \neq \overline{X}$. We note that by 4.2.2(ii), 4.2.4(ii) and ([AT], page 163) we get that d is not conjugate to z in G. So $\overline{Y}$ is not transitive on $P(M)$. Since $GL_{11}(2)$

is a $\{2,3,5,7,11,17,23,31,73,89,127\}$-group we conclude that the orbit of $\overline{Y}$ on $P(M)$ containing $\langle z\rangle$ has length 1+44 or 1+231+44. Set $I = \{\langle z\rangle^{\overline{Y}}\}$. Assume that $|I| = 45$. We note that $C_{\overline{Y}}(A) \leq \overline{F}$. We have $C_{\overline{Y}}(A) \cong L_3(4)$ and by ([As3], lemma 30.2) under the action of $C_{\overline{Y}}(A)$ on $P(M/A)$ we have three orbits of lengths 21, 210 and 280. By this and the lengths of the orbits N_i, $i = 1, ..., 7$, we get that $C_{\overline{Y}}(A)$ has 7 orbits V_i, $i = 1, ..., 7$ on $P(M)$ such that $|V_1| = 21 = |V_2| = |V_6| = |V_7|$, $|V_3| = 1 = |V_4| = |V_5|$ and other orbits of $C_{\overline{Y}}(A)$ on $P(M)$ have at least length 210. Since $N_{\overline{X}}(A) \cong L_3(4) : 2$ and all involutions in A are conjugate, we have $N_{\overline{Y}}(A)/C_{\overline{Y}}(A) \cong S_3$. Set $\beta = \{V_1, V_2, V_6, V_7\}$. Since $|I| = 45$, we get that $|I \cap \beta| = 2$. But $I \cap \beta$ is $N_{\overline{Y}}(A)/C_{\overline{Y}}(A)$-invariant and hence $|I \cap \beta| \geq 3$, a contradiction. This contradiction shows that the orbit of $\overline{Y}$ on $P(M)$ containing $\langle z\rangle$ has length 1+231+44=276 and hence $|\overline{Y}| = |M_{24}|$.

We have $|I| = 276$ and $N_2 \cup N_7 \subset I$. Also one of the orbits N_3 or N_6 is contained in I. We note that 11 does not divide the order of $GL_9(2)$, so $C_M(\overline{F}) = \langle z\rangle$. Since $|I| = 276$, three of the orbits V_i, $i = 1, 2, 6, 7$ are contained in $N_2 \cup N_7 \cup N_i \subset I$ where N_i is one of the orbits N_3 or N_6. We may assume that V_1, V_2 and V_7 are contained in I. Let $\overline{y}_2$ and $\overline{y}_7$ be two elements in $N_{\overline{Y}}(A)$ such that $z^{\overline{y}_2} = t$ and $z^{\overline{y}_7} = zt$, $\overline{y}_i^2 \in C_{\overline{Y}}(A)$, $i = 2, 7$ and $N_{\overline{Y}}(A) = C_{\overline{Y}}(A)\,\langle \overline{y}_2, \overline{y}_7\rangle$. Then as I is an orbit of $\overline{Y}$, we get that $\langle \overline{y}_2, \overline{y}_7\rangle$ acts on the set $\{V_1, V_2, V_7\}$. This gives us that $V_i^{\overline{y}_2} = V_i$ and $V_j^{\overline{y}_1} = V_j$, $i \neq j$ for some $i, j = 1, 2, 7$. We may choose notations such that $V_2^{\overline{y}_2} = V_2$ and $V_7^{\overline{y}_7} = V_7$. At least one of the orbits V_2 or V_7 is contained in $N_2 \cup N_7$. Assume that $V_s \subset N_2 \cup N_7$ where $s = 2$ or 7. Assume also that $V_s \subset N_s$. Set $\overline{D} = \left\langle \overline{F}, \overline{y_s}\right\rangle$. Then as $N_s = V_s \cup \{\langle x\rangle\}$ where $x = zt$ if $s = 2$ and $x = t$ if $s = 7$, we get that N_s is $\overline{D}$-invariant. We note that $C_{\overline{F}}(x)$ contains $C_{\overline{Y}}(A) \cong L_3(4)$. Since $|N_s| = 22$ we get that $|\overline{D}| = 22|C_{\overline{Y}}(x)|$. We note that x is conjugate to z under the action of $\overline{Y}$, so $C_{\overline{Y}}(x) \cong M_{22} : 2$. As 11^2 does not divide the order of $\overline{Y}$ and $L_3(4)$ is a maximal subgroup in M_{22} (see [AT], page 39), we deduce that $|C_{\overline{D}}(x)| = |L_3(4)|$ or $|L_3(4) : 2|$ and so $|\overline{D}| \leq 2|M_{22}|$. This gives us that $\overline{y}_s$ normalizes $\overline{F}$ and so $\overline{y}_s$ centralizes z, a contradiction. Therefore $V_s \subset N_r$ where $r = 2$ or 7 and $r \neq s$. Set $W = \{\langle z\rangle\} \cup N_r$. Then W is an orbit of $\overline{D}$ on $P(M)$ of length 23. Further $C_{\overline{D}}(z) \cong M_{22}$ acts 3-transitively on $W - \{\langle z\rangle\} = N_r$ of length 22. Therefore $\overline{D}$ is 4-transitive on W and $\overline{D} \cong M_{23}$. Since $C_{\overline{Y}}(\overline{F}) \leq \overline{X}$ we get that $C_{\overline{Y}}(\overline{F}) = 1$. Since $Out(M_{23}) = 1$ (see [AT], page 71), we have $N_{\overline{Y}}(\overline{D}) = \overline{D}$. This gives us that $\overline{Y}$ is a transitive extension of $\overline{D}$ and hence $\overline{Y} \cong M_{24}$. Now i) holds and ii) follows from i) and ([As3], lemma 22.1) and the lemma is proved.$\Box$

Lemma 4.3.3 *i)* $N_G(M)$ *contains a Sylow 2-subgroup of* G *and* $N_G(M)$ *controls the fusion in* M *with respect to* G. *Thus* z *and* d *are not conjugate in* G *and* d *is a 2-central involution in* G.

ii) $O_2(C_G(d))$ *is an extraspecial group of order* 2^{13} *and* $C_G(d)/O_2(C_G(d)) \cong 3U_4(3) : 2$.

Proof: By 4.3.2(i) and 4.2.4(ii) we get that z is a non 2-central involution in G. We recall that $A = \langle z, t\rangle$ is an elementary abelian group of order 4 and all involution of A are conjugate in G. Let $1 \neq r \in G$ be a 2-central involution. We may assume that $C_G(r)$ contains a Sylow 2-subgroup of $C_G(z)$. By 4.2.4(i) and ([AT], page 115) we get that $O(C_G(A, r)) = 1$. Also by 4.2.4(ii) and ([AT], page 163) we get that $O(C_G(z, r)) = 1$. This and coprime action give us that $O(C_G(r)) = 1$. We note that 4.3.1 we have $C_G(M, z) = M$. Hence $C_G(M) = M$. Now by 4.3.2(i),(ii) ([Re], theorem B) we get that the groups G and $N_G(M)$ satisfy the conditions of theorem B in [Re]. Of course $G \neq N_G(M)$ and so i) follows from ([Re], lemma 3 and the corollary after lemma 3). Also by ([Re], lemmas 4, 8 and 9) we get that $O_2(C_G(d))$ is an extraspecial group of order 2^{13}, $O(C_G(d)/O_2(C_G(d)))$ is of order 3 and either $C_G(d)/O_2(C_G(d)) \cong 3U_4(3) : 2$ or $C_G(d)/O_{2,2'}(C_G(d)) \cong Aut(M_{22})$. Since by 4.2.2(ii) $|C_G(d)|_3 \geq 3^7$, we get that $O_2(C_G(d))$ is an extraspecial group of order 2^{13} and $C_G(d)/O_2(C_G(d)) \cong 3U_4(3) : 2$ and the lemma is proved. $\Box$

Now we can prove the **Theorem 5**.

Proof: By 4.3.3(ii) $O_2(C_G(d))$ is an extraspecial group of order 2^{13} and $C_G(d)/O_2(C_G(d)) \cong 3U_4(3) : 2$. By 4.2.4(ii) and ([AT], page 163) we get that $z \in O_2(C_G(d))$ and hence $a = zd \in O_2(C_G(d))$. Since a is conjugate to d in H_1, we get that d is not weakly closed in $O_2(C_G(d))$ with respect to G. Now by ([As3], theorem 34.1) we conclude that $G \cong M(24)'$ and the theorem is proved.$\Box$

The proof of **Corollary 7.**

Proof: We adopte the notations D, D_1, D_2 and α as in corollary 7. Set $D_{12} = D_1 \cap D_2$ and $L = O_3(D_2)$. We have $D_{12}/O_3(D_{12}) \cong U_4(2) : 2$. Since the order of a maximal elementary abelian subgroup in $O_3(D_1)$ is 3^6 and L and $O_3(D_1)$ are two subgroups of $O_3(D_{12})$, we get that $|O_3(D_1) \cap L| = 3^6$ and $D_{12}/O_3(D_1) \cong 3 \times U_4(2)$. By 4.1.1(vi),(ii) there is an elementary abelian subgroup U_1 of order 16 in $O^2(D_{12})$ such that $O_2(N_{D_1}(U_1)O_3(D_1)/O_3(D_1))$ is a special group of order 2^8 with center $U_1O_3(D_1)/O_3(D_1)$ and $N_{D_1}(U_1)O_3(D_1)/O_3(D_1)$ is an extension of $O_2(N_{D_1}(U_1)O_3(D_1)/O_3(D_1))$ by $(3 \times A_5) : 2$. By 4.1.1(iv) there are some elementary abelian subgroups $V \leq U_1 \leq C_{D_{12}}(\alpha)$ of order 4 such that all involutions in V are non 2-central in D_1 and all such subgroups of U_1 are conjugate in D_1. If we show that $\langle \alpha \rangle$ is not weakly closed in $C_{D_1}(V)$ with respect to $C_D(V)$ then the corollary follows from theorem 5. So it is enough to show that $\langle \alpha \rangle$ is not weakly closed in $C_{D_1}(V)$ with respect to $C_D(V)$. By 4.1.2(iii) we get that a Sylow 2-subgroup of $C_{D_{12}}(V)/C_L(V)$ is an extension of an elementary abelian group of order 16 by an element of order 2 and a Sylow 2-subgroup of $C_{D_{12}}(V)$ is nonabelian. Now by 4.1.3(iv) we get that $|C_{D_2/L}(V)| \geq 2^6$. Since $C_L(V)$ is $C_{D_2}(V)/C_L(V)$-invariant, $|C_{D_2/L}(V)| > |C_{D_{12}/L}(V)|$ and $\alpha \in C_L(V)$, we get that $\langle \alpha \rangle$ is not weakly closed in $C_L(V)$ with respect to $C_{D_2}(V)$. As $L \leq D_1$, we have shown that $\langle \alpha \rangle$ is not weakly close in $C_{D_1}(V)$ with respect to $C_D(V)$ and the corollary holds.□

Remark: In theorem 5 we could replace $H_1/R \cong U_5(2) : 2$ by some 2-local information about $U_5(2) : 2$. In fact to prove theorem 5 we have just used of some 2-local information about H_1/R. But we should remark that all 2-local information which are used about H_1/R identify $U_5(2) : 2$.

Chapter 5

Characterization of the Monster group

In this chapter we will prove theorem 6. So in this chapter G is of Monster type and we keep the notations S, H_1 and H_2 in definition 3. We gave a sketch of the proof for theorem 6 in 1.3.3. Our strategy for identifying the Monster group is to determine the structures of the centralizers of involutions in G. We find two involutions z and t in G such that $C_G(z)$ is a faithful extension of an extraspecial 2-group of order 2^{25} by Co_1 and $C_G(t) \cong 2F_2$ where F_2 is the baby monster group . Then the main result follows by applying theorem 2.3.1 which is proved by Griess, Meierfrankenfeld and Segev.

5.1 The centralizer of a non 3-central element

In this section we select an element s of order three in the group G and we shall show that $C_G(s) \cong 3M(24)'$. We make use of the following information about the action of $H_1/O_{3,2}(H_1) \cong Suz : 2$ on $P(O_3(H_1)/Z(O_3(H_1)))$ (see [AT], page 131).

Lemma 5.1.1 *Set* $\overline{M} = H_1/O_{3,2}(H_1)$. *Then* $\overline{M}$ *has two orbits* L *and* K *on* $P(O_3(H_1)/Z(O_3(H_1)))$. *Moreover;*

a) If $X \in L$*, then* $C_{\overline{M}}(X)$ *is an extension of an elementary abelian group of order* 3^5 *by* $Z_2 \times M_{11}$.

b) If $X \in K$*, then* $C_{\overline{M}}(X) \cong U_5(2) : 2$.

Notations: Set $Q \cong O_3(H_2)$, $Y = O_3(H_1)$, $H_{12} = H_1 \cap H_2$ and let $\langle \tau \rangle = Z(Y)$. Then by assumption Q is a natural H_2/Q-module. So we adopte the notations A, B and C from 2.4.9 for the orbits of H_2/Q on $P(Q)$. We remark that $\langle \tau \rangle \in A$.

Lemma 5.1.2 *i)* $|Q \cap Y| = 3^7$ *and* $O_3(H_{12}) = QY$.

ii) $P(Q \cap Y) \cap B \neq \varnothing$ *and* $P(Q \cap Y) \cap C \neq \varnothing$.

iii) Let $z \in H_1$ *be an involution such that* $zY \leq Z(H_1/Y)$. *Then* $C_Y(z) = Z(Y)$.

Proof: By the general assumption Q and Y are two subgroups of $O_3(H_{12})$ and $|O_3(H_{12})| = 3^{14}$. Since Y is an extraspecial group of order 3^{13}, we have $|Q \cap Y| \leq 3^7$. As $|O_3(H_{12})| = 3^{14}$, we get that $|Q \cap Y| = 3^7$ and $O_3(H_{12}) = QY$ and i) holds. Let $z \in H_1$ be an involution such that $zY \leq Z(H_1/Y)$. Since by assumption $C_G(Y) = Z(Y)$ and $zY \in Z(H_1/Y)$, we get that z acts fixed point freely on $Y/Z(Y)$ and iii) holds. Now by iii) and coprime action we have that $Q \cap Y = Z(Y) \oplus [z, Q \cap Y]$. By assumption $H_{12}/QY \cong 2U_4(3) : 2 \cong 2O_6^-(3)$. Therefore $[z, Q \cap Y]$ is a natural H_{12}/QY-module. Hence $P(Q \cap Y) \cap B \neq \varnothing$ and $P(Q \cap Y) \cap C \neq \varnothing$ and ii) holds.□

Further notations: By 5.1.2(ii), let $\langle s \rangle \in P(Q \cap Y) \setminus A$. Then by 2.4.9(ii), $C_{H_2}(s)/Q \cong \Omega_7(3)$.

Theorem 5.1.3 $C_G(s) \cong 3M(24)'$.

Proof: By our assumption $\langle s \rangle \in P(Q \cap Y) \setminus A$. So by 2.4.9(ii), $C_{H_2}(s)/Q \cong \Omega_7(3)$. We have $Q/\langle s \rangle$ is a $C_{H_2}(s)/Q$-module. Since $Z(Y) \in A$ and the elements in A are isotropic elements, by ([AT], page 106) we get that $C_G(s, \tau)/Q$ is an extension of an elementary abelian group of order 3^5 by $U_4(2)$. This gives us that $C_{H_1}(s)$ contains a section isomorphic to $U_4(2)$. We remark that by 5.1.2(iii), involutions $zY \in Z(H_1/Y)$ act fixed point freely on $Y/Z(Y)$. Since there is no section isomorphic to $U_4(2)$ in M_{11} (see [AT], page 18), by 5.1.1(i),(ii) we get that $C_{H_1}(s)/C_Y(s) \cong U_5(2) : 2$. Since Y is an extraspecial group of order 3^{13} and $s \in Y$, we deduce that $C_Y(s)/\langle s \rangle$ is an extraspecial group of order 3^{11}. This gives us that $C_{H_1}(s)$ contains a Sylow 3-subgroup of $C_G(s)$. Set $\overline{L} = C_G(s)/\langle s \rangle$, $\overline{L}_i = C_{H_i}(s)/\langle s \rangle$, $i = 1, 2$, and $\overline{L}_3 = C_{H_{12}}(s)/\langle s \rangle$. Now we see that the groups $\overline{L}$ and $\overline{L}_i$, $i = 1, 2, 3$, satisfy the conditions of corollary 7. Hence by corollary 7, $\overline{L} \cong M(24)'$ and the theorem is proved.□

5.2 2-central involution

In this section we show that G has an involution z such that the centralizer of z in G is a faithful extension of an extraspecial group of order 2^{25} by Co_1.

Further notations: Let $z \in H_1$ be an involution such that $zY \leq Z(H_1/Y)$. Set $L = C_Q(z)$.

Lemma 5.2.1 *i)* $C_{H_1}(z) \cong 6Suz : 2$.

ii) z *acts fixed point freely on* $\tau^{\perp}/\langle\tau\rangle$.

iii) $C_{H_{12}}(z)/L \cong 2U_4(3) : 2$ *and* $C_{H_2}(z)/L \cong 2U_4(3) : D_8$.

iv) $|L| = 9$ *and there is an element* ϵ *in* L *such that* ϵ *is conjugate to* τ *in* $C_{H_2}(z)$ *and* $\epsilon \notin \tau^{\perp}$.

v) $\tau^{-1}\epsilon$ *and* $\tau\epsilon$ *are non isotropic elements in* L.

Proof: By 5.1.2(iii), $C_{H_1}(z) \cong 6Suz : 2$ and i) holds. Since $zY \in Z(H_1/Y)$, we have $zO_3(H_{12}) \in Z(H_{12}/O_3(H_{12})$. By 5.1.2(i), we get that $H_{12}/Y \cong 6U_4(3) : 2$. Therefore by (i) and 5.1.2(i), L is of order 9 and $C_{H_{12}}(z)/L \cong 2U_4(3) : 2$. This and ([AT], page 141) give us that $C_{H_2}(z)/L \cong 2U_4(3) : D_8$ and iii) holds. By ([AT], page 128), we get that there is no subgroup isomorphic to $C_{H_2}(z)$ in H_1. Hence there is an element $\epsilon \in L$ conjugate to τ in $C_{H_2}(z)$. Since $H_{12} = N_{H_2}(Z(Y))$, we conclude that $\tau^{\perp}/\langle\tau\rangle$ is a natural $H_{12}/O_3(H_{12})$-module. This gives us that z acts fixed point freely on $\tau^{\perp}/\langle\tau\rangle$. Tehrefore $\epsilon \notin \tau^{\perp}$. Since τ and ϵ are isotropic elements and $\epsilon \notin \tau^{\perp}$, we get that $\tau^{-1}\epsilon$ and $\tau\epsilon$ are non isotropic elements in L. Now the lemma holds.□

Further notations: By 5.2.1(v), there are some non isotropic elements in L. So by conjugations in G we may, and do, assume that $s \in L$. By 5.2.1(iv), there is an element $\epsilon \in L$ conjugate to τ in $C_{H_2}(z)$ and $\epsilon \notin \tau^{\perp}$. So we may assume that $s = \tau\epsilon$. We keep these notations s and ϵ in the remainder of this chapter.

Lemma 5.2.2 *i)* s *and* $\tau^{-1}\epsilon$ *are conjugate in* $C_{H_2}(z)$.

ii) For $t = s$ *or* $\tau^{-1}\epsilon$ *we have* $C_G(t,z)/\langle t\rangle$ *is an extension of an extraspecial 2-group of order* 2^{13} *by* $3U_4(3) : 2$. *Further* $O_2(C_G(t,z))/\langle z\rangle$ *is an irreducible* $C_G(t,z)/O_{2,3}(C_G(z,t))$*-module.*

Proof: By 5.2.1(iii), $C_{H_2}(z)/L \cong 2U_4(3) : D_8$. By 5.1.3 and ([AT], page 200), we get that there is no subgroup isomorphic to $2U_4(3) : D_8$ or $2U_4(3) : 2^2$ in $C_G(s)$.

This and as L is normal in $C_{H_2}(z)$ give us that there is an element conjugate to s in L. Since by 5.2.1(iv), L is of order 9, s is conjugate to $\tau^{-1}\epsilon$ in $C_{H_2}(z)$. We remark that by 5.1.3 and the structure of H_1 we get that s is not conjugate to τ in G. Since $C_{H_2}(z,s)$ contains a section isomorphic to $3U_4(3):2$, by 5.1.3 and ([AT], page 200), we get that $C_G(z,s)/\langle s\rangle$ is an extenstion of an extraspecial group of order 2^{13} by $3U_4(3):2$. By ([As3], 29.9), $O_2(C_G(z,s))/\langle z\rangle$ is an irreducible $C_G(t,z)/O_{2,3}(C_G(z,s))$-module and the lemma is proved.□

Lemma 5.2.3 *Let* $K = O_{\acute{3}}(C_G(z))$. *Then either* $K = \langle z\rangle$ *or is an extraspecial group of order* 2^{25}.

Proof: Set $K = O_{\acute{3}}(C_G(z))$. We recall that $L = \langle\tau, s\rangle$. By coprime action we have that

$$K = \left\langle C_K(t); t \in L^{\sharp}\right\rangle.$$

We recall that by 5.2.1(iv) and 5.2.2(i), ϵ is conjugate to τ in $C_{H_2}(z)$ and s is conjugate to $\tau^{-1}\epsilon$ in $C_{H_2}(z)$. Assume that $t=\tau$. Then by 5.2.1(i), $C_G(z,t)\cong 6Suz$. Since $C_G(z,t)\cap K$ is normal in $C_G(z,t)$, we get that $C_K(t)=\langle z\rangle$. Suppose that $t=s$. Then by 5.2.2(ii), $C_K(t)$ is isomorphic to a subgroup of $O_2(C_G(t,z))$ which is extraspecial of order 2^{13}. Therefore K is 2-group. Since by 5.2.2(ii), $C_G(t,z)/\langle t\rangle$ acts irreducibly on $O_2(C_G(t,z))/\langle z\rangle$, we get that $|C_K(t)/\langle z\rangle| = 2^{12}$ or 1. Hence either for all $t\in L$ of order three we have $C_K(t)=\langle z\rangle$ and then $K=\langle z\rangle$ or for both t we have $C_K(t)$ is extraspecial of order 2^{13}. In the later case by Wielandt's order formula ([BH],XI.12.4) $|K| = 2^{25}$ and $K = C_K(s)C_K(\tau^{-1}\epsilon)$. Therefore K is an extraspecial group of order 2^{25} and the lemma hold.□

Further notations: Set $W = C_G(z)$. We use the bar notation for $W/O_2(W)$.

We are going to show that $\overline{W}$ is isomorphic to Co_1. Since by 5.2.1(iii),(iv), $C_{H_{12}}(z)$ is an extension of an elementary abelian group of order 9 by $2U_4(3):2$, by ([AT], page 52) we get that there is an elementary abelian subgroup E of order 3^6 in $C_{H_{12}}(z)$.

Further notations: We keep the notation E for an elementary abelian subgroup in $C_{H_{12}}(z)$ of order 3^6.

Lemma 5.2.4 $N_{\overline{W}}(\overline{E})/\overline{E} \cong 2M_{12}$ *and the extension splits.*

Proof: By 5.2.1(i) and ([AT], page 128), we get that $N_{\overline{H_1 \cap W}}(\overline{E})/\overline{E} \cong (2 \times M_{11})$. On the other hand by 5.2.1(iii) and ([AT], page 52), we get that $N_{\overline{H_2 \cap W}}(\overline{E})/\overline{E} \cong (2 \times A_6)2^2$. Since there is no subgroup isomorphic to $(2 \times A_6)2^2$ in $2 \times M_{11}$ (see [AT], page 18), we have $N_{\overline{W}}(\overline{E}) \neq N_{\overline{C_{H_1}(z)}}(\overline{E})$.

Set $\widetilde{X} = N_{\overline{H_1 \cap W}}(\overline{E})/\overline{E}$. Then $\widetilde{X}$ has seven orbits $L_0, L_1, L_2, L_3, L_4, L_5, L_6$ on $P(\overline{E})$ such that $L_0 = \{\langle\overline{\tau}\rangle\}$ and $|L_1| = |L_2| = |L_3| = 11$ and $|L_4| = |L_5| = |L_6| = 110$. Further $L_1 = \{\left\langle\overline{\epsilon}^{\widetilde{X}}\right\rangle\}$, $\widetilde{X}$ acts 4-transitively on L_1, $L_4 = \{\langle\overline{x_1 x_2}\rangle^{\widetilde{X}}$, where $\langle\overline{x_1}\rangle$ and $\langle\overline{x_2}\rangle$ are two distinct elements from the orbit $L_1\}$, $L_{1+i} = \{\langle\overline{\tau x}\rangle^{\widetilde{X}}$, where $\langle\overline{x}\rangle \in L_i\}$, $L_{2+i} = \{\langle\overline{\tau}^{-1}\overline{x}\rangle^{\widetilde{X}}$, where $\langle\overline{x}\rangle \in L_i\}$, $i = 1, 4$. We note that $\langle\overline{s}\rangle \in L_2$ and $\langle\overline{\tau}\rangle$ and $\langle\overline{s}\rangle$ are conjugate to $\langle\overline{\epsilon}\rangle$ and $\langle\overline{\tau}^{-1}\overline{\epsilon}\rangle$ in $\overline{H_2 \cap W}$, respectively. By 2.4.4(iii), $\overline{E} = J(\overline{F})$ where $\overline{F} \in Syl_3(\overline{H_2 \cap W})$. Therefore by 2.2.2, $\langle\overline{\tau}\rangle$ is conjugate to $\langle\overline{\epsilon}\rangle$ and $\langle\overline{s}\rangle$ is conjugate to $\langle\overline{\tau}^{-1}\overline{\epsilon}\rangle$ in $N_{\overline{H_2 \cap W}}(\overline{E})$. Hence under the action of $N_{\overline{W}}(\overline{E})/\overline{E}$ on $P(\overline{E})$ we get that L_0 and L_1 are in the same orbit and L_2 and L_3 are in the same orbit. We recall that by 5.1.3 and the structure of H_1, we get that τ is not conjugate to s in G. Thus, L_0 and L_2 or L_3 are not in the same orbit. If $M = L_0 \cup L_1$ and one, two or three of L_4, L_5 and L_6 be in the same orbit, then 61, 29 or 19 divides $\left|N_{\overline{W}}(\overline{E})/\overline{E}\right|$, respectively. But $GL_6(3)$ is a $\{2, 3, 5, 7, 11, 13\}$-group. Therefore M is an orbit of $N_{\overline{W}}(\overline{E})/\overline{E}$ of length 12 that the stabilizer of every element of M in $N_{\overline{W}}(\overline{E})/\overline{E}$ is $2 \times M_{11}$ and $N_{\overline{W}}(\overline{E})/\overline{E}$ is 5-transitive on M. Hence $N_{\overline{W}}(\overline{E})/\overline{E}$ is an extension of M_{12}. As M_{12} does not have a faithful representation on $GF(3)$ of dimension 6 (see [JLPW]), $N_{\overline{W}}(\overline{E})/\overline{E}$ is not isomorphic to $2 \times M_{12}$. Hence $N_{\overline{W}}(\overline{E})/\overline{E} \cong 2M_{12}$. Let $\overline{r}\overline{E} \in Z(N_{\overline{W}}(\overline{E})/\overline{E})$ be an involution. Since a faithful, irreducible, $GF(3)N_{\overline{W}}(\overline{E})/\overline{E}$-module has dimension at lease 6, we get that $\overline{r}$ acts fixed point freely on $\overline{E}$. Hence $C_{N_{\overline{W}}(\overline{E})}(\overline{r}) \cong 2M_{12}$ and the extension splits. This completes the proof.□

Further notations: In the remainder of this chapter $\overline{X}$ is a fixed complement of $\overline{E}$ in $N_{\overline{W}}(\overline{E})$. By 2.4.2, $\overline{X}$ has three orbits M, I and J on $P(\overline{E})$ such that $|M| = 12$, $|I| = 132$ and $|J| = 220$. We keep these notations M, I and J for the orbits of $\overline{X}$ on $P(\overline{E})$. We recall that $\langle\overline{\tau}\rangle$ and $\langle\overline{\epsilon}\rangle$ are in M and $\langle\overline{s}\rangle \in I$. Set $\overline{T} = \left\langle\overline{s}\overline{\delta}\right\rangle$, where $\overline{\tau} \neq \overline{\delta} \neq \overline{\epsilon}$ and $\left\langle\overline{\delta}\right\rangle \in M$. Then $\overline{T} \in J$. Set $\overline{D} = \overline{C_G(z, \tau)}$ and $\overline{N}_2 = N_{\overline{D}}(\overline{T})$.

Lemma 5.2.5 *i)* $O_3(C_{\overline{D}}(\overline{T}))$ *is a special group with center* $\langle\overline{\tau},\overline{\epsilon},\overline{\delta}\rangle$, $C_{\overline{D}}(\overline{T})/O_3(C_{\overline{D}}(\overline{T}))\cong SL_2(3)\times 2$ *and the extension splits.*

ii) Let $\overline{B}_1$ *be a fixed complement of* $O_3(C_{\overline{D}}(\overline{T}))$ *in* $C_{\overline{D}}(\overline{T})$ *and* $\overline{i}\in Z(O^2(\overline{B}_1))$ *be an involution. Then* $|C_{\overline{E}}(i)|=3^4$.

iii) There is a subgroup $\overline{B_3}=\langle\overline{X}_1,\overline{k}\rangle\cong SL_2(3)$ *in* $C_{\overline{D}}(Z(O_3(\overline{N}_2)))$ *such that* $\overline{k}\in\overline{E}$ *is of order three and* $\overline{X}_1\cong Q_8$ *is a quaternion group.*

Proof: We recall that $L=\langle\tau,s\rangle$. By 5.2.1(iii), $C_{\overline{W}}(\overline{L})$ contains a section isomorphic to $U_4(3)$. So by 5.2.1(i) and [Wi2] we get that $C_{\overline{D}}(\overline{L})/\overline{L}\cong U_4(3)$. Also by 5.2.1(i), 2.4.2(iv),(vii) and [Wi2] we get that $O_3(C_{\overline{D}}(\overline{T}))$ is a special group with center $\langle\overline{\tau},\overline{\epsilon},\overline{\delta}\rangle$, $C_{\overline{D}}(\overline{T})/O_3(C_{\overline{D}}(\overline{T}))\cong SL_2(3)\times 2$ and the extension splits. Now i) holds. Let $\overline{B}_1$ be a fixed complement of $O_3(C_{\overline{D}}(\overline{T}))$ in $C_{\overline{D}}(\overline{T})$ and $\overline{i}\in Z(O^2(\overline{B}_1))$ be an involution. We remark that $\overline{i}$ acts fixed point freely on $O_3(C_{\overline{D}}(\overline{T}))/Z(O_3(C_{\overline{D}}(\overline{T})))$ and $O^2(C_{\overline{D}}(\overline{T}))/O_3(C_{\overline{D}}(\overline{T}))$ acts trivially on $Z(O_3(C_{\overline{D}}(\overline{T})))$. Since $\overline{i}\in Z(\overline{B}_1)$, we have $\overline{i}$ normalizes $\overline{E}$. By 2.4.2(i) and ([AT], page 18), we get that $C_{\overline{D}\cap\overline{X}}(\overline{i})\cong GL_2(3)$. Therefore $\overline{i}$ is of determinat 1. Hence $\left|C_{\overline{E}/\langle\overline{\tau}\rangle}(\overline{i})\right|=3$ or $\left|C_{\overline{E}/\langle\overline{\tau}\rangle}(\overline{i})\right|=3^3$. By 2.4.3, $N_{C_{\overline{X}}(\overline{\tau})}(\langle\widehat{r}\rangle)$ is a subgroup of $M_{10}\times 2$ for each element $\widehat{r}\in\overline{E}/\langle\overline{\tau}\rangle$ of order three. As there is no subgroup isomorphic to $C_{\overline{X}\cap\overline{D}}(\overline{i})$ in $M_{10}\times 2$, we get that $\left|C_{\overline{E}/\langle\overline{\tau}\rangle}(i)\right|\neq 3$ and then $\left|C_{\overline{E}}(\overline{i})\right|=3^4$ and ii) holds. By i) and ii) and since $\overline{i}$ acts fixed point freely on $O_3(C_{\overline{D}}(\overline{T}))/Z(O_3(C_{\overline{D}}(\overline{T})))$, we get that there is a subgroup $\overline{B_2}=\langle\overline{X}_1,\overline{k}\rangle\cong SL_2(3)$ in $C_{\overline{D}}(Z(O_3(\overline{N}_2)))$ such that $\overline{k}\in\overline{E}$ is of order three and $\overline{X}_1\cong Q_8$ is a quaternion group. Now the lemma is proved.□

Further notations: By 5.2.5(iii), we keep the notation $\overline{B}_1$ for a fixed complement of $O_3(C_{\overline{D}}(\overline{T}))$ in $C_{\overline{D}}(\overline{T})$ such that $\overline{B_1}=\langle\overline{X}_1,\overline{k}\rangle\cong SL_2(3)$ in $C_{\overline{D}}(Z(O_3(\overline{N}_2)))$, where $\overline{k}\in\overline{E}$ is of order three and $\overline{X}_1\cong Q_8$ is a quaternion group.

Lemma 5.2.6 $N_{\overline{W}}(\overline{T})$ *is a faithful extension of an extraspecial group of order* 3^5 *by* $Sp_4(3):2$ *and contains a Sylow 3-subgroup of* $\overline{W}$.

Proof: Set $\overline{N}_1=N_{N_{\overline{W}}(\overline{E})}(\overline{T})$, $\overline{U}=O_3(\overline{X}\cap\overline{N}_1)$ and $\overline{R}=[\overline{E},\overline{U}]\overline{U}$. By 2.4.2(iii), $\overline{N}_1/\overline{E}$ is an extension of $\overline{U}$ by $GL_2(3)\times 2$. By 2.4.2(v), $\overline{R}$ is the unique extraspecial normal subgroup in $\overline{N}_1$ of order 3^5. We remark that By ([AT], page 32) $\overline{N}_1\cap\overline{X}=N_{\overline{X}}(\overline{U})$. Let $\overline{K}=\overline{N}_1\cap\overline{N}_2$. Then $\overline{K}=\overline{E}(\overline{K}\cap N_{\overline{X}}(\overline{U}))$. By 2.4.2(v), $\overline{U}$ is not a subgroup of $\overline{K}$. By 2.4.2(v)(i) and ([AT], page 18), $\overline{K}\cap N_{\overline{X}}(\overline{U})$ contains a Sylow

3-subgroup $\overline{U}_1$ of $C_{\overline{X}}(\overline{\tau})$. Since $\overline{U}_1 \neq \overline{U}$, we have $|U \cap U_1| = 3$. Hence $\overline{U}_1$ has index at most 2^3 in $\overline{K} \cap N_{\overline{X}}(\overline{U})$. We remark that by [Wi2], $\overline{N}_2$ is a split extension of $O_3(\overline{N}_2)$ by $SL_2(3) * D_8$ and $O_3(\overline{N}_2)$ is a special group of order 3^7 and exponent 3 with center of order 3^3. Let $\overline{P} \in Syl_3(\overline{K})$. Then by considering orders we have $\overline{P} \in Syl_3(\overline{N}_2)$. Let $\overline{F} \in Syl_3(\overline{N}_1)$ containing $\overline{P}$. Then as $\overline{P}$ is of index three in $\overline{F}$, we conclude that $\overline{P} \triangleleft F$ and hence $\overline{R} \leq N_{\overline{W}}(Z(\overline{P}))$. We have $\overline{U} \leq \overline{R}$ and $\overline{U}$ is not contained in $\overline{P}$. Therefore $\overline{F} = \overline{PR}$. By 2.4.2(vi), $\overline{E}$ is a characteristic subgroup of $\overline{F}$ and hence $\overline{F} \in Syl_3(\overline{W})$. We note that from the structure of $\overline{N}_2$ we get that $Z(\overline{P}) = Z(O_3(\overline{N}_2))$. Since $\overline{R} \leq N_{\overline{W}}(Z(O_3(\overline{N}_2)))$, we get that $C_{\overline{D}}(Z(O_3(\overline{N}_2)))\overline{R}$ is a group. Set $\overline{N}_3 = C_{\overline{D}}(Z(O_3(\overline{N}_2)))\overline{R}$. We are going to show that $\overline{R} \triangleleft \overline{N}_3$.

Lemma (5.2.6.1) There is an extraspecial normal subgroup $\overline{R}_1$ of order 3^5 in $\overline{N}_3$.

Proof: We have $\overline{N}_3/O_3(\overline{N}_2) \cong SL_2(3) \times \langle \widehat{a} \rangle$ where $\widehat{a}$ is of order 3. So $\overline{N}_3 = O_3(\overline{N}_3)\overline{B}_1$. Let $\bar{i}$ be an involution in $Z(\overline{B}_1)$. Then $C_{O_3(\overline{N}_3)}(i) = \langle Z(O_3(\overline{N}_2)), \overline{a} \rangle$ where $\overline{a}$ is a preimage of $\widehat{a}$. Therefore $[\overline{X}_1, C_{O_3(\overline{N}_3)}(\bar{i})] = 1$. By coprime action we have $O_3(\overline{N}_2) = C_{O_3(\overline{N}_2)}(\bar{i})[\bar{i}, O_3(\overline{N}_2)]$. Set $\overline{E}_1 = C_{O_3(\overline{N}_2)}(\overline{a})$. As $C_{O_3(\overline{N}_3)}(\bar{i})$ is $\overline{B}_1$-invariant, we deduce that $[\overline{a}, \overline{k}] \in \overline{E} \cap C_{O_3(\overline{N}_3)}(\bar{i})$. Let $\overline{y} \in \overline{E}_1$. Then by the three subgroup lemma, $[\overline{y}, \overline{k}] \in \overline{E}_1$. Therefore $\overline{E}_1$ is $\overline{B}_1$-invariant. On the other hand, under the action of $\overline{B}_1$ on $O_3(\overline{N}_2)/Z(O_3(\overline{N}_2))$ we have $O_3(\overline{N}_2)/Z(O_3(\overline{N}_2)) = \widetilde{V}_1 \oplus \widetilde{V}_2$, where $\left|\widetilde{V}_1\right| = \left|\widetilde{V}_2\right| = 3^2$. Clearly $\overline{E}_1 \cap Z(O_3(\overline{N}_2)) = Z(\overline{F})$ is of order three. If $\overline{a}$ acts trivially on $O_3(\overline{N}_2)/Z(O_3(\overline{N}_2))$ then $O_3(\overline{N}_3)\overline{E}/\overline{E}$ is an abelian subgroup of order 3^3 in $\overline{F}/\overline{E}$. As by ([AT], page 32), $\overline{F}/\overline{E}$ is an extraspecial group of order 3^3, we get that $\overline{a}$ does not act trivially on $O_3(\overline{N}_2)/Z(O_3(\overline{N}_2))$. Since $C_{O_3(\overline{N}_2)/Z(O_3(\overline{N}_2))}(\overline{a}) \neq 1$ and $\overline{a}$ does not act trivially on $O_3(\overline{N}_2)/Z(O_3(\overline{N}_2))$, we get that $\left|\overline{E}_1\right| = 3^3$. Let $\overline{E}_1 \leq \overline{E}$. Then $\overline{E} \cap O_3(\overline{N}_2) = \langle Z(O_3(\overline{N}_2)), \overline{E}_1 \rangle$ and $\overline{E} \cap O_3(\overline{N}_2)$ is $\overline{B}_1$-invariant. Therefore there is a subgroup $\overline{U}_1$ in $O_3(\overline{N}_2)$ of order 3^2 such that $\left|\overline{U}_1 \cap \overline{E}\right| = 1$ and $\overline{U}_1 Z(O_3(\overline{N}_2))/Z(O_3(\overline{N}_2))$ is $\overline{B}_1$-invariant. Since $[\overline{U}_1, \overline{k}] \leq \overline{E}$, we get that $\overline{k}$ acts trivially on $\overline{U}_1 Z(O_3(\overline{N}_2))/Z(O_3(\overline{N}_2))$. As $\overline{k}^{\bar{i}} = \overline{k}$ and $\widehat{\overline{u}_1}^{\bar{i}} = \widehat{\overline{u}_1}^{-1}$ for each $\widehat{\overline{u}_1} \in \overline{U}_1 Z(O_3(\overline{N}_2))/Z(O_3(\overline{N}_2))$ of order three, we get that $[\overline{U}_1, \overline{k}] = 1$. Therefore $\langle \overline{U}_1, \overline{k}, Z(O_3(\overline{N}_2)) \rangle$ is an abelian subgroup of $\overline{F}$ of order 3^6 and this is a contradiction to 2.4.2(v). Assume that $\overline{E}_1$ is abelian. Let $\overline{r} \in \overline{E}_1$ such that $\overline{r} \notin \overline{E}$. Then $[\overline{k}, \overline{r}] \in \overline{E}_1$, therefore $[\overline{k}, \overline{r}, \overline{r}] = 1$. As $[\overline{E} \cap O_3(\overline{N}_2), \overline{r}] \leq Z(O_3(\overline{N}_2))$, we have

$\left[\overline{E}\cap O_3(\overline{N}_2),\overline{r},\overline{r}\right]=1$. Since $\overline{E}=\left\langle\overline{k},\overline{E}\cap O_3(\overline{N}_2)\right\rangle$, we deduce that $\left[\overline{E},\overline{r},\overline{r}\right]=1$. As by Theorem A in [Ch] no element of order three in $C_{\overline{X}}(\overline{\tau})$ acts quadratically on $\overline{E}$, we have a contradiction. Hence $\overline{E}_1$ is not abelian and then $\overline{E}_1$ is an extraspecial group of order 3^3. Let $\overline{u}\in Z(O_3(\overline{N}_2))$ such that $1\neq[\overline{u},\overline{a}]\in Z(\overline{F})$. Set $\overline{R}_1=\left\langle\overline{E}_1,\overline{u},\overline{a}\right\rangle$. Then $\overline{R}_1$ is an extraspecial group of order 3^5 and $Z(\overline{R}_1)=Z(\overline{F})$.

We have $O_3(\overline{N}_3)=\left\langle O_3(\overline{N}_2),\overline{a}\right\rangle$. Since $\overline{a}$ normalizes $Z(O_3(\overline{N}_2))$ and $\widehat{a}$ acts quadratically on $O_3(\overline{N}_2)/Z(O_3(\overline{N}_2))$ so $\left[\overline{R}_1,O_3(\overline{N}_3)\right]\leq Z(O_3(\overline{N}_2))\overline{E}_1$. Let $\overline{x}\in O_3(\overline{N}_2)$ and $\overline{a}^{\overline{x}}=\overline{ae}$, where $\overline{e}\in\overline{E}_1Z(O_3(\overline{N}_2))$. Let $\overline{f}\in\overline{E}_1\times\langle\overline{u}\rangle$. Then $\overline{f}^{\overline{x}}=\overline{f}\overline{y}\in\overline{R}_1^{\overline{x}}$, where $\overline{y}\in\overline{E}_1Z(O_3(\overline{N}_2))$. Therefore $\left[\overline{y}\overline{f},\overline{ae}\right]\in\left\langle Z(\overline{F}),[\overline{y},\overline{a}]\right\rangle$. As $\overline{R}_1^{\overline{x}}$ is an extraspecial group with center $Z(\overline{F})$, we conclude that $[\overline{y},\overline{a}]\in Z(\overline{F})$ and this gives us that $\overline{y}\in\overline{E}_1\times\langle\overline{u}\rangle$ and then $\overline{E}_1\times\langle\overline{u}\rangle\leq\overline{R}_1^{\overline{x}}\cap\overline{R}_1$. Let $\overline{a}^{\overline{x}}=\overline{aem}$, where $\overline{e}\in\overline{E}_1\times\langle\overline{u}\rangle$, $\overline{m}\in Z(O_3(\overline{N}_2))$ and $[\overline{m},\overline{a}]\notin Z(\overline{F})$. Then $(\overline{a}^{\overline{x}})^3=(\overline{am})^3\notin Z(\overline{F})$ and this is a contradiction to $\overline{R}_1^{\overline{x}}$ being an extraspecial group. Hence $\overline{R}_1^{\overline{x}}=\overline{R}_1$ and $\overline{R}_1$ is normal in $O_3(\overline{N}_3)$. By 2.4.2(v) $[\overline{k},\overline{a},\overline{a},\overline{a}]=1$. So $[\overline{k},\overline{a}]\in\left\langle\overline{u},Z(\overline{F})\right\rangle$. Now as $\overline{R}_1$ is $\overline{X}_1$-invariant, we get that $\overline{R}_1$ is normal in $O_3(\overline{N}_3)\overline{B}_1=\overline{N}_3$.□

We are going to show that $\overline{R}$ is a subgroup of $O_3(\overline{N}_3)$. If not, let $\overline{x}\in\overline{R}$ with $\overline{x}\notin O_3(\overline{N}_3)$. Then we have $\overline{R}\cap\overline{R}_1=\left\langle\overline{u},Z(\overline{F})\right\rangle$ and $[\overline{x},\overline{E}_1]\leq\overline{R}\cap\overline{R}_1$. as $[\overline{x},\overline{E}_1]$ is not in $Z(O_3(\overline{N}_2))$, we have $\overline{R}\leq O_3(\overline{N}_3)$.

Now we show that $\overline{R}_1=\overline{R}$. We have $\overline{R}_1\triangleleft\overline{N}_3$ and $\overline{N}_3/\overline{R}_1$ is an extension of an extraspecial group of order 3^3 by $SL_2(3)$. Let $Z(O_3(\overline{N}_3/\overline{R}_1))=\left\langle\overline{d}\overline{R}_1\right\rangle$ where $\overline{d}\in Z(O_3(\overline{N}_2))$ and let $\overline{i}O_3(\overline{N}_3)\in Z(\overline{N}_3/O_3(\overline{N}_3))$ being an involution, then $\overline{i}$ normalizes $\overline{F}$. So by 2.4.2(vi), $\overline{i}$ normalizes $\overline{E}$. As $\overline{i}$ centralizes $\overline{T}$, we deduce that $\overline{i}\in\overline{N}_1$. Since $\overline{R}$ is normal in $\overline{N}_1$, we have $\overline{R}^{\overline{i}}=\overline{R}$. Let $\overline{x}\in\overline{R}\cap\overline{R}_1\overline{d}$ and $\overline{x}=\overline{df}$, where $\overline{f}\notin\overline{R}$ (as $\overline{d}\notin\overline{R}$) and $\overline{f}^{\overline{i}}=\overline{f}^{-1}Z(\overline{F})$. Since $\overline{d}^{\overline{i}}=\overline{d}$, we get that $\overline{x}^{\overline{i}}=\overline{df}^{\overline{i}}$. Now we have $\overline{x}^{-1}\overline{x}^{\overline{i}}=\overline{f}^{-1}\overline{f}^{\overline{i}}\in\overline{R}$, so $\overline{f}\in\overline{R}$ and this is a contradiction. Therefore $\overline{R}\cap\overline{R}_1\overline{d}=\varnothing$ and hence $\overline{R}=\overline{R}_1$.

Finally $\overline{R}$ is normal in $\overline{N}=\left\langle\overline{N}_1,\overline{N}_3\right\rangle$. By 2.2.3(ii), $Aut(\overline{R})/Inn(\overline{R})\cong Sp_4(3):2$. We note that $Z(O_3(\overline{N}_2))=\left\langle\overline{\tau},\overline{\delta},\overline{\epsilon}\right\rangle$, $\overline{u}\in Z(O_3(\overline{N}_2))$, $\overline{a}\notin\overline{N}_3$ and $\overline{T}=\left\langle\overline{\tau}\overline{\delta}\overline{\epsilon}\right\rangle$. Now as $[\overline{a},\overline{u}]\in\overline{T}$, $\overline{T}\in J$, $\overline{a}\in\overline{X}$ and by the representations of the elements in the orbits M, J and I in 2.4.2, we get that $\langle\overline{u}\rangle\in I$. Therefore $\overline{u}$ is conjugate to $\overline{s}$ in $\overline{W}$. Now by 5.2.2(ii), ([AT], page 52) and as $C_{\overline{W}}(Z(O_3(\overline{N}_2)))\leq C_{\overline{W}}(\overline{T},\overline{u})$, we get that $C_{\overline{W}}(\overline{R})\leq C_{\overline{N}_1}(\overline{T},\overline{U},\overline{u})$. Since $C_{\overline{N}_1}(\overline{U})=\overline{T}$, we deduce that $C_{\overline{W}}(\overline{R})=\overline{T}$.

Hence $\overline{N}/Z(\overline{R})$ is a subgroup of $Aut(\overline{R})$. So $\overline{N}/\overline{R}$ is isomorphic to a subgroup of $Sp_4(3):2$. Since $\overline{N}_1/\overline{R}$ is a maximal subgroup of $Sp_4(3):2$ ([AT], page 26), we get that $\overline{N}/\overline{R} \cong Sp_4(3):2$ and the lemma is proved.□

Lemma 5.2.7 $\overline{W} \cong Co_1$.

Proof: By 5.2.4, 5.2.6 and definition 1, we get that $\overline{W}$ is of Co_1-type. So by theorem 4 $\overline{W} \cong Co_1$ and the lemma is proved.□

Theorem 5.2.8 *$C_G(z)$ is a faithful extension of an extraspecial group of order 2^{25} by Co_1.*

Proof: By theorem 5.2.7, $C_G(z)/O_2(C_G(z))$ is isomorphic to Co_1. By 5.2.3 $K = O_2(C_G(z)) = \langle z \rangle$ or K is an extraspecial group of order 2^{25}. By 5.2.2(ii), $O_2(C_G(s,z))$ is an extraspecial group of order 2^{13} and $C_G(s,z)/O_2(C_G(s,z))$ is an extension of an elementary abelian group of order 3^2 by $U_4(3):2$. If $K = \langle z \rangle$, then Co_1 must have a subgroup isomorphic to $C_G(s,z)/\langle z \rangle$. But Co_1 does not have an elementary abelian subgroup of order 2^{12} ([AT], page 180). Hence K is an extraspecial 2-group of order 2^{25} and the theorem is proved.□

5.3 Proof of theorem 6

In this short section we prove the theorem 6. By ([As3] 34.13) there is an involution $t \in O_2(C_G(s,z))$ such that t is not conjugate to z in $C_G(s)$ and $C_G(s,t)/\langle s \rangle \cong 2M(22):2$. Set $\widetilde{W_1} = C_G(t)/\langle t \rangle$.

Lemma 5.3.1 $\widetilde{W_1} \cong F_2$*, the baby monster group.*

Proof: As $C_G(s,t)/\langle s \rangle \cong 2M(22):2$, by 5.2.7 and Lagrange theorem t is not conjugate to z in G. By 5.2.7 and ([AS2] lemmas 32.1 and 32.2) we get that $\widetilde{W_2} = C_G(z,t)/\langle t \rangle$ is an extension of an extraspecial 2-group of order 2^{23} by Co_2. Now applying 2.3.2 we have either $\widetilde{W_1} = O(\widetilde{W_1})\widetilde{W_2}$ or $\widetilde{W_1} \cong F_2$. Since $C_G(s,t)/\langle s \rangle \cong 2M(22):2$, the case $\widetilde{W_1} = O(\widetilde{W_1})\widetilde{W_2}$ does not happen. Hence $\widetilde{W_1} \cong F_2$ and the lemma is proved.□

Now we can prove the **Theorem 6:**

Proof: It follows from 5.2.7, 5.3.1 and 2.3.1.□

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