CONVERGENCE TO STEADY STATE FOR A PHASE FIELD SYSTEM WITH MEMORY

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Introduction

The study of long time behaviour of solutions to nonlinear dissipative evolutionary equations has attracted the interest of many mathematicians for a long time. The research in this field has been focused principally on two aspects. One is concerned with the asymptotic behaviour of families of global solutions for initial data starting from any bounded set in certain Sobolev space with the aim to find a compact invariant set which absorbs these solutions, that is, an attractor. We refer to Temam [Tem88], Hale [Hal88], and Babin and Vishik [BV92] for a comprehensive study of this subject.

Another important aspect is the study of the convergence to an equilibrium of global bounded solutions as time goes to infinity. In the ODE case, the classical theory of Lyapunov functions and La Salle's invariance principle allow to prove convergence of global bounded solutions to an equilibrium provided that the set of equilibrium points is finite or discrete. This condition can be checked in many problems arising in applications. The same technique can be used for PDEs, but in this case, it is not so easy to describe the structure of the set of equilibrium points. Therefore one should look for new methods to establish convergence to steady state for such problems.

For nonlinear dissipative evolutionary equations there exist some papers which develop new techniques in different settings and provide positive results in this direction, we refer to [Zel68, Mat78, Sim83, Lio84, HR92, HP92, BP97, Jen98, RH99]. A seminal contribution was made by Simon [Sim83], who was the first to observe that in case of analytic nonlinearities and under suitable growth conditions any global bounded solution of a gradient-like evolution equation converges to an equilibrium. His idea relies on a generalization of the so-called Łojasiewicz inequality for analytic functions defined in finite dimensional space $\mathbb{R}^n$. Jendouby [Jen98] simplified Simon's proof and obtained a corresponding convergence result for a class of hyperbolic evolution equations. Since then the Łojasiewicz-Simon inequality has been used by many authors to prove convergence to steady state of bounded solutions of several types of evolution equations, see for example [AFIR01, AF01, HT01, AP03, Chi03, FIRP04, WZ04, CF05, PW06].

Actually, the problems studied in the aforementioned papers are related to the first order equation

$$\dot{u}(t) + \mathcal{E}'(u(t)) = 0, \; t > 0, \tag{0.0.1}$$

and the second order equation

$$\ddot{u}(t) + \dot{u}(t) + \mathcal{E}'(u(t)) = 0, \; t > 0, \tag{0.0.2}$$

respectively, where the nonlinear term $\mathcal{E}'$ is the Fréchet derivative of a functional $\mathcal{E} \in C^1(V)$, and V is a Hilbert space which is densely and continuously embedded into another Hilbert space H. The main assumption in all of the above papers to prove convergence to single steady state is that the functional $\mathcal{E}$ satisfies the Łojasiewicz-Simon inequality near some point $\vartheta \in V$ in the ω-limit set, that is, there exist constants $\theta \in (0, 1/2]$, $C \geqslant 0$ and $\sigma > 0$ such that for all $v \in V$ with $| v - \vartheta |_V < \sigma$, there holds

$$| \mathcal{E}(v) - \mathcal{E}(\vartheta) |^{1-\theta} \leqslant C \, | \mathcal{E}'(v) |_{V'},$$

where V' denotes the topological dual of V.

A typical functional $\mathcal{E}$, which satisfies the Łojasiewicz-Simon inequality and often appears in applications, is given by

$$\mathcal{E}(v) = \frac{1}{2}\alpha(v, v) + \int_\Omega \Phi(x, v)dx, \; v \in V,$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$, $\alpha : V \times V \to \mathbb{R}$ is a bilinear, continuous, symmetric, and coercive form, and the nonlinear term $\Phi(x, \cdot)$ is a $C^2(V)$ function with suitable growth conditions. We refer to Chill [Chi03] for a comprehensive study of this subject.

As to nonlinear evolutionary equations with memory term there has been only some progress concerning convergence to steady state. The reason for this lies essentially in the fact that these problems do not generate in general a semi-flow in the natural phase space. Another difficulty consists in finding Lyapunov functions for such problems which are appropriate to investigate the asymptotic behaviour of global bounded solutions.

We will now describe some positive results in this direction. There are some papers which deal with the equation

$$\ddot{u} + B_0\dot{u} + \int_0^t a(t-s)B_1\dot{u}(s)ds + \mathcal{E}'(u) = 0, \ t > 0, \tag{0.0.3}$$

where B_0 and B_1 are closed, linear, self-adjoint, positive operators on a Hilbert space H. The first positive result was obtained by Fašangová and Prüss in [FP99, FP01], where the authors develop a method which combines techniques from nonlinear Volterra equations in finite dimensions (cf. [GLS90]) and harmonic analysis of vector-valued functions (cf. [Chi98]). The main problem of this approach is that in order to establish convergence to an equilibrium one has to assume that the set of stationary points of (0.0.3) is discrete, a condition that is not easy to verify and not fulfilled in general. Recently, Chill and Fašangová [CF05], using ideas from Dafermos [Daf70] and [AF01], were able to prove that under suitable conditions on the kernel a any global bounded solution u of (0.0.3) converges to a steady state, provided that the functional $\mathcal{E}$ satisfies the Łojasiewicz-Simon inequality near some $\vartheta \in \omega(u)$. Note that the latter allows to avoid additional assumptions on the set of equilibria.

A series of papers is concerned with non-conserved phase field models with memory of the form

$$u_t + \phi_t = \int_0^t a(t-s)\Delta u(s)ds + f, \tag{0.0.4}$$

$$\phi_t = \Delta\phi - \Phi'(\phi) + u, \tag{0.0.5}$$

complemented by Neumann boundary and initial conditions, and the corresponding variants in the conserved case. Concerning convergence to steady state, we refer to the pioneering works [AF01] and [AP03], in which the approach via the Łojasiewicz-Simon inequality is used for the first time in the context of phase field models.

During the last years many papers have also addressed the problem of global existence and dynamic properties such as existence of attractors for the model (0.0.4)-(0.0.5) (and variants of it) in different settings. We refer to Giorgi et. al. [GGP99] and Grasselli et. al. [GP04] and the references given therein.

Further, there exist some results for the system

$$u_t + \phi_t = \int_0^t a_1(t-s)\Delta u(s)ds + f_1, \tag{0.0.6}$$

$$\phi_t = \int_0^t a_2(t-s)(\Delta\phi - \Phi'(\phi) + u)(s)ds + f_2, \tag{0.0.7}$$

on $[0,\infty)\times\Omega$, $\Omega\subset\mathbb{R}^n$ a bounded domain, together with Neumann boundary and initial conditions. This system was proposed by Rotstein et. al. in [RBNCN01] as a phenomenological model to describe phase transitions in the presence of a slowly relaxing internal variable. Novick-Cohen [NC02] obtained global well-posedness of it in a weak sense in the case $n \leqslant 3$, by means of the Galerkin method and energy estimates, where $\Phi(s) = (s^2-1)^2$, the well-known double-well potential. In [GP04] existence of a uniform attractor is shown for the system (0.0.6)-(0.0.7) with a quadratic potential.

The purpose of the present thesis is twofold. The first objective is to establish the global strong well-posedness of (0.0.6)-(0.0.7) in the L_p-setting in the case $n \leqslant 3$, as well as of its conserved version, that is,

$$u_t + \phi_t = \gamma\Delta u + \int_0^t a_1(t-s)\Delta u(s)ds + f_1, \tag{0.0.8}$$

$$\phi_t = -\int_0^t a_2(t-s)\Delta\left[\Delta\phi - \Phi'(\phi) + u\right](s)ds + f_2. \tag{0.0.9}$$

on $[0,\infty)\times\Omega$, $\Omega\subset\mathbb{R}^n$ $(n\leqslant 3)$ a bounded domain, together with Neumann boundary condition for u, ϕ, and $\Delta\phi$ as well as initial conditions.

The second and main goal of this thesis consists in proving convergence to steady state for the conserved phase field model with memory (0.0.8)-(0.0.9). To achieve this, it is crucial to understand a simplified model which in abstract form can be written as a nonlinear evolutionary equation in a real Hilbert space H of the form

$$\dot{u}(t) + \int_0^t a(t-s)\mathcal{E}'(u(s))\,ds = f(t),\ t > 0. \tag{0.0.10}$$

Here $\mathcal{E}'$ is the Fréchet derivative of a functional $\mathcal{E} \in C^1(V)$, where V is a Hilbert space which densely and continuously injects into H. The scalar kernel a belongs to a certain kernel class whose prototypical example is given by

$$a(t) = Ce^{-wt}t^{-\alpha}, \quad t > 0,$$

where C, w, α are positive constants with $\alpha \in (0, 1)$.

Problems of the form (0.0.10) also arise in several other applications such as e.g. nonlinear heat conduction with memory and nonlinear viscoelasticity. For these reasons, a separate section of this thesis is devoted to the study of convergence to steady state of global bounded solutions of the abstract equation (0.0.10).

This thesis is organized as follows. In *Chapter 1*, we describe a theoretical framework and tools to solve abstract linear problems of parabolic type in Banach spaces. The chapter consists of two parts. The first part is devoted to the class of sectorial operators and subclasses of it, which play an important role in the theory of maximal regularity. This will be the subject of the second part, where we will recall fundamental results in the context of maximal regularity, such as a version of the well-known Dore-Venni theorem due to Prüss and Sohr [PS90], the operator-valued version of the famous Mikhlin Fourier multiplier theorem due to Weis [Wei01], and a resent result in the theory of abstract parabolic Volterra equations due to Zacher [Zac05]. These results will be used in Chapter 3 and 4 to obtain optimal regularity estimates for linearized versions of the phase field models to be studied.

Chapter 2 gives an outline of the physical background of heat conduction in materials with memory. On the basis of the discussion in [JF85, BFJ86] and [RBNCN01] we propose a conserved phase field model, which can be interpreted as a non-isothermal Cahn-Hilliard equation with memory and relaxing chemical potential.

In *Chapter 3* we prove the global strong well-posedness of the non-conserved phase field system with memory (0.0.6)-(0.0.7) in an L_p-setting. Assuming enough regularity of the kernels a_1 and a_2, we apply a recent result in the theory of abstract parabolic Volterra equations, which was proved in [Zac05], to obtain a local strong solution in the framework of Bessel potential spaces. To solve (0.0.6)-(0.0.7), we first show that this system is equivalent to a semilinear problem of Volterra type of the form

$$v = \int_0^t b(t-s)\Delta v(s)ds + H(v) + f(t), \tag{0.0.11}$$

where $H(v)$ is a non-local nonlinear term. Maximal regularity of an appropriate linearization and the contraction mapping principle then yield the local well-posedness of (0.0.11). Finally, global well-posedness of (0.0.6)-(0.0.7) (in the case of trivial history) is obtained by means of energy estimates and the Gagliardo-Nirenberg inequality. The main result of this chapter is stated in Theorem 3.2.2.

Chapter 4 is concerned with the conserved phase field model (0.0.8)-(0.0.9). Our proof of the local strong well-posedness is again based on linearization and the contraction mapping principle. However, our approach to obtain maximal regularity for the linearized problem differs from that in the previous chapter. Using inversion of the convolution (cf. [Prü93, Thm. 8.6]) we reformulate the linear version of the system (0.0.8)-(0.0.9) as an abstract system

$$(B_1 + \mathcal{A})v = -B_1\varphi + B_1h_1, \tag{0.0.12}$$

$$(B_2 + \mathcal{A}^2)\varphi = \mathcal{A}v + B_2h_2, \tag{0.0.13}$$

where the operator $\mathcal{A}$ is the canonical extension of the negative Laplacian in $L_p(\Omega)$ to

$L_p(\mathbb{R}; L_p(\Omega))$, and the operators B_i are the Volterra operators defined in (1.1.3) that correspond to the kernels $1 * a_1$ and $1 * a_2$, respectively. Further, if we assume that φ is known in (0.0.12)-(0.0.13) then by the method of sums of operators, the unknown function v in (0.0.12) can be represented as

$$v = -(B_1 + \mathcal{A})^{-1} B_1 \varphi + (B_1 + \mathcal{A})^{-1} B_1 h_1. \tag{0.0.14}$$

Inserting this into equation (0.0.13) leads to the problem

$$(B_2 + \mathcal{A}^2 + \mathcal{A}(B_1 + \mathcal{A})^{-1} B_1)\varphi = h. \tag{0.0.15}$$

If φ solves (0.0.15), then this together with (0.0.14) yields the solution of the system (0.0.12)-(0.0.13).

As to (0.0.15), note that the method of sums is not applicable since the power angles of the operators B_i, $i = 1, 2$, are in general greater than $\pi/2$ and therefore the parabolicity condition is not satisfied. However, by imposing an extra assumption that roughly speaking says that the imaginary parts of the Laplace transforms of $1 * a_1$ and $1 * a_2$ have the same sign, we are able to use the operator-valued version of the Mikhlin Fourier multiplier theorem in one variable to obtain existence and uniqueness for (0.0.15).

Having solved (0.0.8)-(0.0.9) locally, global strong well-posedness (in the case of trivial history), Theorem 4.3.1, is obtained in the same fashion as for the non-conserved model by using energy estimates and the Gagliardo-Nirenberg inequality.

Finally, in *Chapter 5* we investigate convergence to steady state for the abstract model (0.0.10) and use the ideas from this first part to prove convergence to steady state for the conserved phase field model (0.0.8)-(0.0.9). To achieve this, we construct appropriate Lyapunov functions and employ the Łojasiewicz-Simon inequality for the energy functional associated with the corresponding stationary problem. In the case of the phase field model considered, this inequality has already been verified and used in the literature, while in the case of the abstract model (0.0.10) it constitutes an assumption.

We point out that due to the presence of the convolution term(s), the finding of suitable Lyapunov functions in either case is a nontrivial task.

To describe how to tackle this problem, let us first consider the abstract models (0.0.1), and (0.0.2) of first and second order, respectively. For the equation (0.0.1), a canonical Lyapunov function is given by

$$\Upsilon_1(t) = \mathcal{E}(u(t)),$$

while in the case of the second order equation (0.0.2),

$$\Upsilon_2(t) = \frac{1}{2} \mid \dot{u}(t) \mid_H^2 + \mathcal{E}(u(t)),$$

is a Lyapunov function. In the first case, $\Upsilon_1(t)$ is good enough for the approach via Łojasiewicz-Simon inequality, whereas in the second case, one has to modify $\Upsilon_2(t)$, e.g. by adding the term of mixed type $\delta \langle \dot{u}, \mathcal{E}'(u) \rangle_H$, where $\delta > 0$ is chosen sufficiently small.

Now, to find an appropriate Lyapunov function for the problem

$$\dot{u} + \int_0^t a(t-s)\mathcal{E}'(u(s))ds = 0, \tag{0.0.16}$$

our first idea is to isolate the nonlinear term $\mathcal{E}'(\cdot)$. To this purpose, we assume that there exists a nonnegative, nonincreasing kernel k such that

$$\int_0^t a(t-s)k(s)ds = 1, \text{ for all } t > 0.$$

Then (0.0.16) can be written in equivalent form as

$$\frac{d}{dt}(k * \dot{u})(t) + \mathcal{E}'(u(t)) = 0, \tag{0.0.17}$$

where the symbol $*$ means the convolution of two functions supported in $\mathbb{R}_+$. Observe that (0.0.17) interpolates (0.0.1) and (0.0.2) in the sense that $k = 1$ leads to (0.0.1) while $k = \delta_0 + 1$, δ_0 denoting the Dirac delta, formally gives (0.0.2). The last observation suggests to consider sums $k = e + \gamma 1 * e$ with a constant $\gamma > 0$ and e positive, decreasing. In this case (0.0.17) becomes

$$\frac{d}{dt}(e * \dot{u})(t) + \gamma(e * \dot{u})(t) + \mathcal{E}'(u(t)) = 0, \tag{0.0.18}$$

and it turns out that indeed one can find a proper Lyapunov function for (0.0.18), namely

$$\Upsilon(t) = \frac{1}{2}(e* \mid \dot{u} \mid_H^2)(t) + \mathcal{E}(u(t)).$$

Similarly as in the case of second order, it is then possible to modify this function to produce a new Lyapunov function which combined with the Łojasiewicz-Simon inequality allows to prove convergence to single steady state in V for equation (0.0.18), provided that the range of the solution u is relatively compact in V. This result can be extended to the case where a function $f(t)$ appears on the right-hand side of equation (0.0.10). The assumption on such f to make this work is essentially the same as in Huang and Takáč [HT01].

The main results obtained in this chapter are Theorem 5.2.4 (abstract model) and Theorem 5.3.4 (phase field model).

Chapter 1

Mathematical Preliminaries

In this chapter we describe a general theoretical framework, which is necessary to understand this thesis. We begin by fixing some of the notations used throughout this thesis, recall some basic definitions and give references concerning function spaces.

By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ we denote the sets of natural numbers, integers, real and complex numbers respectively. Let further $\mathbb{R}_+ = [0,\infty)$, $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$. The capital letters X, Y, Z will usually stand for Banach spaces; $|\cdot|_X$ designates the norm of the Banach space X. Also, we denote by X' the topological dual space of X and by $(\cdot,\cdot)$ the duality relation. The norm in X' is denoted by $|\cdot|_{X'}$, and is defined by $|x'|_{X'} = \sup\{|(x',x)| : x \in X : |x|_X = 1\}$.

For a Hilbert space H we denote by $\langle\cdot,\cdot\rangle_H$ its scalar product. The symbol $B(X,Y)$ means the space of all bounded linear operators from X to Y, we write $B(X) = B(X,X)$ for short. If A is a linear operator in X, $D(A)$, $R(A)$, $N(A)$ stand for domain, range, and null space of A, respectively, while $\rho(A)$, $\sigma(A)$ designate resolvent set and spectrum of A. For a closed operator A we denote by D_A the domain of A equipped with the graph norm.

If (Ω,Σ,μ) is a measure space then $L_p(\Omega,\Sigma,\mu;X)$, $1 \leqslant p < \infty$, denotes the space of all Bochner-measurable functions $f : \Omega \to X$ such that $|f(\cdot)|^p$ is integrable. This space is also a well-known Banach space when endowed with the norm

$$|f|_p = \left(\int_\Omega |f(t)|^p \, d\mu(t)\right)^{1/p},$$

and functions equal a.e. are identified. Similarly, $L_\infty(\Omega,\Sigma,\mu;X)$ denotes the space of (equivalence classes of) Bochner-measurable essentially bounded functions $f : \Omega \to X$, and the norm is defined according to

$$| f |_\infty = \operatorname{ess\,sup}_{t\in\Omega} | f(t) | .$$

For $\Omega \subset \mathbb{R}^n$ open, Σ the Lebesgue σ-algebra, μ the Lebesgue measure, we abbreviate $L_p(\Omega, \Sigma, \mu; X)$ to $L_p(\Omega; X)$. In this case $W^m_p(\Omega; X)$ is the space of all functions $f : \Omega \to X$ having distributional derivatives $D^\alpha f \in L_p(\Omega; X)$ of order $| \alpha |\leqslant m$; the norm in $W^m_p(\Omega; X)$ is given by

$$| f |_{W^m_p(\Omega;X)} = \left(\sum_{|\alpha|\leqslant m} | D^\alpha f |^p_p \right)^{1/p} \quad \text{for } 1 \leqslant p < \infty,$$

and

$$| f |_{W^m_\infty(\Omega;X)} = \max_{|\alpha|\leqslant m} | D^\alpha f |_\infty \quad \text{for } p = \infty.$$

The spaces $W^m_p(\Omega; X)$ are the well-known Sobolev spaces. Further, we define the Bessel potential spaces $H^{sm}_p(\Omega; X)$, by means of complex interpolation, i.e.

$$H^{sm}_p(\Omega; X) = \left[L_p(\Omega; X), W^m_p(\Omega; X)\right]_s \quad \text{for } s \in (0, 1).$$

We will also set $H^{sm}_p(\Omega; X) = L_p(\Omega; X)$ if $s = 0$, and $H^{sm}_p(\Omega; X) = W^m_p(\Omega; X)$ if $s = 1$, and we denote by ${}_0H^s_p(\Omega; X)$ the completion of $C^\infty_0(\Omega; X)$ in $H^s_p(\Omega; X)$, where $C^\infty_0(\Omega; X)$ is the space of test functions (see [Tri92] for the scalar case and [Ama95] for the vector valued case).

As usual, $C^{(k+1)-}$ is the space of all C^k functions whose k^{th} derivative are locally Lipschitz continuous.

In the sequel we denote by $\hat{f}$ and $\tilde{f}$ the Laplace transform and the Fourier transform of a function f, respectively. The symbol $*$ means the convolution of two functions supported on the half line, i.e. $(a * b)(t) = \int_0^t a(t - s)b(s)ds$.

1.1 Sectorial operators

Definition 1.1.1. *Let* X *be a complex Banach space, and* A *be a closed linear operator in* X*. We say that* A *is sectorial if* $\overline{D(A)} = X$, $\overline{R(A)} = X$, $N(A) = \{0\}$, $(-\infty, 0) \subset \rho(A)$, *and*

$$| t(t + A)^{-1} |\leqslant M \;\; t > 0, \; \textit{for some constant } M < \infty.$$

We denote the class of sectorial operators in X by $S(X)$. Let further $\Sigma_\theta \subset \mathbb{C}$ stand for the open sector with vertex 0, opening angle 2θ, which is symmetric with respect to the positive halfaxis $\mathbb{R}_+$, i.e.

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : \;\; | \arg \lambda |< \theta\}.$$

If $A \in S(X)$ then $\rho(-A) \supset \Sigma_\theta$, for some $\theta > 0$ and $\sup\{|\lambda(\lambda + A)^{-1}| : |\arg\lambda| < \theta\} < \infty$. Therefore, we may define the spectral angle ϕ_A of $A \in S(X)$ by

$$\phi_A = \inf\{\phi : \rho(-A) \supset \Sigma_{\pi-\phi}, \sup_{\lambda \in \Sigma_{\pi-\phi}} |\lambda(\lambda + A)^{-1}| < \infty\}.$$

We consider some important subclasses of $S(X)$. A sectorial operator A in X is said to admit bounded imaginary powers, if $A^{is} \in \mathcal{B}(X)$ for each $s \in \mathbb{R}$ and there is a constant $C > 0$ such that $|A^{is}| \leqslant C$ for $|s| \leqslant 1$. The class of such operators will be denoted by $BIP(X)$ and we will call

$$\theta_A = \overline{\lim}_{|s|\to\infty} \frac{1}{|s|} \log |A^{is}|$$

the power angle of A. The class of operators that admit bounded imaginary powers was introduced by Prüss and Sohr in [PS90]. An important application of the class $BIP(X)$ concerns the fractional power spaces

$$X_\alpha = X_{A^\alpha} = (D(A^\alpha), |\cdot|_\alpha), \ |x|_\alpha = |x| + |A^\alpha x|, \ 0 < \alpha < 1,$$

where $A \in S(X)$. If A belongs to $BIP(X)$, a characterization of X_α in terms of complex interpolation spaces can be derived.

Theorem 1.1.2. *Assume that* $A \in BIP(X)$. *Then*

$$X_\alpha = [X, D_A]_\alpha, \alpha \in (0, 1),$$

the complex interpolation space between X *and* $D_A \hookrightarrow X$ *of order* α.

For a proof we refer to Triebel [Tri78, pp. 103-104], or Yagi [Yag84].

Recall that for $A \in S(X)$, $1 \leqslant p \leqslant \infty$, and $\gamma \in (0, 1)$, the real interpolation space $(X, D_A)_{\gamma,p}$ defined e.g. by the K-method, coincides with the space $D_A(\gamma, p)$ which is defined by means of

$$D_A(\gamma, p) := \{x \in X : [x]_{D_A(\gamma,p)} < \infty\},$$

where

$$[x]_{D_A(\gamma,p)} = \begin{cases} \left(\int_0^\infty [t^\gamma |A(t + A)^{-1}x|_X]^p d/dt\right)^{1/p}, & 1 \leqslant p < \infty \\ \sup_{t>0} t^\gamma |A(t + A)^{-1}x|_X, & p = \infty, \end{cases}$$

see e.g. [CGH00, Prop. 3].

For $\phi \in (0, \pi]$ we define the space of holomorphic functions on Σ_ϕ by $\mathcal{H}(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic}\}$, and

$$\mathcal{H}^\infty(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic and bounded}\}.$$

The space $\mathcal{H}^\infty(\Sigma_\phi)$ with norm $|\, f\,|_\infty^\phi = \sup\{|\, f(\lambda)\,| : |\arg\lambda| < \phi\}$ forms a Banach algebra. We also set $\mathcal{H}_0(\Sigma_\phi) := \cup_{\alpha,\beta<0}\mathcal{H}_{\alpha,\beta}(\Sigma_\phi)$, where $\mathcal{H}_{\alpha,\beta}(\Sigma_\phi) := \{f \in \mathcal{H}(\Sigma_\phi) \ : \ |\, f\,|_{\alpha,\beta}^\phi < \infty\}$, and

$$|\, f\,|_{\alpha,\beta}^\phi := \sup_{|\lambda|\leqslant 1} |\,\lambda^\alpha f(\lambda)\,| + \sup_{|\lambda|\geqslant 1} |\,\lambda^{-\beta} f(\lambda)\,|\,.$$

Given $A \in S(X)$, fix any $\phi \in (\phi_A, \pi]$ and let $\Gamma = (\infty, 0]e^{i\psi} \cup [0,\infty)e^{-i\psi}$ with $\phi_A < \psi < \phi$. Then

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - A)^{-1}d\lambda, \quad f \in \mathcal{H}_0(\Sigma_\phi),$$

defines via $\Phi_A(f) = f(A)$ a functional calculus $\Phi_A : \mathcal{H}_0(\Sigma_\phi) \to \mathcal{B}(X)$ which is a bounded algebra homomorphism. We say that a sectorial operator A admits a bounded $\mathcal{H}^\infty$-calculus if there are $\phi > \phi_A$ and a constant $K_\phi > 0$ such that

$$|\, f(A)\,| \leqslant K_\phi\ |\, f\,|_\infty^\phi, \text{ for all } f \in \mathcal{H}_0(\Sigma_\phi). \tag{1.1.1}$$

The class of sectorial operators A which admit an $\mathcal{H}^\infty$-calculus will be denoted by $\mathcal{H}^\infty(X)$ and the $\mathcal{H}^\infty$-angle of $A \in \mathcal{H}^\infty(X)$ is defined by

$$\phi_A^\infty = \inf\{\phi > \phi_A : \text{ (1.1.1) is valid }\}.$$

If $A \in \mathcal{H}^\infty(X)$, the functional calculus for A on $\mathcal{H}_0(\Sigma_\phi)$ extends uniquely to $\mathcal{H}^\infty(\Sigma_\phi)$. See [DHP03, Lemma 2.10].

We come now to $\mathcal{R}$-sectorial operators. Let X, Y be complex Banach spaces. We recall that a family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called $\mathcal{R}$-bounded, if there is a constant $C > 0$ and $p \in [1,\infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric $\{-1, 1\}$-valued random variables ε_j on a probability space $(\Sigma, \mathcal{M}, \mu)$ the inequality

$$\left|\sum_{j=1}^N \varepsilon_j T_j x_j\right|_{L_p(\Sigma;Y)} \leqslant C \left|\sum_{j=1}^N \varepsilon_j x_j\right|_{L_p(\Sigma;X)}$$

is valid. The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, we denote it by $\mathcal{R}(\mathcal{T})$. The concept of $\mathcal{R}$-bounded families of operators leads to the two important notions of $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus and $\mathcal{R}$-sectorial operators, replacing bounded with $\mathcal{R}$-bounded in the definitions of $\mathcal{H}^\infty$-calculus and sectorial operators.

Definition 1.1.3. *Let X be a Banach space and suppose that $A \in \mathcal{H}^\infty(X)$. The operator A is said to admit an $\mathcal{R}$-bounded $\mathcal{H}^\infty$- calculus if*

$$\mathcal{R}\left\{h(A) : \ h \in \mathcal{H}^\infty(\Sigma_\theta), \ \ |\, h\,|_\infty^\theta \leqslant 1\right\} < \infty$$

for some $\theta > 0$. We denote the class of such operators by $\mathcal{RH}^\infty(X)$ and define the $\mathcal{RH}^\infty$-angle $\phi_A^{\mathcal{R}_\infty}$ of A as the infimum of such angles θ.

Definition 1.1.4. *Let* X *be a complex Banach space, and assume that* A *is a sectorial operator in* X. *Then* A *is called* $\mathcal{R}$- *sectorial if*

$$\mathcal{R}_A(0) := \mathcal{R}\left\{t(t+A)^{-1} : \ t>0\right\} < \infty.$$

The $\mathcal{R}$-*angle* $\phi_A^{\mathcal{R}}$ *of* A *is defined by means of*

$$\phi_A^{\mathcal{R}} = \inf\{\theta \in (0,\pi) : \ \mathcal{R}_A(\pi-\theta) < \infty\},$$

where

$$\mathcal{R}_A(\theta) := \mathcal{R}\left\{\lambda(\lambda+A)^{-1} : \ |\arg\lambda| \leqslant \theta\right\}.$$

The class of $\mathcal{R}$-sectorial operators will be denoted by $\mathcal{RS}(X)$. The class of $\mathcal{R}$-sectorial operators was introduced by Clément and Prüss in [CP01], where the inclusion

$$\mathsf{BIP}(X) \subset \mathcal{RS}(X),$$

and the inequality

$$\phi_A^{\mathcal{R}} \leqslant \theta_A$$

were obtained in the special case, when the space X is such that the Hilbert transform defined by

$$(Hf)(t) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon\leqslant|s|\leqslant 1/\varepsilon} f(t-s)\frac{ds}{s}, \ t\in\mathbb{R},$$

is bounded in $L_p(\mathbb{R};X)$ for some $p\in(1,\infty)$. The class of spaces with this property will be denoted by $\mathcal{HT}$.

There is a well known theorem which says that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces, where UMD stands for unconditional martingale difference property. It is further known that $\mathcal{HT}$-spaces are reflexive. Every Hilbert space belongs to the class $\mathcal{HT}$, and if $(\Sigma,\mathcal{M},\mu)$ is a measure space and $X\in\mathcal{HT}$, then $L_p(\Sigma,\mathcal{M},\mu;X)$ is an $\mathcal{HT}$-space for $1<p<\infty$. For all of these results see the survey article by Burkholder [Bur86].

Summarizing, if X is a Banach space of class $\mathcal{HT}$ we have the inclusions

$$\mathcal{RH}^\infty(X) \subseteq \mathcal{H}^\infty(X) \subseteq \mathsf{BIP}(X) \subset \mathcal{RS}(X) \subseteq \mathcal{S}(X),$$

and the corresponding inequalities

$$\phi_A^{\mathcal{R}_\infty} \geqslant \phi_A^\infty \geqslant \theta_A \geqslant \phi_A^{\mathcal{R}} \geqslant \phi_A \geqslant \sup\{|\arg\lambda| : \ \lambda\in\sigma(A)\}.$$

For a detailed study of the mentioned topics, see for instance [DHP03], and also [DDH+04].

1.1.1 Examples

Volterra Operators

Definition 1.1.5. *Let* $a \in L_{1,loc}(\mathbb{R}_+)$ *be of subexponential growth and suppose* $\hat{a}(\lambda) \neq 0$ *for all* $\mathrm{Re}\,\lambda > 0$. a *is called* sectorial *with angle* $\theta > 0$ *(or merely* θ*-sectorial) if*

$$| \arg \hat{a}(\lambda) | \leqslant \theta$$

for all $\mathrm{Re}\lambda > 0$.

Definition 1.1.6. *Let* $a \in L_{1,loc}(\mathbb{R}_+)$ *be of subexponential growth and* $k \in \mathbb{N}$. $a(t)$ *is called* k-*regular, if there is a constant* $c > 0$ *such that*

$$| \lambda^n \hat{a}^{(n)}(\lambda) | \leqslant c \, | \hat{a}(\lambda) |$$

for all $\mathrm{Re}\lambda > 0$, *and* $1 \leqslant n \leqslant k$.

It is not difficult to see that convolutions of k-regular kernels are again k-regular. Furthermore, k-regularity is preserved by integration and differentiation, while sums and differences of k-regular kernels need not be k-regular. However, if a and b are k-regular and

$$| \arg \hat{a}(\lambda) - \arg \hat{b}(\lambda) | \leqslant \theta < \pi, \; \mathrm{Re}\lambda > 0, \tag{1.1.2}$$

then $a + b$ is k-regular as well, see [Prü93, p.70].

Some important properties of 1-regular kernels are contained in the following lemma.

Lemma 1.1.7. *Suppose* $a \in L_{1,loc}(\mathbb{R}_+)$ *is of subexponential growth and 1-regular. Then*

(i) $\hat{a}(i\rho) := \lim_{\lambda \to i\rho} \hat{a}(\lambda)$ *exists for each* $\rho \neq 0$;

(ii) $\hat{a}(\lambda) \neq 0$ *for each* $\mathrm{Re}\lambda \geqslant 0$;

(iii) $\hat{a}(i\cdot) \in W^{\infty}_{1,loc}(\mathbb{R} \setminus \{0\})$;

(iv) $| \rho \hat{a}'(i\rho) | \leqslant c \, | \hat{a}(i\rho) |$ *for a.a.* $\rho \in \mathbb{R}$;

(v) *there is a constant* $c > 0$ *such that*

$$c \, | \hat{a}(| \lambda |) | \leqslant | \hat{a}(\lambda) | \leqslant c^{-1} \, | \hat{a}(| \lambda |) |, \quad \mathrm{Re}\lambda \geqslant 0, \; \lambda \neq 0;$$

(vi) $\lim_{r \to \infty} \hat{a}(re^{i\phi}) = 0$ *uniformly for* $| \phi | \leqslant \frac{\pi}{2}$.

The following result expresses the fact that the inverse of an convolution operator associated with a 1-regular and sectorial kernel belongs to the class $\mathrm{BIP}(L_p(\mathbb{R}; X))$, for each Banach space X of class $\mathcal{HT}$, and $p \in (1, \infty)$.

Theorem 1.1.8 ([Prü93]). *Suppose* X *belongs to the class* $\mathcal{HT}$, $p \in (1,\infty)$, *and let* $a \in L_{1,loc}(\mathbb{R}_+)$ *be of subexponential growth. Assume that* a *is 1-regular and* θ*-sectorial, where* $\theta < \pi$. *Then there is a unique operator* $B \in S(L_p(\mathbb{R};X))$ *such that*

$$(Bf\tilde{)}(\rho) = \frac{1}{\hat{a}(i\rho)}\tilde{f}(\rho), \;\; \rho \in \mathbb{R}, \;\; \tilde{f} \in C_0^\infty(\mathbb{R} \setminus \{0\}; X). \tag{1.1.3}$$

Moreover, B *has the following properties:*

(i) B *commutes with the group of translations;*

(ii) $(\mu + B)^{-1} L_p(\mathbb{R}_+; X) \subset L_p(\mathbb{R}_+; X)$ *for each* $\mu > 0$, *i.e.* B *is causal;*

(iii) $B \in BIP(L_p(\mathbb{R};X))$, *and the power angle* $\theta_B = \theta_a$, *where* $\theta_a = \sup\{| \arg \hat{a}(\lambda) | : \; \mathrm{Re}\lambda > 0\}$*;*

(iv) $\sigma(B) = \overline{\{1/\hat{a}(i\rho) : \;\; \rho \in \mathbb{R} \setminus \{0\}\}}$.

The next result provides information about the domain of the operator B in Theorem 1.1.8.

Proposition 1.1.9 ([Prü93]). *Let the assumptions of Theorem 1.1.8 hold, let* B *be defined by* (1.1.3), *and let* $\alpha, \beta \geqslant 0$. *Then*

(i) $\limsup_{\mu \to \infty} | \hat{a}(\mu) | \mu^\alpha < \infty$ *implies* $D(B) \hookrightarrow H_p^\alpha(\mathbb{R};X)$*;*

(ii) $\liminf_{\mu \to \infty} | \hat{a}(\mu) | \mu^\beta > 0$ *and* $\liminf_{\mu \to 0} | \hat{a}(\mu) | > 0$ *imply* $H_p^\beta(\mathbb{R};X) \hookrightarrow D(B)$.

Elliptic Operators

Let E be a Banach space and $\mathcal{A}(\xi)$ denote a $B(E)$-valued polynomial on $\mathbb{R}^n$, which homogeneous of degree $m \in \mathbb{N}$, i.e.

$$\mathcal{A}(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha, \;\; \xi \in \mathbb{R}^n,$$

where we use multi-index notation, and $a_\alpha \in B(E)$.

Definition 1.1.10. *The* $B(E)$*-valued polynomial* $\mathcal{A}(\xi)$ *is called* parameter-elliptic *if there is an angle* $\phi \in [0,\pi)$ *such that the spectrum* $\sigma(\mathcal{A}(\xi))$ *of* $\mathcal{A}(\xi)$ *in* $B(E)$ *satisfies*

$$\sigma(\mathcal{A}(\xi)) \subset \Sigma_\phi \text{ for all } \xi \in \mathbb{R}^n, \;\; | \xi |= 1. \tag{1.1.4}$$

We then call

$$\phi_{\mathcal{A}} := \inf\{\phi : \;\; (1.1.4) \text{ holds}\} = \sup_{|\xi|=1} | \arg \sigma(\mathcal{A}(\xi)) |$$

angle of ellipticity of $\mathcal{A}$.

The following result shows that differential operators

$$\mathcal{A}(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha,$$

where $D := -i(\partial_1, ..., \partial_n)$, with parameter-elliptic symbols $\mathcal{A}(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ admit a bounded $\mathcal{H}^\infty$-calculus.

Theorem 1.1.11 ([DHP03]). *Let E be a Banach space of class $\mathcal{HT}$, $n, m \in \mathbb{N}$, and $1 < p < \infty$. Suppose $\mathcal{A}(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ with $a_\alpha \in B(E)$ is a homogeneous differential operator of order m whose symbol is parameter-elliptic with angle of ellipticity $\phi_{\mathcal{A}}$. Let A denotes its realization in $X = L_p(\mathbb{R}^n; E)$ with domain $D(A) = H_p^m(\mathbb{R}^n; E)$. Then $A \in \mathcal{H}^\infty(X)$ with $\mathcal{H}^\infty$-angle $\phi_A^\infty \leqslant \phi_{\mathcal{A}}$, in particular A is $\mathcal{R}$-sectorial with $\phi_A^{\mathcal{R}} \leqslant \phi_{\mathcal{A}}$.*

1.2 Operator-valued Fourier Multipliers

Let X be a Banach space and consider the spaces $L_p(\mathbb{R}; X)$ for $1 < p < \infty$. We denote by $\mathcal{D}(\mathbb{R}; X)$ the space of X-valued C^∞-functions with compact support and we let $\mathcal{D}'(\mathbb{R}; X) := \mathcal{B}(\mathcal{D}(\mathbb{R}); X)$ denote the space of X-valued distributions. The X-valued Schwartz space $\mathcal{S}(\mathbb{R}; X)$ and the space of X-valued temperate distributions $\mathcal{S}'(\mathbb{R}; X)$ are defined similarly. Let Y be another Banach space. Then, given $M \in L^1_{loc}(\mathbb{R}; \mathcal{B}(X, Y))$, we may define an operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}; X) \to \mathcal{S}'(\mathbb{R}; Y)$ by means of

$$T_M\phi := \mathcal{F}^{-1}M\mathcal{F}\phi, \text{ for all } \mathcal{F}\phi \in \mathcal{D}(\mathbb{R}; X), \tag{1.2.1}$$

where $\mathcal{F}$ denotes the Fourier transform. Since $\mathcal{F}^{-1}\mathcal{D}(\mathbb{R}; X)$ is dense in $L_p(\mathbb{R}; X)$, we see that T_M is well-defined and linear on a dense subset of $L_p(\mathbb{R}; X)$.

The following theorem, which is due to Weis [Wei01], contains the operator-valued version of the famous Mikhlin Fourier multiplier theorem in one variable.

Theorem 1.2.1. *Suppose X and Y are Banach spaces of class $\mathcal{HT}$ and let $1 < p < \infty$. Let $M \in C^1(\mathbb{R} \setminus \{0\}; B(X, Y))$ be such that the following conditions are satisfied.*

(i) $\mathcal{R}\{M(\rho) : \rho \in \mathbb{R} \setminus \{0\}\} := \kappa_0 < \infty$*;*

(ii) $\mathcal{R}\left\{\rho M'(\rho) : \rho \in \mathbb{R} \setminus \{0\}\right\} := \kappa_1 < \infty$.

Then the operator T_M defined by (1.2.1) *is bounded from $L_p(\mathbb{R}; X)$ into $L_p(\mathbb{R}; Y)$ with norm*

$$| T_M |_{B(L_p(\mathbb{R}; X); L_p(\mathbb{R}; Y))} \leqslant C(\kappa_0 + \kappa_1),$$

where $C > 0$ depends only on p, X, and Y.

A rather short and elegant proof of this theorem is given in [DHP03].

An important result due to Kalton and Weis [KW01], which can be applied together with Theorem 1.2.1, gives necessary conditions for the $\mathcal{R}$-boundedness of the symbol of the form $M(\rho, A)$, which is often encountered in applications.

Theorem 1.2.2. *Let* X *be a Banach space,* $A \in \mathcal{RH}^\infty(X)$ *and suppose that* $\{h_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{H}^\infty(\Sigma_\theta)$ *is uniformly bounded, for some* $\theta > \phi_A^{\mathcal{R}_\infty}$*, where* Λ *is an arbitrary index set. Then* $\{h_\lambda(A) : \lambda \in \Lambda\}$ *is* $\mathcal{R}$*-bounded.*

Actually, the strong condition $A \in \mathcal{RH}^\infty(X)$ in this result may not be easy to check in a general Banach space X. However, if $X = L_p$ with $1 < p < \infty$ then from Kalton and Weis [KW01, Thm. 5.3], it follows that

$$\mathcal{RH}^\infty(X) = \mathcal{H}^\infty(X) \text{ and } \phi_A^{\mathcal{R}_\infty} = \phi_A^\infty.$$

1.3 Sums of closed linear operators

The following result, which is an extension of the well-known Dore-Venni theorem [DV87], is due to Prüss-Sohr [PS90].

Theorem 1.3.1. *Suppose* X *belongs to the class* $\mathcal{HT}$*, and assume* $A, B \in BIP(X)$ *commute in the resolvent sense and satisfy the strong parabolicity condition* $\theta_A + \theta_B < \pi$*. Then*

(i) $A + B$ *is closed and sectorial;*

(ii) $A + B \in BIP(X)$ *with* $\theta_{A+B} \leqslant \max\{\theta_A, \theta_B\}$*;*

(iii) *there is a constant* $C > 0$ *such that*

$$| Ax | + | Bx | \leqslant C | Ax + Bx |, \; x \in D(A) \cap D(B).$$

In particular, if A *or* B *is invertible, then* $A + B$ *is invertible as well.*

The next result is known as the mixed derivative theorem and is due to Sobolevskii [Sob64].

Theorem 1.3.2. *Suppose* A*,* B *are sectorial operators in a Banach space* X*, commuting in the resolvent sense. Assume that their spectral angles satisfy the parabolicity condition* $\phi_A + \phi_B < \pi$*. Further suppose that* $A + \mu B$ *with natural domain* $D(A + \mu B) = D(A) \cap D(B)$ *is closed for each* $\mu > 0$ *and there is a constant* $M > 0$ *such that*

$$| Ax |_X + \mu | Bx |_X \leqslant M | Ax + \mu Bx |_X, \; \textit{for all } x \in D(A) \cap D(B), \; \mu > 0.$$

Then there exists a constant $C > 0$ *such that*

$$| A^{\alpha}B^{1-\alpha}x |_X \leqslant C | Ax + Bx |_X, \ \textit{for all } x \in D(A) \cap D(B), \ \alpha \in [0,1].$$

In particular, if A *or* B *is invertible, then* $A^{\alpha}B^{1-\alpha}(A+B)^{-1}$ *is bounded in* X*, for each* $\alpha \in [0,1]$.

1.4 Abstract parabolic Volterra equations

In this section, the basic theory of an parabolic Volterra equation is stated. This is done by making use of the monograph of Prüss [Prü93]. This section is divided into two parts. The first one is devoted to the concept of the resolvent, which is central for the theory of linear Volterra equations. In the second part, a recent result in the theory of maximal L_p-regularity for a parabolic Volterra equation due to Zacher [Zac05] is stated. Here we will also cite a result due to Clément and Prüss [CP90], which is very useful to obtain a-priori estimates.

We begin by giving the notions of solutions of abstract Volterra equations. Let X be a complex Banach space, A a closed linear in general unbounded operator in X with dense domain $D(A)$, and $a \in L^1_{loc}(\mathbb{R}_+)$ a scalar kernel. We consider the Volterra equation

$$u(t) + \int_0^t a(t-s)Au(s)ds = f(t), \ t \in J, \tag{1.4.1}$$

where $f \in C(J;X)$, $J = [0,T]$.

1.4.1 Resolvent families

Definition 1.4.1. *A family* $\{S(t)\}_{t\geqslant 0} \subset B(X)$ *of bounded linear operators in* X *is called a* resolvent *for* (1.4.1) *if the following conditions are satisfied.*

(S1) $S(t)$ *is strongly continuous on* $\mathbb{R}_+$ *and* $S(0) = I$*;*

(S2) $S(t)$ *commutes with* A*, which means that* $S(t)D(A) \subset D(A)$ *and* $AS(t)x = S(t)Ax$ *for all* $x \in D(A)$ *and* $t \geqslant 0$*;*

(S3) *the* resolvent equation *holds*

$$S(t)x = x + \int_0^t a(t-s)AS(s)xds, \ \textit{for all } x \in D(A), \ t \geqslant 0.$$

Suppose $S(t)$ is a resolvent for (1.4.1) and let $u(t)$ be a mild solution of (1.4.1). If we convole (1.4.1) with $S(t)$, then from (S1)-(S3), it follows that

$$1 * u = S * f,$$

i.e. $S * f$ is continuously differentiable and

$$u(t) = \frac{d}{dt}\int_0^t S(t-s)f(s)ds, \ t \in J. \tag{1.4.2}$$

This is the variation of parameters formula for Volterra equation (1.4.1).

Definition 1.4.2. *Equation* (1.4.1) *is called* parabolic, *if the following conditions hold.*

(P1) $\hat{a}(\lambda) \neq 0$, $1/\hat{a}(\lambda) \in \rho(A)$ *for all* $\operatorname{Re} \lambda > 0$.

(P2) *There is a constant* $M \geqslant 1$ *such that* $H(\lambda) = (I + \hat{a}(\lambda)A)^{-1}/\lambda$ *satisfies*

$$| H(\lambda) | \leqslant \frac{M}{| \lambda |}, \text{ for all } \operatorname{Re} \lambda > 0.$$

The notion of parabolicity yields to the following result Prüss [Prü93, Thm. 3.1].

Theorem 1.4.3. *Let* X *be a Banach space,* A *a closed linear operator in* X *with dense domain* $D(A)$, $a \in L^1_{loc}(\mathbb{R}_+)$. *Assume* (1.4.1) *is parabolic, and* $a(t)$ *is* k*-regular, for some* $k \geqslant 1$.
Then there is a resolvent $S \in C^{k-1}(\mathbb{R}_+; B(X))$ *for* (1.4.1), *and there is a constant* $M > 0$ *such that the estimates*

$$| t^n S^{(n)}(t) | \leqslant M, \qquad \text{for all } t \geqslant 0, \ n \leqslant k-1, \tag{1.4.3}$$

$$| t^k S^{(k-1)}(t) - s^k S^{(k-1)}(s) | \leqslant M \, | t-s | \, [1 + \log \frac{t}{t-s}], \qquad 0 \leqslant s < t < \infty, \tag{1.4.4}$$

are valid.

Remark 1.4.1. If $A \in S(X)$ with spectral angle $\phi_A < \pi$ and the kernel a is 1-regular and θ-sectorial with $\theta < \pi$, such that the condition of parabolicity $\theta + \phi_A < \pi$ holds, then there is a resolvent operator $S \in C((0, +\infty); B(X))$ for (1.4.1), which is also uniformly bounded in $\mathbb{R}_+$.

1.4.2 Maximal regularity in L_p

The following definition introduced by Zacher [Zac05] collects the notions of sectoriality, k-regularity and the conditions of Proposition 1.1.9.

Definition 1.4.4. *Let* $a \in L^1_{loc}(\mathbb{R}_+)$ *be of subexponential growth, and assume* $r \in \mathbb{N}$, $\theta_a > 0$, *and* $\alpha \geqslant 0$. *Then* a *is said to belong to the class* $\mathcal{K}^r(\alpha, \theta_a)$ *if*

($\mathcal{K}$1) a *is* r*-regular;*

($\mathcal{K}$2) a *is* θ_a*-sectorial;*

($\mathcal{K}3$) $\limsup_{\mu\to\infty} |\hat{a}(\mu)|\mu^{\alpha} < \infty$, $\liminf_{\mu\to\infty} |\hat{a}(\mu)|\mu^{\alpha} > 0$, $\liminf_{\mu\to 0} |\hat{a}(\mu)| > 0$.

Further, $\mathcal{K}^{\infty}(\alpha,\theta_a) := \{a \in L^1_{loc}(\mathbb{R}_+) \ : \ a \in \mathcal{K}^r(\alpha,\theta_a) \text{ for all } r \in \mathbb{N}\}$. *The kernel* a *is called a* $\mathcal{K}$*-kernel if there exist* $r \in \mathbb{N}$, $\theta_a > 0$, *and* $\alpha \geqslant 0$, *such that* $a \in \mathcal{K}^r(\alpha,\theta_a)$.

A typical example of a $\mathcal{K}$-kernel is given by

$$a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-\eta t}, \quad t > 0,$$

which belongs to the class $\mathcal{K}^{\infty}(\alpha, \alpha\frac{\pi}{2})$ for every $\alpha > 0$ and $\eta \geqslant 0$.
The concept of $\mathcal{K}$-kernels is very useful when working with Bessel potential spaces, since it connects the order of the kernels with the order of the Bessel potential spaces. The following result due to Zacher [Zac05] expresses this fact.

Corollary 1.4.5. *Let* X *be a Banach space of class* $\mathcal{HT}$, $p \in (1,\infty)$, *and* $J = [0,T]$ *or* $J = \mathbb{R}_+$. *Suppose* $a \in \mathcal{K}^1(\alpha,\theta)$ *with* $\theta < \pi$, *and assume in addition* $a \in L_1(\mathbb{R}_+)$ *in the case* $J = \mathbb{R}_+$. *Then the restriction* $\mathcal{B} := B|_{L_p(J;X)}$ *of the operator* B *constructed in Theorem 1.1.8 to* $L_p(J;X)$ *is well-defined. The operator* $\mathcal{B}$ *belongs to the class* $\mathcal{BIP}(L_p(J;X))$ *with power angle* $\theta_{\mathcal{B}} \leqslant \theta_B = \theta_a$ *and is invertible satisfying* $\mathcal{B}^{-1}w = a * w$ *for all* $w \in L_p(J;X)$. *Moreover* $D(\mathcal{B}) = {}_0H^{\alpha}_p(J;X)$.

The next result gives necessary and sufficient conditions for the existence of a unique solution u of (1.4.1) in the space

$$H^{\alpha+\kappa}_p(J;X) \cap H^{\kappa}_p(J;D_A).$$

Theorem 1.4.6 (Zacher, [Zac05]). *Let* X *be a Banach space of class* $\mathcal{HT}$, $p \in (1,\infty)$, $J = [0,T]$ *or* $\mathbb{R}_+$, *and* A *an* $\mathcal{R}$*-sectorial operator in* X *with* $\mathcal{R}$*-angle* $\phi^{\mathcal{R}}_A$. *Suppose that* a *belongs to* $\mathcal{K}^1(\alpha,\theta_a)$ *with* $\alpha \in (0,2)$ *and that in addition* $a \in L_1(\mathbb{R}_+)$ *in the case* $J = \mathbb{R}_+$. *Further let* $\kappa \in [0,1/p)$ *and* $\alpha+\kappa \neq \{1/p, 1+1/p\}$. *Assume the parabolicity condition* $\theta_a + \phi^{\mathcal{R}}_A < \pi$. *Then (1.4.1) has a unique solution in* $H^{\alpha+\kappa}_p(J;X) \cap H^{\kappa}_p(J;D_A)$ *if only if the function* f *satisfies the subsequent conditions:*

(i) $f \in H^{\alpha+\kappa}_p(J;X)$;

(ii) $f(0) \in D_A(1+\frac{\kappa}{\alpha}-\frac{1}{p\alpha}, p)$, *if* $\alpha+\kappa > 1/p$;

(iii) $\dot{f}(0) \in D_A(1+\frac{\kappa}{\alpha}-\frac{1}{\alpha}-\frac{1}{p\alpha}, p)$, *if* $\alpha+\kappa > 1+1/p$.

The next result is due to Clément and Prüss [CP90].

Theorem 1.4.7. *Let* X *be a Banach space,* $1 \leqslant p < \infty$*,* $\nu \in L_{1,loc}(\mathbb{R}_+)$ *nonnegative, nonincreasing, and let* B_p *be defined in* $L_p(\mathbb{R}_+;X)$ *by*

$$(B_p u)(t) = \frac{d}{dt}(\nu * u)(t), \quad t \geqslant 0, \quad u \in D(B_p),$$

with domain

$$D(B_p) = \{u \in L_p(\mathbb{R}_+;X) : \ \nu * u \in \ {}_0W^1_p(\mathbb{R}_+;X)\}.$$

Then B_p *is* m*-accretive. In particular, if* $X = H$ *is a Hilbert space, then*

$$\int_0^T \langle B_p u(t), u(t)\rangle \mid u \mid_H^{p-2} dt \geqslant 0, \quad T > 0,$$

for each $u \in D(B_p)$.

Remark 1.4.2. Let a be a 1-regular and θ-sectorial kernel with $\theta < \pi$. Let B be the operator from Proposition 1.1.9 associated with a, and assume that there exists $\nu \in L_{1,loc}(\mathbb{R}_+)$ nonnegative, and nonincreasing, such that $a * \nu = 1$. Then from Theorem 1.4.7, it follows that $(Bu)(t) = (B_p u)(t) = \frac{d}{dt}\nu * u(t)$, for each $u \in D(B) \cap D(B_p)$. In particular for $p = 2$ and $D(B) = \ {}_0H^\alpha_2(J, L_2(\Omega))$, it follows that

$$\int_0^T \langle Bu, u\rangle \, dt = \int_0^T \left\langle \frac{d}{dt}\nu * u, u \right\rangle dt \geqslant 0.$$

Chapter 2

Physical background

2.1 Heat conduction

In this section we discuss a mathematical model for the process of heat conduction in materials with memory. We begin our discussion with a constitutive relation between the heat flux $\mathbf{q}$ and the temperature u. A simple relation for it is given by

$$\mathbf{q} = -\lambda \nabla u, \tag{2.1.1}$$

where $\lambda > 0$. (2.1.1) is well-know as Fourier's law for the heat flux. Assuming that $u_t = -\mathrm{div}\mathbf{q}$ then it follows from (2.1.1) the diffusion equation

$$u_t - \lambda \Delta u = 0, \tag{2.1.2}$$

where λ is the thermal diffusivity. The diffusion equation has the unphysical property that if a sudden change of temperature is made at some point on the body, it will be felt instantly everywhere, though with exponentially small amplitudes at distant points, i.e. the diffusion gives rise to infinite speeds of propagations. The problem of infinite speeds of propagation generated by diffusion were first discussed in the work of Cattaneo [Cat49]. Later [Cat58] proposed the equation

$$\tau \frac{d}{dt}\mathbf{q} + \mathbf{q} = -\lambda \nabla u \tag{2.1.3}$$

for the heat flux (see also Maxwell [Max67]), where $\tau > 0$. From (2.1.3), we obtain a telegraph equation

$$u_{tt} + \frac{1}{\tau} u_t = r^2 \Delta u, \tag{2.1.4}$$

with $r^2 = \lambda/\tau$. Equation (2.1.4) is hyperbolic and it transmits waves of temperature with speed r. The waves are attenuated as a result of relaxation, and steady heat flow may be induced by temperature gradients. Equation (2.1.3) can be expressed as an integral over the history of the temperature gradient,

$$\mathbf{q}(t,x) = -\frac{\lambda}{\tau}\int_{-\infty}^{t} \exp\left(-\frac{t-s}{\tau}\right) \nabla u(s,x)ds.$$

A more general form for the heat flux is

$$\mathbf{q}(t,x) = -\int_{-\infty}^{t} a(t-s)\nabla u(s,x)ds, \tag{2.1.5}$$

where $a(t)$ is a positive, decreasing relaxation function that tends to zero as $t \to \infty$. Integral expressions like (2.1.5) are also used in Boltzmann's theory of linear viscoelasticity to express the present value of the stress in term of past values of the strain or strain of rate (see Boltzmann [Bol76], Maxwell [Max67], and Volterra [Vol09a], [Vol09b], for the early history of linear viscoelasticity).

Using Cattaneo-Maxwell's equation (2.1.3) and the works of Coleman and collaborators [CN60, Col64, CM66, CG67], Gurtin and Pipkin [GC68] give a general constitutive theory for rigid heat conductors that propagate waves. They consider after linearization, the expression for the internal energy and the heat flux $\mathbf{q}$ as follows

$$e(t,x) = c + \nu u(t,x) + \int_{-\infty}^{t} b(t-s)u(s,x)ds \tag{2.1.6}$$

and

$$\mathbf{q}(t,x) = -\int_{-\infty}^{t} a(t-s)\nabla u(s,x)ds, \tag{2.1.7}$$

where $\nu \neq 0$, $a(t)$ and $b(t)$ are positve, decreasing relaxation functions that tend to zero as $t \to \infty$. By Coleman and Gurtin [CG67] we could also consider the heat flux as a perturbation of Fourier's law, that is

$$\mathbf{q}(t,x) = -\gamma\nabla u - \int_{-\infty}^{t} a(t-s)\nabla u(s,x)ds, \tag{2.1.8}$$

where γ is a positive constant, which represent an instantaneous conductivity of heat. Equations (2.1.6) and (2.1.8) yield the heat equation with memory:

$$\nu\dot{u} + \int_{-\infty}^{t} b(t-s)\dot{u}(s)ds = \gamma\Delta u + \int_{-\infty}^{t} a(t-s)\Delta u(s)ds. \tag{2.1.9}$$

The prototype of relaxation functions, that we consider throughout this work, is given by

$$a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}e^{-\beta t}, \text{ and } b(t) = 0, \ t > 0,$$

where $\alpha > 0$ and $\beta \geqslant 0$. Observe that some of these kernels enjoy the property of having a fast and slow relaxation (e.g. if $\alpha < 1$). The fast relaxation at time t near to zero corresponds to an instantaneous thermal conductivity. We refer the reader to [JP89, JP90, JCVL96] for a modern discussion of these topics.

2.2 Phase field systems with memory

In this section we discuss a non-conserved as well as conserved phase field model with memory. For the non-conserved phase field model with memory reads as

$$u_t + \frac{l}{2}\phi_t = \int_0^t a_1(t-s)\Delta u(s)ds + f_1, \qquad \text{in } J \times \Omega; \tag{2.2.1}$$

$$\tau\phi_t = \int_0^t a_2(t-s)\left[\xi^2\Delta\phi + \frac{\phi - \phi^3}{\eta} + u\right] ds + f_2, \qquad \text{in } J \times \Omega; \tag{2.2.2}$$

$$\partial_n u = \partial_n \phi = 0, \qquad \text{on } J \times \partial\Omega;$$

$$u(0,x) = u_0(x), \ \phi(0,x) = \phi_0(x), \qquad \text{in } \Omega,$$

where

$$f_1(t,x) = \int_{-\infty}^0 a_1(t-s)\Delta u(s,x)ds, \qquad (t,x) \in J \times \Omega;$$

$$f_2(t,x) = \int_{-\infty}^0 a_2(t-s)\left[\xi^2\Delta\phi + \frac{\phi - \phi^3}{\eta} + u\right](s,x)ds, \qquad (t,x) \in J \times \Omega,$$

contains the history of the system; we refer to [RBNCN01] for the physical background.

2.2.1 Conserved model

We denote by ϕ the concentration of one of the two components in the alloy, and by $\mathbf{j}$ the concentration flux. The corresponding physical law at constant temperature u is given by

$$\tau\phi_t = -\text{div } \mathbf{j}. \tag{2.2.3}$$

Classical theory assumes $\mathbf{j}$ to be proportional to the gradient of the local chemical potential μ, i.e.,

$$\mathbf{j} = -\xi^2\nabla\mu. \tag{2.2.4}$$

The free-energy $\mathcal{F}_u$ at constant temperature u is assumed to be given by an expression of the form

$$\mathcal{F}_u(\phi) = \int_\Omega \left[\frac{\xi^2}{2} \mid \nabla\phi \mid^2 + \Phi(\phi) - \rho u\phi - \frac{1}{2}u^2\right] dx,$$

where $\rho \geqslant 0$ denotes an entropy coefficient (see Caginalp-Fife [CF88]) and the term $-\rho u \phi$ corresponds to the entropic contribution to the free-energy, due to the difference in the entropy densities of the two components of the alloy. The functional derivative of $\mathcal{F}_u$ with respect to ϕ is then given by

$$\frac{\delta \mathcal{F}_u}{\delta \phi} = -\xi^2 \Delta \phi + \Phi'(\phi) - \rho u.$$

By Cahn-Hilliard [CH58], it follows that

$$\mu \equiv \frac{\delta \mathcal{F}_u}{\delta \phi} = -\xi^2 \Delta \phi + \Phi'(\phi) - \rho u. \tag{2.2.5}$$

So, at time t, μ is completely determined by the concentration ϕ and temperature u. In the isothermal case equations (2.2.3)-(2.2.5) yield the standard Cahn-Hilliard equation

$$\phi_t = \xi^2 \Delta \left(-\xi^2 \Delta \phi + \Phi'(\phi) \right),$$

where $\Phi'(\phi) = k(\phi^3 - \phi)$, which represents a double-well potential.

If we assume that the temperature also varies in time and space (that is $u = u(t, x)$), then the internal energy e of the system is given by

$$e = -\frac{\delta \mathcal{F}_u}{\delta u},$$

where the presence of ϕ is due to the fact that it may also be considered as a form of energy. From the energy equation it follows that

$$u_t + \rho \phi_t = -\text{div } \mathbf{q}, \tag{2.2.6}$$

where $\mathbf{q}$ is the heat flux in the alloy.

Equations (2.2.3)-(2.2.6) yield the non-isothermal Cahn-Hilliard equation

$$\begin{aligned} u_t + \rho \phi_t &= -\text{div } \mathbf{q} \\ \phi_t &= \xi^2 \Delta \left(-\xi^2 \Delta \phi + \Phi'(\phi) - \rho u \right). \end{aligned}$$

Using the argument given in [RBNCN01], the relaxed chemical potential can be written as

$$\mu \mid_{rel} = \int_{-\infty}^{t} a_2(t-s) \frac{\delta \mathcal{F}_u}{\delta \phi}(s) ds,$$

where a_2 denotes a history kernel. If we assume that μ contains only a relaxing chemical potential $\mu \mid_{rel}$ and $a_2(0)$ is bounded, then there is no instantaneous contribution from the history of the system to the chemical potential to $\mu \mid_{rel} (0)$. This can be avoided by considering relaxation functions of the form

$$a_2(t) = \frac{t^{\alpha_2 - 1}}{\Gamma(\alpha_2)} e^{-\beta t}, \ t > 0,$$

where $\alpha_2 > 0$ and $\beta \geqslant 0$. This way, for $\alpha_2 < 1$ we have a fast and a slow relaxation. The fast relaxation near $t = 0+$ responses to an instantaneous contribution of the concentration history. Finally, equations (2.2.3) and (2.2.4) yield

$$\tau\phi_t = \xi^2 \Delta\mu \,|_{rel}\, . \tag{2.2.7}$$

Finally, if the alloy is contained in a region $\Omega \subset \mathbb{R}^n$ equation (2.2.7) should be supplemented with boundary conditions on the boundary $\partial\Omega$. These are usually of the form

$$\partial_n\phi = \partial_n\mu \,|_{rel}= 0, \tag{2.2.8}$$

where ∂_n means the normal derivative at $\partial\Omega$. The physical meaning of the second of these two conditions is that none of the mixture can pass through the wall of the container, while the first means a neutral wall, which does not interact with the substances. In addition, a usual boundary condition for u is given by

$$\partial_n e = \partial_n u = 0, \tag{2.2.9}$$

which means an insulated wall.

Since $\partial_n\Phi'(\phi) = \Phi''(\phi)\partial_n\phi = 0$, the boundary conditions (2.2.8) and (2.2.9) take the equivalent form

$$\partial_n u = \partial_n\phi = \partial_n(\Delta\phi) = 0.$$

With these boundary conditions, equation (2.2.7) truly ensures conservation of mass and energy, as can be seen by the divergence theorem, integrating (2.2.3) and (2.2.6) over Ω

Now we can write the equations of the conserved model

$$\begin{aligned}
&u_t + \rho\phi_t = \gamma\Delta u + \int_{-\infty}^{t} a_1(t-s)\Delta u(s)ds, && \text{in } J\times\Omega; && (2.2.10)\\
&\tau\phi_t = -\xi^2 \int_{-\infty}^{t} a_2(t-s)\Delta\left[\xi^2\Delta\phi - \Phi'(\phi) + \rho u\right](s)ds, && \text{in } J\times\Omega; && (2.2.11)\\
&\partial_n u = \partial_n\phi = \partial_n(\Delta\phi) = 0, && \text{on } J\times\partial\Omega; &&\\
&u(0,x) = u_0(x),\;\; \phi(0,x) = \phi_0(x), && \text{in } \Omega. &&
\end{aligned}$$

Here J is an interval of the form $[0,T]$ with $T > 0$, and Ω is a smooth bounded domain in $\mathbb{R}^n$. The constants ρ, τ, and ξ are all positive and represent the latent heat, a relaxation time, and a correlation length, respectively. The nonlinearity $\Phi : \mathbb{R} \to \mathbb{R}$ is a given potential, which satisfies certain growth conditions. In particular, Φ can be the double-well potential $\Phi(s) = k(s^2-1)^2$ $(k > 0)$, which is considered frequently in the literature. The kernels a_1 and a_2 are scalar kernels, which satisfy properties discussed bellow.

In the sequel, we will assume w.l.o.g. that all constants in the models (2.2.1)-(2.2.2) and (2.2.10)-(2.2.11) are equal to one.

Chapter 3

A non-conserved phase field model

In this chapter we obtain the global well-posedness in the strong sense in the L_p-setting for a phase field model with memory

$$u_t + \phi_t = \int_0^t a_1(t-s)\Delta u(s)ds + f_1, \quad \text{in } J \times \Omega; \tag{3.0.1}$$

$$\phi_t = \int_0^t a_2(t-s)\left[\Delta\phi + \phi - \phi^3 + u\right] ds + f_2, \quad \text{in } J \times \Omega; \tag{3.0.2}$$

$$\partial_n u = \partial_n \phi = 0, \quad \text{on } J \times \partial\Omega; \tag{3.0.3}$$

$$u(0,x) = u_0(x), \ \phi(0,x) = \phi_0(x), \quad \text{in } \Omega, \tag{3.0.4}$$

where

$$f_1(t,x) = \int_{-\infty}^0 a_1(t-s)\Delta u(s,x)ds, \quad (t,x) \in J \times \Omega; \tag{3.0.5}$$

$$f_2(t,x) = \int_{-\infty}^0 a_2(t-s)\left[\Delta\phi + \phi - \phi^3 + u\right](s,x)ds, \quad (t,x) \in J \times \Omega, \tag{3.0.6}$$

$J = [0,T]$ is an interval on $\mathbb{R}$, and Ω a smooth bounded domain in $\mathbb{R}^n$.

3.1 Local well-posedness

This section is devoted to the local well-posedness of (3.0.1)-(3.0.4). To achieve this, we will reduce the system (3.0.1)-(3.0.4) to a semilinear equation of Volterra type. Our strategy to solve this semilinear equation consists of two steps. Firstly we solve the linear version of it using maximal regularity tools (Theorem 1.4.6), and secondly we apply the contraction principle to solve nonlinear problem by means of linearization and results from first step and the contraction mapping principle.

We would like to begin with some definitions. Let $T > 0$ be given and fixed and let Ω be a smooth bounded domain in $\mathbb{R}^n$. For $0 < \delta \leqslant T$ and $1 < p < \infty$, we define the spaces

$$\begin{aligned} Z(\delta) &= H_p^{\alpha+\kappa}([0,\delta];X) \cap H_p^{\kappa}([0,\delta];D_A); \\ Z_i(\delta) &= H_p^{1+\alpha_i+\kappa_i}([0,\delta];X) \cap H_p^{\kappa_i}([0,\delta];D_A); \\ \tilde{X}_i(\delta) &= H_p^{\alpha_i+\kappa_i}([0,\delta];X); \\ X_i(\delta) &= H_p^{1+\alpha_i+\kappa_i}([0,\delta];X), \end{aligned}$$

for $i = 1, 2$, where $\alpha,\ \alpha_i > 0$, and $\kappa,\ \kappa_i \geqslant 0$, and $X := L_p(\Omega)$, and A is a closed linear operator in X with dense domain $D(A)$. The spaces ${}_0Z(\delta)$ and ${}_0Z_i(\delta)$ denote the corresponding spaces $Z(\delta)$ and $Z_i(\delta)$ resp., with zero trace at $t = 0$. A similar definition holds for ${}_0\tilde{X}_i(\delta)$ and ${}_0X_i(\delta)$. Whenever no confusion may arise, we shall simply write Z, Z_i, etc., resp. ${}_0Z$, ${}_0Z_i$, etc. if $\delta = T$. Furthermore, in case that $\kappa_i \in [0, 1/p)$ and $\alpha_i + \kappa_i \neq 1/p$, we define the natural phase spaces for Z_i by

$$\begin{aligned} Y_p^i &= (X; D_A)_{\gamma_i,p}, \quad \text{with } \gamma_i = 1 + \frac{\kappa_i}{1+\alpha_i} - \frac{1}{p(1+\alpha_i)}, \quad \text{for } i = 1, 2; \\ \widetilde{Y}_p^i &= (X; D_A)_{\sigma_i,p}, \quad \text{with } \sigma_i = 1 + \frac{\kappa_i}{1+\alpha_i} - \frac{1}{1+\alpha_i} - \frac{1}{p(1+\alpha_i)}, \quad \text{for } i = 1, 2. \end{aligned}$$

Let $J = [0, T]$ be an interval on $\mathbb{R}$, and let Ω be a smooth bounded domain in $\mathbb{R}^n$. We consider the system

$$u_t + \phi_t = a_1 * \Delta u + f_1, \quad \text{in } J \times \Omega; \tag{3.1.1}$$

$$\phi_t = a_2 * \Delta\phi + a_2 * (\phi - \phi^3) + a_2 * u + f_2, \quad \text{in } J \times \Omega; \tag{3.1.2}$$

$$\partial_n u = \partial_n \phi = 0, \quad \text{on } J \times \partial\Omega; \tag{3.1.3}$$

$$u(0,x) = u_0(x),\ \phi(0,x) = \phi_0(x), \quad \text{in } \Omega, \tag{3.1.4}$$

where f_1 and f_2 are as in (3.0.5)-(3.0.6).

For the discussion of equations (3.1.1)-(3.1.4), we will assume that the kernels a_i belong to $\mathcal{K}^1(\alpha_i, \theta_i)$, with $\theta_i \in (0, \frac{\pi}{2})$ and $\alpha_i \in (0,1)$ for $i = 1, 2$, and we will set $A = -\Delta$ equipped with Neumann boundary condition in X.

If we consider ϕ as known then equation (3.1.1) is equivalent to the two problems

$$\text{(I)} \begin{cases} u_t^* = -a_1 * Au^* + f_1, & \text{in} \quad J \times \Omega; \\ u^*(0) = u_0, & \text{in} \quad \Omega, \end{cases}$$

and

$$(\mathrm{II})\begin{cases} w_t = -a_1 * Aw - \phi_t, & \text{in} \quad J \times \Omega; \\ w(0) = 0, & \text{in} \quad \Omega, \end{cases}$$

by means of the relation $u = u^* + w$. Observe that Theorem 1.4.6 gives necessary and sufficient conditions to obtain a strong solution of (I) and also for (II). Indeed, integrating the equation (I) over $[0, t]$, we have

$$u^* = -1 * a_1 * Au^* + 1 * f_1 + u_0.$$

It is easy to show that $a := 1 * a_1$ is a kernel that belongs to the class $\mathcal{K}^1(1+\alpha_1, \theta_1 + \frac{\pi}{2})$. In addition, it is well-known that $A = -\Delta$ with Dirichlet- or Neumann- or Robin-boundary conditions belongs to the class $\mathsf{BIP}(X)$ with power angle $\theta_A = 0$. Moreover, from [CP01] it follows that $A \in \mathcal{RS}(X)$ too, with $\mathcal{R}$-angle $\phi_A^{\mathcal{R}} = 0$. Hence, (I) transforms into the equation (1.4.1), with $f = 1 * f_1 + u_0$. Therefore, we may apply Theorem 1.4.6. A similar argument holds for (II).

Now we want to have a representation formula for the mild solution of (II). For this, we take $f = -1 * \phi_t$ and $a = 1 * a_1$ in (1.4.1). On the other hand, since $A \in S(X)$ with spectral angle $\phi_A = 0$, it follows from Remark 1.4.1 that (1.4.1) admits a resolvent operator S. Using this fact and the variation of parameters formula, it follows that the mild solution w of equation (II) can be represented as

$$w = \frac{d}{dt}(-S * 1 * \phi_t) = -S * \phi_t. \tag{3.1.5}$$

Now substituting $u = u^* + w$ in (3.1.2) and using (3.1.5) it follows that

$$\phi_t = -a_2 * A\phi + a_2 * (\phi - \phi^3) + a_2 * u^* - a_2 * S * \phi_t + f_2, \quad \text{in} \quad J \times \Omega. \tag{3.1.6}$$

Defining

$$g(t) = 1 * a_2 * u^* + 1 * f_2 + \phi_0 \text{ and } H(\phi) = 1 * a_2 * (\phi - \phi^3) - 1 * a_2 * S * \phi_t$$

then (3.1.6) can be rewritten as

$$\phi = -1 * a_2 * A\phi + H(\phi) + g(t). \tag{3.1.7}$$

Now we will establish the equivalence between system (3.1.1)-(3.1.4) and equation (3.1.7). To do so, we will first assume that the functions in (3.1.1)-(3.1.4) and (3.1.7) enjoy enough regularity (later, we will make precise this aspect).

We begin assuming that u^* as well as ϕ are known in (I) and (3.1.7), respectively. Using ϕ in equation (II) we obtain a function w, and by defining a new function $u = u^* + w$ one can show (after an easy computation) that the pair (u, ϕ) is a solution of (3.1.1)-(3.1.4). The converse direction is trivial.

We will now make precise the type of regularity which we will give to the solutions.

A natural choice for the regularity class of the solution (u, ϕ) of (3.1.1)-(3.1.4) is delivered by Theorem 1.4.6, therefore we can assume that (u, ϕ) belongs to $Z_1 \times Z_2$. In addition, by applying the contraction mapping principle, we see that the solution ϕ of (3.1.7) belongs to Z_2, if and only if $H(\phi) + g(t) \in X_2$. From Corollary 1.4.5 we have that for each function $u^* \in L_p(J; X)$ (in particular in Z_1) the function $1 * a_2 * u^*$ is in ${}_0X_2$, hence $g \in X_2$, provided that $u^* \in L_p(J; X)$ and $1 * f_2 + \phi_0 \in X_2$.

From equation (II) and Theorem 1.4.6, it follows that the solution w of (II) belongs to ${}_0Z_2$. Since $u = u^* + w$ is a solution of (3.1.1), we have $u \in Z_1$. On the other hand, since $u^* \in Z_1$ and $w \in Z_2$, we have to impose a condition which relates the spaces Z_1 and Z_2. In fact, the embedding $Z_2 \hookrightarrow Z_1$ is an admissible condition, which is equivalent to

$$\alpha_2 - \alpha_1 \geqslant \kappa_1 - \kappa_2 \text{ and } \kappa_2 \geqslant \kappa_1. \tag{3.1.8}$$

The following auxiliary results are needed to estimate the nonlinear term $H(\phi)$ in equation (3.1.7) in X_2. To this purpose we begin with an estimate for products of functions in Bessel potential spaces.

Lemma 3.1.1. *Let* $0 \leqslant \kappa < 1$, $\alpha > 0$, $n \in \mathbb{N}$. *Suppose that* $p > \frac{n}{3} + \frac{2}{3\alpha}$. *Then there is a constant* $C > 0$ *and an* $\varepsilon > 0$ *such that*

$$| uvw |_{H_p^{\kappa+\varepsilon}(L_p)} \leqslant C \, | u |_Z | v |_Z | w |_Z \tag{3.1.9}$$

is valid for all $u, v, w \in Z$.

Proof. Let $\rho_i > 1$ for $i = 1, ..., 4$ such that

$$1 = \frac{1}{\rho_1} + \frac{2}{\rho_3} = \frac{1}{\rho_2} + \frac{2}{\rho_4},$$

which in particular mean that ρ_3 and ρ_4 are greater than 2. Let $\varepsilon > 0$ such that $0 < \kappa + \varepsilon < 1$, then from the characterization of $H_p^{\kappa+\varepsilon}$ via differences (see [Tri92]) and with the aid of Hölder's inequality, it follows that

$$| uvw |_{H_p^{\kappa+\varepsilon}(L_p)} \leqslant C \, | u |_{H_{p\rho_1}^{\kappa+\varepsilon}(L_{p\rho_2})} | v |_{H_{p\rho_3}^{\kappa+\varepsilon}(L_{p\rho_4})} | w |_{H_{p\rho_3}^{\kappa+\varepsilon}(L_{p\rho_4})} \,. \tag{3.1.10}$$

Observe that (3.1.10) is valid for $\kappa = \varepsilon = 0$ too.

On the other hand, the mixed derivative theorem yields

$$Z \hookrightarrow H_p^{(1-\theta)\alpha+\kappa}(H_p^{2\theta}).$$

Then for completion of the proof, we have to check the validity of the Sobolev embeddings

$$H_p^{(1-\theta)\alpha+\kappa}(H_p^{2\theta}) \hookrightarrow H_{p\rho_1}^{\kappa+\varepsilon}(L_{p\rho_2}) \text{ and } H_p^{(1-\theta)\alpha+\kappa}(H_p^{2\theta}) \hookrightarrow H_{p\rho_3}^{\kappa+\varepsilon}(L_{p\rho_4}).$$

Is easy to verify that the first embedding is valid for some $\theta \in (0,1)$, provided

$$p \geqslant \frac{\alpha n}{2(\alpha-\varepsilon)}\left(1-\frac{1}{\rho_2}\right)+\frac{1}{\alpha-\varepsilon}\left(1-\frac{1}{\rho_1}\right)=\frac{\alpha n}{2(\alpha-\varepsilon)}\left(\frac{2}{\rho_4}\right)+\frac{1}{\alpha-\varepsilon}\left(\frac{2}{\rho_3}\right) \tag{3.1.11}$$

and the second one is valid for some $\theta \in (0,1)$, provided

$$p \geqslant \frac{\alpha n}{2(\alpha-\varepsilon)}\left(1-\frac{1}{\rho_4}\right)+\frac{1}{\alpha-\varepsilon}\left(1-\frac{1}{\rho_3}\right). \tag{3.1.12}$$

Taking $\rho_3 = \rho_4 = 3$, (3.1.11) and (3.1.12) are equivalent to

$$p \geqslant \frac{\alpha n}{3(\alpha-\varepsilon)}+\frac{2}{3(\alpha-\varepsilon)}.$$

Then the claim follows from the strict inequality

$$\frac{\alpha n}{3(\alpha-\varepsilon)}+\frac{2}{3(\alpha-\varepsilon)} > \frac{n}{3}+\frac{2}{3\alpha},$$

since $\varepsilon > 0$. □

Lemma 3.1.2. *Let X be a Banach space of class $\mathcal{HT}$, and let $J = [0,T]$, $T > 0$. Further let $b \in \mathcal{K}^1(\beta,\theta)$, $\beta > 1$, $\theta < \pi$. Assume that the constants $\kappa \geqslant 0$ and $\varepsilon \in (0,1)$ are given and suppose further that $1 < \beta+\kappa < 2$. Then for all $u \in H_p^{\kappa+\varepsilon}(J;X)$ there is a constant $c(T) > 0$, such that*

$$|\, b * u \,|_{{}_0H_p^{\beta+\kappa}(J;X)} \leqslant c(T)\,|\, u \,|_{H_p^{\kappa+\varepsilon}(J;X)}\ . \tag{3.1.13}$$

Moreover, $c(T) \to 0$ as $T \to 0$.

***Proof*.** We begin by recalling the notion of fractional derivatives. Let $\alpha > 0$. The fractional derivative of order α of a function $f \in {}_0H_p^{\alpha}(J;X)$ is defined by

$$D_t^{\alpha} f(t) = \frac{d^m}{dt^m}\int_0^t g_{m-\alpha}(t-s) f(s)\,ds,$$

where $m = [\alpha] \in \mathbb{N}$, and $g_{\alpha}(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$.
Observe that by Corollary 1.4.5 the operator D_t^{α} coincides with the operator given there, if $\alpha \in (0,2)$. Moreover, it defines an isometrical isomorphism from ${}_0H_p^{\alpha}(J;X)$ to $L_p(J;X)$. On the other hand, since $f \in {}_0H_p^{\alpha}(J;X)$, it follows that

$$|\, g_{\varepsilon} * f \,|_{{}_0H_p^{\alpha}(J;X)} \leqslant c(T)\,|\, f \,|_{{}_0H_p^{\alpha}(J;X)}, \tag{3.1.14}$$

where $c(T) > 0$ and $c(T) \to 0$ as $T \to 0$. Indeed, observing that the operators D_t^α and $g_\varepsilon * \cdot$ commute in ${}_0H_p^\alpha(J;X)$, we have

$$| g_\varepsilon * f |_{{}_0H_p^\alpha(J;X)} = | D_t^\alpha(g_\varepsilon * f) |_{L_p(J;X)} = | g_\varepsilon * D_t^\alpha f |_{L_p(J;X)} .$$

Using this and Young's inequality the claim follows with $c(T) := | g_\varepsilon |_{L_1(J)}$.
Now, since $b * g_\varepsilon$ and $\frac{d}{dt} b * g_\varepsilon$ are of order $t^{\beta+\varepsilon}$ and $t^{\beta+\varepsilon-1}$ respectively, it follows that the operator $D_t^\varepsilon(b * \cdot) : H_p^{\kappa+\varepsilon}(J;X) \to {}_0H_p^{\beta+\kappa}(J;X)$ is well-defined, linear and bounded. On the other hand, since $\varepsilon < 1$ and the identity $g_\varepsilon * D_t^\varepsilon = I$ is valid in ${}_0H_p^\varepsilon(J;X)$, we obtain

$$| b * u |_{{}_0H_p^{\beta+\kappa}(J;X)} = | g_\varepsilon * D_t^\varepsilon(b * u) |_{{}_0H_p^{\beta+\kappa}(J;X)} . \tag{3.1.15}$$

Therefore, (3.1.13) follows from (3.1.14) and (3.1.15) with $\alpha = \beta + \kappa$, since the operator $D_t^\varepsilon(b * \cdot)$ is bounded in $H_p^{\kappa+\varepsilon}(J;X)$. □

We can now estimate $H(\phi)$ in X_2.

Corollary 3.1.3. *Let $\alpha_1, \alpha_2 \in (0,1)$ and $\kappa_1, \kappa_2 \in [0, 1/p)$ such that the condition (3.1.8) holds. Let $a_i \in \mathcal{K}^1(\alpha_i, \theta_i)$, with $\theta_i < \pi/2$, for $i = 1,2$ and let S be the operator given in (3.1.5). Suppose that $p > \frac{n}{3} + \frac{2}{3(\alpha_2+1)}$. Then the map $H : Z_2 \to {}_0X_2$, defined as*

$$H(\phi) = 1 * a_2 * (\phi - \phi^3) - 1 * a_2 * S * \phi_t$$

is continuous and bounded in Z_2. Moreover, there is a constant $K(T) > 0$, with $K(T) \to 0$ as $T \to 0$, such that

$$| H(v) |_{{}_0X_2} \leqslant K(T) \cdot \left[| v |_{Z_2}^3 + | v |_{Z_2} + | v - v(0) |_{{}_0X_2} \right] . \tag{3.1.16}$$

is valid for all $v \in Z_2$.

Proof. Let $v \in Z_2$, then $1 * v_t \in {}_0X_2$. From Lemma 3.1.2 with $b = 1 * a_2$ and $\beta = 1 + \alpha_2$, it follows that there is a constant $c(T) > 0$, such that

$$| 1 * a_2 * S * v_t |_{{}_0X_2} \leqslant c(T) | S * v_t |_{H_p^{\kappa_2+\varepsilon}(L_p)} . \tag{3.1.17}$$

On the other hand, from the embedding $Z_2 \hookrightarrow H_p^{\kappa_2+\varepsilon}$ ($\varepsilon < \alpha_2$) and maximal regularity of equation (II), we obtain the existence of a constant $C > 0$, such that

$$| S * v_t |_{H_p^{\kappa_2+\varepsilon}(L_p)} \leqslant | S * v_t |_{Z_2} \leqslant C \cdot | 1 * v_t |_{{}_0X_2} = C \cdot | v - v(0) |_{{}_0X_2} . \tag{3.1.18}$$

Therefore, from (3.1.17) and (3.1.18), there exists a constant $K(T) > 0$ with

$$| 1 * a_2 * S * v_t |_{{}_0X_2} \leqslant K(T) | v - v(0) |_{{}_0X_2} . \tag{3.1.19}$$

Finally, Lemma 3.1.2, yields

$$| 1 * a_2 * (v - v^3) |_{{}_0X_2} \leqslant c(T) \left(| v |_{H_p^{\kappa_2+\varepsilon}(L_p)} + | v^3 |_{H_p^{\kappa_2+\varepsilon}(L_p)} \right) . \tag{3.1.20}$$

Hence, using the embedding $Z_2 \hookrightarrow H_p^{\kappa_2+\varepsilon}(L_p)$ ($\varepsilon < \alpha_2$) and Lemma 3.1.1, the proof is complete. □

3.1.1 Contraction mapping principle

In this section we solve the equation

$$\phi = -1 * a_2 * A\phi + H(\phi) + g(t), \ t \in J, \tag{3.1.21}$$

in Z_2, where the nonlinearity $H(\phi)$ and the function $g(t)$ are defined by

$$H(\phi) = 1 * a_2 * (\phi - \phi^3) - 1 * a_2 * S * \phi_t, \quad t \in J, \text{ and} \tag{3.1.22}$$

$$g(t) = 1 * a_2 * u^* + 1 * f_2 + \phi_0, \qquad t \in J. \tag{3.1.23}$$

We begin with the linear version of (3.1.21), that is

$$v^* = -1 * a_2 * Av^* + g(t), \ t \in J. \tag{3.1.24}$$

Theorem 1.4.6 allows us to define an operator $\mathcal{L}$ in Z_2 by

$$\mathcal{L}v = v + 1 * a_2 * Av, \ v \in Z_2,$$

which is an isomorphism between Z_2 and the space

$$\mathbb{E} := \left\{ g \in X_2 : \ g(0) \in Y_p^2 \text{ and } g_t(0) \in \widetilde{Y}_p^2, \text{ if } \alpha_2 + \kappa_2 > \frac{1}{p} \right\}.$$

Observe that the function g defined by (3.1.23) belongs to $\mathbb{E}$, if and only if

(i) $f_i \in \widetilde{X}_i$ for $i = 1, 2$,

(ii) $\phi_0 \in Y_p^2$,

(iii) $f_2(0) \in \widetilde{Y}_p^2$, if $\alpha_2 + \kappa_2 > \frac{1}{p}$.

On the other hand, from Corollary 3.1.3, it follows $H(w) \in X_2$, for each $w \in Z_2$. Furthermore, it easy to check that $H(w) \in \mathbb{E}$ too, actually $H(w)(0) = d/dt H(w)(t)|_{t=0} = 0$. Now, let $v^* \in Z_2$ denote the solution of $\mathcal{L}v^* = g$ and assume that in equation (3.1.21) $\phi \in Z_2$ is known. By defining $v = \phi - v^*$, equation (3.1.21) is equivalent to a fix point problem

$$v = \mathcal{L}^{-1} H(v + v^*) =: \mathcal{T}v \text{ in } {}_0Z_2.$$

We have now the following result concerning the solution of equation (3.1.21).

Theorem 3.1.4. *Let $\alpha_i \in (0,1)$, $0 < \theta_i < \pi/2$, $\kappa_i \in [0, 1/p)$ for $p > 1$, and let $a_i \in \mathcal{K}^1(\alpha_i, \theta_i)$ for $i = 1, 2$. Suppose that $p > \frac{n}{3} + \frac{2}{3(\alpha_2+1)}$, $\alpha_i + \kappa_i \neq 1/p$, $i = 1, 2$ and that the condition (3.1.8) holds. Then for some $0 < \delta \leqslant T$, equation (3.1.21) has a unique local solution in $Z_2(\delta)$, if conditions*

(i) $f_i \in \widetilde{X}_i$ *for* $i = 1, 2$,

(ii) $\phi_0 \in Y_p^2$,

(iii) $f_2(0) \in \widetilde{Y}_p^2$, *if* $\alpha_2 + \kappa_2 > \frac{1}{p}$,

are fulfilled.

***Proof*.** Assume that the conditions (i)-(iii) are fulfilled. Defining g by (3.1.23), it follows that $g \in \mathbb{E}$, and from Theorem 1.4.6 there is a unique solution v^* in Z_2 of equation

$$\mathcal{L}v^* = g.$$

Since $H(w) \in \mathbb{E}$, for each $w \in Z_2$ we have that equation (3.1.21) is equivalent to a fix point problem. Consider the ball $\mathbb{B}_r(0) \subset {}_0Z_2(\delta)$, where $r > 0$ is fixed, and define $\mathcal{T} : \mathbb{B}_r(0) \subset {}_0Z_2(\delta) \to {}_0Z_2(\delta)$ by $\mathcal{T}v = \mathcal{L}^{-1}H(v^* + v)$. Furthermore, let $b := 1 * a_2$. We first show that $\mathcal{T}$ is a contraction by using Lemma 3.1.1 and Corollary 3.1.3.

$$\begin{aligned}
| \mathcal{T}v - \mathcal{T}w |_{{}_0Z_2(\delta)} \leqslant & | \mathcal{L}^{-1} || H(v^* + v) - H(v^* + w) |_{{}_0X_2(\delta)} \\
\leqslant & C | b * (v - w) \left[(v^* + w)^2 + (v^* + v)(v^* + w) + (v^* + v)^2\right] |_{{}_0X_2(\delta)} \\
& + C | b * S * (v_t - w_t) |_{{}_0X_2(\delta)} + C | b * (v - w) |_{{}_0X_2(\delta)} \\
\leqslant & CK(\delta) | v - w |_{{}_0Z_2(\delta)} \left[2 | v^* |_{Z_2(\delta)} + | w |_{{}_0Z_2(\delta)} + | v |_{{}_0Z_2(\delta)}\right]^2 \\
& + C | b * S * (v_t - w_t) |_{{}_0X_2(\delta)} + CK(\delta) | v - w |_{{}_0Z_2(\delta)} .
\end{aligned}$$

Using the same argument as in the proof of Corollary 3.1.3, it follows that

$$\begin{aligned}
| \mathcal{T}v - \mathcal{T}w |_{{}_0Z_2(\delta)} \leqslant & CK(\delta) | v - w |_{{}_0Z_2(\delta)} \left[4(| v^* |_{Z_2(\delta)} + r)^2 + C_1\right] \\
\leqslant & \frac{1}{2} | v - w |_{{}_0Z_2(\delta)},
\end{aligned} \tag{3.1.25}$$

since $K(\delta) \to 0$ as $\delta \to 0$.

To show that $\mathcal{T}\mathbb{B}_r(0) \subset \mathbb{B}_r(0)$, in a similar way we obtain that

$$\begin{aligned}
| \mathcal{T}v |_{{}_0Z_2(\delta)} \leqslant & | \mathcal{L}^{-1} || H(v^* + v) |_{X_2(\delta)} \\
\leqslant & CK(\delta) \left[| v^* + v |_{Z_2(\delta)} + | v^* + v |^3_{Z_2(\delta)} + | v^* + v - v^*(0) |_{{}_0X_2(\delta)}\right] \\
\leqslant & CK(\delta) \left[| v^* |_{Z_2(\delta)} + 2r + (| v^* |_{Z_2(\delta)} + r)^3 + | v^* - v^*(0) |_{{}_0X_2(\delta)}\right] \\
< & r,
\end{aligned} \tag{3.1.26}$$

provided $\delta > 0$ is small enough. Note that $| v^* |_{Z_2(\delta)} \to 0$ as $\delta \to 0$, since v^* is a fixed function.

Hence, the contraction mapping principle yields a unique fixed point $v \in \mathbb{B}_r(0)$ of $\mathcal{T}$ and therefore, $\phi = v^* + v$ is the unique strong solution of (3.1.21) in $[0, \delta]$. □

Concerning continuation of the solution ϕ, observe that by Theorem 3.1.4, there exists a $\delta > 0$ and a unique solution $\phi = \nu + \nu^*$ of (3.1.21) in $Z_2(\delta)$. On the other hand, from the embedding $Z_2(\delta) \hookrightarrow C^1([0,\delta];\widetilde{Y}_p^2) \cap C([0,\delta];Y_p^2)$ we have $\phi(\delta) \in Y_p^2$ and $\phi_t(\delta) \in \widetilde{Y}_p^2$. This fact allows us to continue the solution. Indeed, let $\mathcal{T}$ be the map defined in the proof of Theorem 3.1.4, and let $\nu \in {}_0Z_2(\delta)$ be its unique fixed point. For $\eta > 0$ consider the space

$$\mathcal{M}_\nu := \{\psi \in {}_0Z_2(\delta+\eta) \;:\; \psi|_{[0,\delta]} = \nu\}.$$

The set $\mathcal{M}_\nu$ is not empty and with the metric induced by $Z_2(\delta+\eta)$, we have that $(\mathcal{M}_\nu, d)$ is a complete metric space, where

$$d(f,g) := \mid f - g \mid_{Z_2(\delta+\eta)}, \text{ for all } f, g \in \mathcal{M}_\nu.$$

Now we can apply the contraction mapping principle in $\mathcal{M}_\nu$. From (3.1.25) and (3.1.26), it is easy to show that $\mathcal{T}$ has a unique fixed point $\psi \in \mathcal{M}_\nu$, for some $\delta_1 \in (\delta, \delta+\eta)$, provided $\eta > 0$ is chosen sufficiently small. Hence, the function $\phi := \nu^* + \psi$ is the unique solution of (3.1.7) in $Z_2(\delta_1)$. Successive application of this argument yields a solution ϕ on a maximal time interval $[0, t_{max})$, which is characterized by the two equivalent conditions

$$\begin{cases} \lim\limits_{\delta \to t_{max}} \mid \phi(\delta) \mid_{Y_p^2} \text{ does not exist, or} \\ \lim\limits_{\delta \to t_{max}} \mid \phi_t(\delta) \mid_{\widetilde{Y}_p^2} \text{ does not exist, if } \alpha_2 > 1/p, \end{cases}$$

and

$$\mid \phi \mid_{Z_2(t_{max})} = \infty.$$

As we already proved in this section, (3.1.21) and the system (3.1.1)-(3.1.2) are equivalent. Therefore, we obtain the following result.

Theorem 3.1.5. *Let* $\alpha_i \in (0,1)$, $0 < \theta_i < \pi/2$, $\kappa_i \in [0, 1/p)$ *for* $p > 1$, *and let* $a_i \in \mathcal{K}^1(\alpha_i, \theta_i)$ *for* $i = 1, 2$. *Suppose that* $p > \frac{n}{3} + \frac{2}{3(\alpha_2+1)}$, $\alpha_i + \kappa_i \neq 1/p$ *for* $i = 1, 2$, *and that the condition* (3.1.8) *holds. Then for some* $0 < \delta < t_{max}$ *the system (3.1.1)-(3.1.4) has a unique solution* $(u, \phi) \in Z_1(\delta) \times Z_2(\delta)$, *if the data are subject to the following conditions.*

(i) $f_1 \in \widetilde{X}_1$ *and* $f_2 \in \widetilde{X}_2$,

(ii) $u_0 \in Y_p^1$ *and* $\phi_0 \in Y_p^2$,

(iii) $f_i(0) \in \widetilde{Y}_p^i$, *if* $\alpha_i + \kappa_i > \frac{1}{p}$ *for* $i = 1, 2$.

3.2 Global well-posedness

In this section we want to solve the nonlinear system

$$
\begin{aligned}
u_t + \phi_t =& \int_{-\infty}^{t} a_1(t-s)\Delta u(s)ds, && \text{in } J \times \Omega; && (3.2.1)\\
\phi_t =& \int_{-\infty}^{t} a_2(t-s)\left[\Delta\phi + \phi - \phi^3 + u\right](s)ds, && \text{in } J \times \Omega; && (3.2.2)\\
\partial_n u =& \partial_n \phi = 0, && \text{on } J \times \partial\Omega; && (3.2.3)\\
u(0,x) =& u_0(x),\ \phi(0,x) = \phi_0(x), && \text{in } \Omega, && (3.2.4)
\end{aligned}
$$

globally in time in the setting used in the previous section. We restrict ourselves to the case where the system has trivial history, i.e.

$$u(t,x) = \phi(t,x) = 0, \quad (t,x) \in (-\infty, 0) \times \Omega. \tag{3.2.5}$$

For the sake of simplicity we also set $\kappa_2 = 0$. In case $\kappa_2 \neq 0$ the global existence result remains true, but the calculation is more length. Observe that from (3.1.8) it follows that $\alpha_2 \geqslant \alpha_1$ if $\kappa_2 = 0$.

We now begin the discussion concerning global existence of (3.2.1)-(3.2.4). From (3.2.5) and the definition of the operator B in Corollary 1.4.5, which is associated with the kernel a_2, (3.2.1)-(3.2.4) can be written as follows

$$
\begin{aligned}
u_t + \phi_t =& \int_0^t a_1(t-s)\Delta u(s)ds, && \text{in } J \times \Omega; && (3.2.6)\\
B\phi_t =& \Delta\phi + \phi - \phi^3 + u, && \text{in } J \times \Omega; && (3.2.7)\\
\partial_n u =& \partial_n \phi = 0, && \text{on } J \times \partial\Omega; && (3.2.8)\\
u(0,x) =& u_0(x),\ \phi(0,x) = \phi_0(x), && \text{in } \Omega. && (3.2.9)
\end{aligned}
$$

The subsequent result gives an a-priori estimate in case that the kernels a_1 and a_2 satisfy the following conditions:

(P1) $a_1 \in L_{1,loc}(\mathbb{R}_+)$, such that

$$\mathrm{Re}\int_0^T [a_1 * \psi](t)\overline{\psi(t)}dt \geqslant 0 \quad \text{for all } \psi \in L_2((0,T);\mathbb{C}), \text{and } T > 0.$$

(P2) $a_2 \in L_{1,loc}(\mathbb{R}_+)$, and there exists $\nu \in L_{1,loc}(\mathbb{R}_+)$ nonnegative, nonincreasing, such that

$$\int_0^t a_2(t-s)\nu(s)ds = 1 \text{ for all } t > 0.$$

Observe that the condition (P1) corresponds to the definition of positive type and (P2) corresponds to an particular case of the definition of completely positive type, see [Prü93].

Lemma 3.2.1. *Let* $(u, \phi) \in Z_1(\delta) \times Z_2(\delta)$ *be the solution of (3.2.6)-(3.2.9), with* $p \geqslant 2$. *Assume that the conditions (P1) and (P2) are fulfilled. Then there is constant* $M > 0$, *independent of* δ, *such that, the inequalities*

$$\begin{aligned} -M \leqslant \sup_{0<\delta<t_{max}} & \left\{ | u(\delta) |^2_{L_2(\Omega)} + | \phi(\delta) |^2_{H^1_2(\Omega)} + \frac{1}{2} | \phi(\delta) |^4_{L_4(\Omega)} - | \phi(\delta) |^2_{L_2(\Omega)} \right\} \\ & + 2\int_0^{t_{max}} \langle a_1 * \nabla u, \nabla u \rangle \, dt + 2\int_0^{t_{max}} \langle B\phi_t, \phi_t \rangle \, dt \\ & \leqslant 2 \left[| u_0 |^2_{L_2(\Omega)} + | \phi_0 |^2_{H^1_2(\Omega)} + | \phi_0 |^4_{L_4(\Omega)} \right] \end{aligned}$$

hold.

***Proof*.** We multiply (3.2.6) by u and (3.2.7) by ϕ_t, add the result and integrate by parts, to obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left\{ \int_\Omega | u |^2 \, dx + \int_\Omega | \nabla\phi |^2 \, dx + \frac{1}{2}\int_\Omega | \phi |^4 \, dx - \int_\Omega | \phi |^2 \, dx \right\} \\ & \quad + \langle a_1 * \nabla u, \nabla u \rangle + \langle B\phi_t, \phi_t \rangle = 0. \end{aligned} \tag{3.2.10}$$

Integrating (3.2.10) over $(0, \delta)$ $(\delta < t_{max})$ one obtains

$$\begin{aligned} & | u |^2_{L_2(\Omega)} + | \phi |^2_{H^1_2(\Omega)} + \frac{1}{2} | \phi |^4_{L_4(\Omega)} - | \phi |^2_{L_2(\Omega)} \\ & + 2\int_0^\delta \langle B\phi_t, \phi_t \rangle \, ds] + 2\int_0^\delta \langle a_1 * \nabla u, \nabla u \rangle \, ds \\ & \leqslant 2 \left[| u_0 |^2_{L_2(\Omega)} + | \phi_0 |^2_{H^1_2(\Omega)} + | \phi_0 |^4_{L_4(\Omega)} \right]. \end{aligned} \tag{3.2.11}$$

Note that the term $\int_0^\delta \langle B\phi_t, \phi_t \rangle \, ds$ is positive, since B is accretive (see Theorem 1.4.7). On the other hand, the parabola $x^4 - 2x^2$ is bounded from below by -1. Therefore, taking the supremum over $(0, t_{max})$ in (3.2.11), the proof is completed. □

We can now state our main result of this chapter.

Theorem 3.2.2. *Let* $\alpha_i \in (0, 1)$, $0 < \theta_i < \pi/2$, *and let* $a_i \in \mathcal{K}^1(\alpha_i, \theta_i)$ *for* $i = 1, 2$. *Suppose that* $p \geqslant 2$ *and* $n \leqslant 3$, $\alpha_i \neq 1/p$ *for* $i = 1, 2$, *and the condition* $\alpha_2 \geqslant \alpha_1$ *holds. If*

(i) *the conditions (P1) and (P2) are fulfilled, and*

(ii) $u_0 \in Y^1_p$ *and* $\phi_0 \in Y^2_p$,

then the system (3.2.6)-(3.2.9) *has a unique global solution* $(u, \phi) \in Z_1 \times Z_2$.

Proof. Let $0 < \delta < t_{max}$ and let $(u, \phi) \in Z_1(\delta) \times Z_2(\delta)$ be the unique local solution of (3.2.6)-(3.2.9), given by Theorem 3.1.5. From Lemma 3.2.1 it follows that

$$\phi \in L_\infty([0, t_{max}); L_6(\Omega)). \tag{3.2.12}$$

If $\rho \in (1/4, 1/3)$, then the inequality

$$-\frac{n}{3p} \leqslant \rho(2 - \frac{n}{p}) - \frac{n(1-\rho)}{6}$$

is valid for $p \geqslant 2$ and $n \leqslant 3$. Therefore, by the Gagliardo-Nirenberg inequality, it follows that there is a constant $C := C(\Omega) > 0$, such that

$$| \phi |_{L_{3p}(\Omega)} \leqslant C | \phi |^{\rho}_{H^2_p(\Omega)} | \phi |^{1-\rho}_{L_6(\Omega)} . \tag{3.2.13}$$

Furthermore, from (3.2.12) and (3.2.13) we obtain

$$| \phi^3 |_{L_p(L_p)} \leqslant C_0 | \phi |^{3\rho}_{L_{3\rho p}(H^2_p)} \leqslant C_0 | \phi |^{3\rho}_{L_p(H^2_p)} \leqslant C_0 | \phi |^{3\rho}_{Z_2(\delta)}, \tag{3.2.14}$$

since $\rho < 1/3$. On the other hand, by maximal L_p-regularity, there is a constant $M := M(T) > 0$, such that

$$| u |_{Z_1(\delta)} + | \phi |_{Z_2(\delta)} \leqslant M(1 + | \phi^3 |_{L_p([0,\delta];L_p)}).$$

Hence (3.2.14) yields

$$| \phi |_{Z_2(\delta)} \leqslant M(1 + | \phi |^{3\rho}_{Z_2(\delta)})$$

with a different constant M, which is independent of $\delta < t_{max}$. Therefore,

$$| \phi |_{Z_2(t_{max})} < \infty.$$

This in turn yields the boundedness of $u \in Z_1(t_{max})$. Hence the global existence of (3.2.6)-(3.2.7) follows. □

Chapter 4

A conserved phase field model

In this chapter we show the global well-posedness in the strong sense in the L_p-setting for the following system

$$u_t + \phi_t = \gamma \Delta u + \int_0^t a_1(t-s)\Delta u(s)ds + f_1, \qquad \text{in } J \times \Omega; \tag{4.0.1}$$

$$\phi_t = -\int_0^t a_2(t-s)\Delta[\Delta\phi - \Phi'(\phi) + u](s)ds + f_2, \qquad \text{in } J \times \Omega; \tag{4.0.2}$$

$$\partial_n u = \partial_n \phi = \partial_n(\Delta\phi) = 0, \qquad \text{on } J \times \partial\Omega; \tag{4.0.3}$$

$$u(0,x) = u_0(x), \;\; \phi(0,x) = \phi_0(x), \qquad \text{in } \Omega, \tag{4.0.4}$$

where

$$f_1(t) = \int_{-\infty}^0 a_1(t-s)\Delta u(s)ds, \qquad \text{in } J \times \Omega; \tag{4.0.5}$$

$$f_2(t) = -\int_{-\infty}^0 a_2(t-s)\Delta[\Delta\phi - \Phi'(\phi) + u](s)ds, \qquad \text{in } J \times \Omega, \tag{4.0.6}$$

J is an interval of the form $[0, T]$ with $T > 0$, and Ω is a smooth bounded domain in $\mathbb{R}^n$.

4.1 Main assumptions on the potential Φ and on the kernels

The basic assumption on the potential function $\Phi : \mathbb{R} \to \mathbb{R}$ to obtain local existence is

$$\Phi \in C^{4-}(\mathbb{R}). \tag{4.1.1}$$

Furthermore, for global existence we suppose that there are constants $m_1, m_2 \in \mathbb{R}$ such that

$$\Phi(s) \geqslant -\frac{m_1}{2}s^2 - m_2, \text{ for each } s \in \mathbb{R}, \text{ and } \lambda_1 > m_1, \tag{4.1.2}$$

where λ_1 is the smallest nontrivial eigenvalue of the negative Laplacian on Ω with homogeneous Neumann boundary conditions. Also, we will assume the growth condition

$$|\ \Phi'''(s)\ | \leqslant C\left(1 + |\ s\ |^\beta\right), \ s \in \mathbb{R}, \tag{4.1.3}$$

with some constants $C, \beta > 0$ for $n \leqslant 2$, and in case $n = 3$ with the restriction $\beta < 3$.

The main assumptions on the kernels a_1 and a_2 are the following:

(P0) $a_i \in \mathcal{K}^1(\alpha_i, \theta_{a_i})$ with $\theta_{a_i} < \frac{\pi}{2}$, and $\alpha_2 \geqslant \alpha_1$ with $\alpha_i \in (0,1)$ for $i = 1, 2$. Further, $\operatorname{Im} \hat{a}_1(i\rho) \cdot \operatorname{Im} \hat{a}_2(i\rho) \geqslant 0$, for all $\rho \in \mathbb{R} \setminus \{0\}$.

(P1) $a_1 \in L_{1,loc}(\mathbb{R}_+)$, such that

$$\operatorname{Re} \int_0^T a_1 * \psi(t)\overline{\psi(t)}dt \geqslant 0 \ \text{ for all } \psi \in L_2((0,T);\mathbb{C}), \text{and } T > 0.$$

(P2) $a_2 \in L_{1,loc}(\mathbb{R}_+)$, and there exists $\nu \in L_{1,loc}(\mathbb{R}_+)$ nonnegative, nonincreasing, such that

$$\int_0^t a_2(t-s)\nu(s)ds = 1 \text{ for all } t > 0.$$

(P0) is the main condition to obtain local well-posedness, and (P1)-(P2) are needed additionally to obtain global well-posedness. Observe that the condition (P1) corresponds to definition of positive type and (P2) corresponds to a particular case of the definition of completely positive type, see [Prü93].

A typical example of kernels that satisfy the properties (P0)-(P2) is

$$a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}e^{-\eta t}, \quad t > 0,$$

for every $\alpha \in (0, 2)$ and $\eta \geqslant 0$.

Important properties of all these types of kernels are discussed in the monograph Prüss [Prü93].

4.2 Local well-posedness

Let $T > 0$ be given and fixed and let Ω be a smooth bounded domain in $\mathbb{R}^n$. For $0 < \delta \leqslant T$ and $1 < p < \infty$, we define the space

$$Z_i^{\beta_i}(\delta) = H_p^{\beta_i}([0,\delta]; X) \cap L_p([0,\delta]; D_{A^i}),$$

for $i = 1, 2$, where $\beta_i > 0$, and $X := L_p(\Omega)$, and A is a closed linear operator in X with dense domain $D(A)$, and D_{A^i} means the domain of A^i equipped with the graph norm. The space ${}_0Z_i^{\beta_i}(\delta)$ denotes the corresponding space $Z_i^{\beta_i}(\delta)$ with zero traces at $t = 0$. Whenever no confusion may arise, we shall simply write $Z_i^{\beta_i}$, and ${}_0Z_i^{\beta_i}$ if $\delta = T$. Furthermore, we will assume that $\beta_i \notin \{1/p, 1 + 1/p\}$, and define the natural phase spaces for $Z_i^{\beta_i}$ by

$$\begin{aligned} Y_p^i(\beta_i) =& (X; D_{A^i})_{\gamma_i,p}, \quad \text{with } \gamma_i = 1 - \frac{1}{p\beta_i}, \quad \text{if } \beta_i > \frac{1}{p}, \quad \text{for } i = 1, 2; \\ \widetilde{Y}_p^i(\beta_i) =& (X; D_{A^i})_{\sigma_i,p}, \quad \text{with } \sigma_i = 1 - \frac{1}{\beta_i} - \frac{1}{p\beta_i}, \quad \text{if } \beta_i > 1 + \frac{1}{p}, \text{ for } i = 1, 2. \end{aligned}$$

When no confusion can arise, we shall simply write $|\cdot|_{p,q}$ to designate the norm on $L_p(J; L_q(\Omega))$, and we write simply $|\cdot|_p$ if $p = q$.

4.2.1 Fourier multipliers and auxiliary results

We begin our discussion assuming that the kernels a_i are 1-regular and θ_{a_i}-sectorial with $\theta_{a_i} \in (0, \frac{\pi}{2})$, for $i = 1, 2$. We consider the integrated version of model (4.0.1)-(4.0.4) which reads

$$u + \phi = b_1 * \Delta u + 1 * f_1 + u_0 + \phi_0, \quad \text{in } J \times \Omega; \tag{4.2.1}$$

$$\phi = - b_2 * \Delta[\Delta\phi - \Phi'(\phi) + u] + 1 * f_2 + \phi_0, \quad \text{in } J \times \Omega; \tag{4.2.2}$$

$$\partial_n u = \partial_n \phi = \partial_n \Delta\phi = 0, \quad \text{on } J \times \partial\Omega; \tag{4.2.3}$$

$$u(0, x) = u_0(x), \ \phi(0, x) = \phi_0(x), \quad \text{in } \Omega, \tag{4.2.4}$$

where the functions f_i are as in (4.0.5)-(4.0.6) and the kernels b_1 and b_2 are given by

$$\begin{aligned} b_1(t) &:= \gamma + \int_0^t a_1(s)ds, \\ b_2(t) &:= \int_0^t a_2(s)ds. \end{aligned} \tag{4.2.5}$$

In order to apply Theorem 1.1.8, we have to show that the kernels b_i are θ_i-sectorial with $\theta_i < \pi$ and also 1-regular. From the sectoriality of a_i, it follows immediately that

$$|\arg \hat{b}_i(\lambda)| \leqslant \left|\frac{\pi}{2} + \arg \hat{a}_i(\lambda)\right| \leqslant \theta_{a_i} + \frac{\pi}{2} < \pi, \ \ \mathrm{Re}\lambda > 0.$$

The 1-regularity of b_1 follows from (1.1.2), i.e.

$$\left|\arg \frac{\gamma}{\lambda} - \arg \widehat{1 * a_1}(\lambda)\right| \leqslant \frac{\pi}{2} + |\arg \hat{a}_1(\lambda)| < \pi, \ \ \mathrm{Re}\lambda > 0.$$

The 1-regularity of b_2 is trivial. Moreover, it is easy to show that $b_1 \in \mathcal{K}^1(1, \pi/2 + \theta_{a_1})$ for $\gamma > 0$ and $b_2 \in \mathcal{K}^1(1 + \alpha_2, \pi/2 + \theta_{a_2})$. In case that $\gamma = 0$, $b_1 \in \mathcal{K}^1(1 + \alpha_1, \pi/2 + \theta_{a_1})$.

We concentrate first on the case of vanishing traces at $t = 0$ linear version of (4.2.1)-(4.2.4), that is

$$u + \phi = -\, b_1 * Au + 1 * (f_1 - f_1(0)), \tag{4.2.6}$$

$$\phi = -\, b_2 * A^2\phi + b_2 * Au + 1 * (f_2 - f_2(0)), \tag{4.2.7}$$

where $A := -\Delta$ with domain

$$D(A) := \{v \in H_p^2(\Omega) : \ \partial_n v = 0 \text{ on } \partial\Omega\},$$

and

$$D(A^2) := \{v \in H_p^4(\Omega) \cap D(A) : \ \partial_n Av = 0 \text{ on } \partial\Omega\}.$$

To solve system (4.2.6)-(4.2.7), will first reformulate it in the following way. Let $v(t) = e^{-\omega t}u(t)$, $\varphi(t) = e^{-\omega t}\phi(t)$, $h_i(t) = e^{-\omega t}1 * (f_i(\cdot) - f_i(0))$, $d_i(t) = e^{-\omega t}b_i(t)$ for $i = 1, 2$, where $\omega > 0$ is fixed. Then (4.2.6)-(4.2.7) is equivalent to

$$v + \varphi = -\, d_1 * Av + h_1, \tag{4.2.8}$$

$$\varphi = -\, d_2 * A^2\varphi + d_2 * Av + h_2. \tag{4.2.9}$$

Observe that the kernels d_i enjoy the same properties of regularity and sectoriality as b_i, moreover, $d_i \in L_1(\mathbb{R}_+)$ for $i = 1, 2$.

Now we associate with the kernels d_i the operators B_i from Theorem 1.1.8 with domain $D(B_i)$ for $i = 1, 2$. So, the system (4.2.8)-(4.2.9) can be written in abstract form as follows:

$$(B_1 + \mathcal{A})v = -B_1\varphi + B_1h_1, \tag{4.2.10}$$

$$(B_2 + \mathcal{A}^2)\varphi = \mathcal{A}v + B_2h_2, \tag{4.2.11}$$

where $(\mathcal{A}^i v)(t) := A^i v(t)$ with domain $D(\mathcal{A}^i) = L_p(\mathbb{R}_+, D_{A^i})$, for $i = 1, 2$. The operators B_i and $\mathcal{A}^j$ commute in the resolvent sense, and they belong to the class $\mathsf{BIP}(L_p(\mathbb{R}_+; X))$ with power angle $\theta_{B_i} = \theta_i$ and $\theta_{A^j} = 0$ $(i, j = 1, 2)$, respectively. Therefore, from Theorem 1.3.1 it follows that $B_i + \mathcal{A}^j$ belongs to $\mathsf{BIP}(L_p(\mathbb{R}_+; X))$, moreover, it is invertible, since B_i is invertible. In particular, observe that the operator $B_1(B_1 + \mathcal{A})^{-1}$ is bounded in $L_p(\mathbb{R}_+; X)$. Now assume that φ is known, then we can represent v in (4.2.10) as

$$v = -(B_1 + \mathcal{A})^{-1}B_1\varphi + (B_1 + \mathcal{A})^{-1}B_1h_1. \tag{4.2.12}$$

In this way, (4.2.10)-(4.2.11) can be associated with an operator G defined in $L_p(\mathbb{R}_+; X)$ by

$$G = B_2 + \mathcal{A}^2 + \mathcal{A}(B_1 + \mathcal{A})^{-1}B_1,$$

with domain $D(G) = D(B_2) \cap D(\mathcal{A}^2)$. Hence, a solution φ of the equation

$$G\varphi = h, \tag{4.2.13}$$

with $h := \mathcal{A}(B_1 + \mathcal{A})^{-1}B_1h_1 + B_2h_2$, combined with (4.2.12) yields a solution $(\nu, \varphi) \in (D(B_1) \cap D(\mathcal{A})) \times D(G)$ of (4.2.10)-(4.2.11). Furthermore, by using the formulation above, we get a solution (u, ϕ) of (4.2.6)-(4.2.7) in an interval $J := [0, T]$, for each $T > 0$ with the same regularity as (ν, φ).

One possibility to prove the existence and uniqueness of a strong solution ϕ of (4.2.13) in $D(G)$ is to show that the symbol of the operator G is invertible and its inverse fulfills the assumptions of Theorem 1.2.1.

Theorem 4.2.1. *Let X be a Banach space of class $\mathcal{HT}$, and let $A \in \mathcal{RH}^\infty(X)$ with $\mathcal{RH}^\infty$-angle $\phi_A^{\mathcal{R}_\infty} < \pi$, and let $\mathcal{A}$ denote the canonical extension of A to $L_p(\mathbb{R}_+; D_A)$ for $1 < p < \infty$. Let d_i be scalar kernels which are 1-regular and θ_i-sectorial with $\theta_i < \pi$ for $i = 1, 2$. Let B_i be the operator defined by* (1.1.3)*, with $\hat{d}_i$ in place of $\hat{a}$, for $i = 1, 2$. Furthermore, assume that the following conditions hold:*

(i) $d_i \in L_1(\mathbb{R}_+)$, *for* $i = 1, 2$;

(ii) *for all* $\rho \in \mathbb{R} \setminus \{0\}$ *it holds that* $\mathrm{Im}\ \hat{d}_1(i\rho) \cdot \mathrm{Im}\ \hat{d}_2(i\rho) \geqslant 0$;

(iii) $D(B_2) \hookrightarrow D(B_1)$;

(iv) $\phi_A^{\mathcal{R}_\infty} < \frac{\pi - \sigma}{2}$ *with* $\sigma := \max\{\theta_1, \theta_2\}$.

Then for each $f \in L_p(\mathbb{R}_+; X)$ there exists a unique solution $\varphi \in D(G) := D(B_2) \cap D(\mathcal{A}^2)$ of the equation

$$G\varphi = f, \tag{4.2.14}$$

where $G = B_2 + \mathcal{A}^2 + \mathcal{A}(B_1 + \mathcal{A})^{-1}B_1$. In particular, if $d_i \in \mathcal{K}^1(\beta_i, \theta_i)$ with $\beta_i > 0$ for $i = 1, 2$, then $\varphi \in {}_0H_p^{\beta_2}(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; D_{A^2}) =: {}_0Z_2^{\beta_2}(\mathbb{R}_+)$.

Proof. Define $m_j(i\rho) = 1/\hat{d}_j(i\rho)$ for $j = 1, 2$, and $\rho \in \mathbb{R} \setminus \{0\}$. Let $\varepsilon \in (\phi_{\mathcal{A}}^\infty, (\pi - \sigma)/2)$ and define $g(i\rho, \mu) = m_2(i\rho) + \mu^2 + \mu m_1(i\rho)(\mu + m_1(i\rho))^{-1}$ for $\mu \in \Sigma_\varepsilon$ and $\rho \in \mathbb{R} \setminus \{0\}$.

Let $\rho \in \mathbb{R} \setminus \{0\}$ fixed and assume that $\mathrm{Im}\ \hat{d}_1(i\rho) \leqslant 0$. Hence from condition (ii) and the sectoriality of d_i, $i = 1, 2$, it follows that $\mathrm{Im}\ \hat{d}_2(i\rho) \leqslant 0$, and

$$0 < \arg(m_j(i\rho)) < \theta_j, \ j = 1, 2.$$

Furthermore, the inequalities

$$-2\varepsilon < \arg\left(m_2(i\rho) + \mu^2\right) < \theta_2,$$
$$-2\varepsilon < \arg\left(\mu m_1(i\rho)(m_1(i\rho) + \mu)^{-1}\right) < \theta_1,$$

hold for all $\mu \in \Sigma_\varepsilon$. Since $\varepsilon \in (\phi_A^{\mathcal{R}\infty}, (\pi - \sigma)/2)$, we obtain that

$$-2\varepsilon < \arg(g(i\rho, \mu)) < (2\varepsilon + \sigma) < \pi.$$

Analogously, we obtain that

$$-(2\varepsilon + \sigma) < \arg(g(i\rho, \mu)) < 2\varepsilon,$$

if we assume that Im $\hat{d}_1(i\rho) \geqslant 0$. Therefore, $g(i\rho, \mu) \neq 0$ for all $\mu \in \Sigma_\varepsilon$ and $\rho \in \mathbb{R} \setminus \{0\}$. Moreover, since $d_j \in L_1(\mathbb{R}_+)$ and by the continuity of the function arg, it follows that $g(0, 0) > 0$. Therefore, $(g(i\rho, \mu))^{-1} =: M(i\rho, \mu)$ is analytic in μ and uniformly bounded for all $\rho \in \mathbb{R} \setminus \{0\}$, and all $\mu \in \Sigma_\varepsilon$, thus we have $\{M(i\rho, \cdot)\}_{\rho \in \mathbb{R} \setminus \{0\}} \in \mathcal{H}^\infty(\Sigma_\varepsilon)$. Moreover, since $\mathcal{A} \in \mathcal{RH}^\infty(L_p(\mathbb{R}_+; X))$ with $\phi_{\mathcal{A}}^{\mathcal{R}\infty} \leqslant \phi_A^{\mathcal{R}\infty}$, it follows from Theorem 1.2.2 that

$$\mathcal{R}\{M(i\rho, \mathcal{A}) : \ \rho \in \mathbb{R} \setminus \{0\}\} < \infty.$$

From the 1-regularity of d_i and Kahane's contraction principle (cf. [DHP03, Lemma 3.5]), it follows that the set

$$\{i\rho\partial_1 g(i\rho, \mathcal{A}) : \ \rho \in \mathbb{R} \setminus \{0\}\}$$

is also $\mathcal{R}$-bounded. Since a product of $\mathcal{R}$-bounded families is also $\mathcal{R}$-bounded, we obtain

$$\mathcal{R}\{\rho\partial_1 M(i\rho, \mathcal{A}) : \ \rho \in \mathbb{R} \setminus \{0\}\} < \infty.$$

Therefore, by Theorem 1.2.1, the operator

$$T_M f := \mathcal{F}^{-1}[M(\cdot, \mathcal{A})\mathcal{F}f]$$

is a Fourier L_p-multiplier in the sense of distributions. Hence, from the uniqueness of Fourier transform, it follows that the function $\varphi := T_M f|_{\mathbb{R}_+}$ is the unique solution of (4.2.14) in $D(G)$. Moreover, from Corollary 1.4.5, it follows that $\varphi \in {}_0Z_2^{\beta_2}(\mathbb{R}_+)$. □

Corollary 4.2.2. *Suppose that the conditions of Theorem 4.2.1 are satisfied. Furthermore, we assume that* $d_i \in \mathcal{K}^1(\beta_i, \theta_i)$ *for* $i = 1, 2$*, with* $\beta_2 \geqslant \beta_1$*. Then the system* (4.2.8)-(4.2.9) *has a unique solution* $(\nu, \varphi) \in {}_0Z_1^{\beta_1}(\mathbb{R}_+) \times {}_0Z_2^{\beta_2}(\mathbb{R}_+)$ *if and only if*

$$h_i \in {}_0H_p^{\beta_i}(\mathbb{R}_+; X) \textit{ for } i = 1, 2.$$

Remark 4.2.1. The condition $\beta_2 \geqslant \beta_1$ in Corollary 4.2.2 ensures the existence of a unique solution (ν, φ) with optimal regularity, whereas in case $\beta_2 < \beta_1$ we obtain also a unique solution (ν, φ) with $\nu + \varphi \in {}_0Z_1^{\beta_1}(\mathbb{R}_+)$ and $\nu \in L_p(\mathbb{R}_+; D_A)$, hence no optimal regularity.

Now we consider the non-homogeneous version of (4.2.6)-(4.2.7) that is

$$u + \phi = -\, b_1 * Au + 1 * f_1 + u_0 + \phi_0, \tag{4.2.15}$$

$$\phi = -\, b_2 * A^2\phi + b_2 * Au + 1 * f_2 + \phi_0. \tag{4.2.16}$$

Corollary 4.2.3. *Suppose that the kernels a_i fulfill the condition (P0), and assume that $\alpha_i \neq 1/p$ for $i = 1, 2$. In case that $\gamma = 0$ in (4.2.5), the system (4.2.15)-(4.2.16) has a unique solution $(u, \phi) \in Z_1^{1+\alpha_1} \times Z_2^{1+\alpha_2}$ if and only if the following conditions hold:*

(i) $f_i \in H_p^{\alpha_i}(J; X)$*, for* $i = 1, 2$*;*

(ii) $u_0 \in Y_p^1(1 + \alpha_1)$ *and* $\phi_0 \in Y_p^2(1 + \alpha_2)$*;*

(iii) $u_t(0) \in \widetilde{Y}_p^1(1 + \alpha_1)$ *and* $\phi_t(0) \in \widetilde{Y}_p^2(1 + \alpha_2)$*, if* $\alpha_i > \frac{1}{p}$*, for* $i = 1, 2$*.*

Proof. We begin with the necessity part. From the condition (P0), we have that $\alpha_2 \geqslant \alpha_1$. Hence, $Z_2^{1+\alpha_2} \hookrightarrow Z_1^{1+\alpha_1}$ therefore, $u - u_0 + \phi - \phi_0 + b_1 * Au = 1 * f_1 \in {}_0H_p^{1+\alpha_1}(J; X)$. In addition, since $Au \in L_p(J; X)$, it follows from Corollary 1.4.5 that $b_2 * Au \in {}_0H_p^{1+\alpha_2}(J; X)$. Therefore, $\phi - \phi_0 + b_2 * A^2\phi - b_2 * Au = 1 * f_2 \in {}_0H_p^{1+\alpha_2}(J; X)$. Hence, the condition (i) is proved. The conditions (ii) and (iii) follow from the embeddings $Z_i^{1+\alpha_i} \hookrightarrow C^1(J; \widetilde{Y}_p^i(1 + \alpha_i)) \cap C(J; Y_p^i(1 + \alpha_i))$, provided $\alpha_i > 1/p$ holds for $i = 1, 2$. If $\alpha_2 > 1/p > \alpha_1$ then we set $u_t(0) = 0$; in this case $H_p^{\alpha_1}(J; X) = {}_0H_p^{\alpha_1}(J; X)$. Analogously, in the case, $1/p > \alpha_2 \geqslant \alpha_1$, we set $u_t(0) = \phi_t(0) = 0$.

Now, we prove the sufficiency part. Firstly, we will discuss the non-homogenous version of (4.2.15)-(4.2.16). For this we consider the problems

$$w_1(t) = -\,(b_1 * Aw_1)(t) + x_1, \tag{4.2.17}$$

$$w_2(t) = -\,(b_2 * A^2 w_2)(t) + x_2. \tag{4.2.18}$$

Moreover, if the kernels b_i for $i = 1, 2$ have more regularity we can also consider the problems

$$z_1(t) = -\,(b_1 * Az_1)(t) + ty_1, \tag{4.2.19}$$

$$z_2(t) = -\,(b_2 * A^2 z_2)(t) + ty_2. \tag{4.2.20}$$

Observe that from the variation of parameters formula (1.4.2) the mild solutions of (4.2.17) and (4.2.18) are given by

$$w_i(t) = S_i(t)x_i \text{ for } i = 1, 2,$$

where S_i corresponds to the resolvent operator of equation (4.2.17) and (4.2.18) respectively. Analogously, we have that the solutions of (4.2.19) and (4.2.20) are given by

$$z_i(t) = (1 * S_i)(t)y_i \text{ for } i = 1, 2.$$

Since $1+\alpha_i > 1/p$ for $i = 1, 2$, it follows from [Zac05, Thm. 3.2] that

$$\text{if } x_i \in Y_p^i(1+\alpha_i) \text{ then } w_i \in Z_i^{1+\alpha_i} \text{ for } i = 1, 2.$$

Similarly,

$$\text{if } \alpha_i > 1/p \text{ and } y_i \in \widetilde{Y}_p^i(1+\alpha_i) \text{ then } z_i \in Z_i^{1+\alpha_i} \text{ for } i = 1, 2.$$

We set $y_1 = f_1(0) - f_2(0)$ and $y_2 = f_2(0)$ in (4.2.19)-(4.2.20) and define $h_1 := 1 * (f_1 - f_1(0)) + b_2 * A^2(w_2 + z_2)$ and $h_2 := 1 * (f_2 - f_2(0)) + b_2 * A(w_1 + z_1)$ if $\alpha_i > 1/p$ for $i = 1, 2$; otherwise we set $f_i(0) = 0$ for $i = 1, 2$. In this case, it follows that $z_i = 0$ for $i = 1, 2$. Clearly, $h_i \in {}_0H_p^{1+\alpha_i}(J; X)$ for $i = 1, 2$ since $\alpha_2 \geqslant \alpha_1$. Therefore, from Corollary 4.2.2 there exists a unique solution $(v, \varphi) \in {}_0Z_1^{1+\alpha_1} \times {}_0Z_2^{1+\alpha_2}$ of (4.2.15)-(4.2.16). Hence, from (4.2.17)-(4.2.18), it follows that the functions $u := v + w_1 + z_1$ and $\phi := \varphi + w_2 + z_2$ satisfy the system

$$\begin{aligned} u + \phi &= - b_1 * Au + 1 * f_1 + x_1 + x_2, \\ \phi &= - b_2 * A^2\phi + b_2 * Au + 1 * f_2 + x_2. \end{aligned}$$

Moreover, if $x_i \in Y_p^i(1+\alpha_i)$ for $i = 1, 2$ and $y_i \in \widetilde{Y}_p^i(1+\alpha_i)$ for $i = 1, 2$, in case $\alpha_i > 1/p$, then $(u, \phi) \in Z_1^{1+\alpha_1} \times Z_2^{1+\alpha_2}$. On the other hand, from $Z_i^{1+\alpha_i} \hookrightarrow C^1(J; \widetilde{Y}_p^i(1+\alpha_i)) \cap C(J; Y_p^i(1+\alpha_i))$ for $i = 1, 2$, it follows that $u(0) = x_1 = u_0$, $u_t(0) = f_1(0) - f_2(0)$, $\phi(0) = x_2 = \phi_0$, and $\phi_t(0) = f_2(0)$. □

Remark 4.2.2. In case that $\gamma > 0$, the result remains true if we set $\alpha_1 = 0$ in this case only second part of (iii).

4.2.2 Contraction mapping principle

Let $(u^*, \phi^*) \in Z_1^{1+\alpha_1} \times Z_2^{1+\alpha_2}$ be the solution of the linear system

$$u^* + \phi^* = - b_1 * Au^* + 1 * f_1 + u^*(0) + \phi^*(0), \tag{4.2.21}$$

$$\phi^* = - b_2 * A^2\phi^* + b_2 * Au^* + 1 * f_2 + \phi^*(0), \tag{4.2.22}$$

and assume that $(u, \phi) \in Z_1^{1+\alpha_1} \times Z_2^{1+\alpha_2}$ is known in the following semilinear system

$$u + \phi = - b_1 * Au + 1 * f_1 + u^*(0) + \phi^*(0), \tag{4.2.23}$$

$$\phi = - b_2 * A^2\phi + b_2 * Au - b_2 * A\Phi'(\phi) + 1 * f_2 + \phi^*(0). \tag{4.2.24}$$

Let $u_* = u - u^*$ and $\phi_* = \phi - \phi^*$, then from (4.2.21)-(4.2.24), it follows that (u_*, ϕ_*) satisfies the equations

$$u_* + \phi_* = - b_1 * Au_*, \tag{4.2.25}$$

$$\phi_* = - b_2 * A^2\phi_* + b_2 * Au_* - b_2 * A\Phi'(\phi_* + \phi^*). \tag{4.2.26}$$

Since $\alpha_2 \geqslant \alpha_1$, we have that $\phi_* \in {}_0H_p^{1+\alpha_1}(J;X)$, therefore the function u_* in (4.2.25) can be represented as

$$u_* = (B_1 + \mathcal{A})^{-1} B_1 \phi_*. \tag{4.2.27}$$

Using this in (4.2.26) we obtain that ϕ_* is a solution of

$$G\phi_* = -\mathcal{A}\Phi'(\phi_* + \phi^*). \tag{4.2.28}$$

Therefore, by means of this formulation, the system (4.2.23)-(4.2.24) is equivalent to solving equation (4.2.28) which in turn is equivalent to the fixed point problem

$$\mathcal{T}\phi_* := T_M(-\mathcal{A}\Phi'(\phi_* + \phi^*)), \tag{4.2.29}$$

where T_M is the L_p-multiplier associated with the operator G. Observe that the map $\mathcal{T}$ is well-defined in $Z_2^{1+\alpha_2}$, provided that $-\mathcal{A}\Phi'(w) \in L_p(J;X)$ for all $w \in Z_2^{1+\alpha_2}$, and some $p > 1$.

We define the closed ball $B_R(0) \subset {}_0Z_2^{1+\alpha_2}$ as the set of all $\phi_* \in {}_0Z_2^{1+\alpha_2}$ such that $| \phi_* |_{{}_0Z_2^{1+\alpha_2}} \leqslant R$. Analogously, we define the shifted ball $B_R(\phi^*) \subset Z_2^{1+\alpha_2}$ by means of

$$B_R(\phi^*) := \{w \in Z_2^{1+\alpha_2} : \ w = \phi_* + \phi^*, \ \phi_* \in B_R(0)\}.$$

In addition, we have the embedding

$$Z_2^{1+\alpha_2} \hookrightarrow C(J, H_p^{4-4/(p(1+\alpha_2))}(\Omega)) \hookrightarrow C(J \times \overline{\Omega}),$$

provided the condition $p \geqslant n/4 + 1/(1+\alpha_2)$ holds. This way, all functions $w \in B_R(\phi^*)$ are uniformly bounded. It follows that the same holds for $\phi_* \in B_R(0)$.

Lemma 4.2.4. *Let* $\alpha_2 \in [0,1)$ *and* $n/4 + 1/(1+\alpha_2) \leqslant p < \infty$. *Let* $\Phi \in C^{4-}(\mathbb{R})$, *and let* $\phi^* \in Z_2^{1+\alpha_2}$ *be fixed. Then the map* $\mathcal{T}$ *defined by* (4.2.29) *has a unique fixed point* $\phi_* \in {}_0Z_2^{1+\alpha_2}(\delta)$ *for some* $0 < \delta \leqslant T$. *Furthermore, there exists a constant* $C(\delta) > 0$ *such that*

$$| \mathcal{T}w - \mathcal{T}z |_{{}_0Z_2^{1+\alpha_2}(\delta)} \leqslant C(\delta) \, | w - z |_{{}_0Z_2^{1+\alpha_2}(\delta)}, \tag{4.2.30}$$

$$| \mathcal{T}w |_{{}_0Z_2^{1+\alpha_2}(\delta)} \leqslant C(\delta)[| w |_{{}_0Z_2^{1+\alpha_2}(\delta)} + | \phi^* |_{Z_2^{1+\alpha_2}(\delta)}], \tag{4.2.31}$$

hold for all $w, \ z \in B_R(0)$. *Moreover,* $C(\delta) \to 0$ *as* $\delta \to 0$.

Proof. Observe that, in order to prove (4.2.30)-(4.2.31), it suffices to check that

$$| \Delta\Phi'(u) - \Delta\Phi'(v) |_p \leqslant c(\delta) \, | u - v |_{Z_2^{1+\alpha_2}(\delta)}, \tag{4.2.32}$$

$$| \Delta\Phi'(u) |_p \leqslant c(\delta) \, | u |_{Z_2^{1+\alpha_2}(\delta)} \tag{4.2.33}$$

are valid for all $u, \ v \in B_R(\phi^*)$, where $c(\delta)$ enjoys the same properties as $C(\delta)$.

Since all functions in $B_R(\phi^*)$ are uniformly bounded, it follows from Hölder's inequality that

$$\begin{aligned}
|\Delta\Phi'(u) - \Delta\Phi'(v)|_p \leqslant &|\Delta u \Phi''(u) - \Delta v \Phi''(v)|_p + \| \nabla u|^2 \Phi'''(u) - |\nabla v|^2 \Phi'''(v)|_p \\
\leqslant &|\Delta u|_{rp} |\Phi''(u) - \Phi''(v)|_{r'p} + |\Delta u - \Delta v|_{rp} |\Phi''(v)|_{r'p} \\
&+ |\nabla u|^2_{2\sigma p} |\Phi'''(u) - \Phi'''(v)|_{\sigma' p} + \| \nabla u|^2 - |\nabla v|^2|_{\sigma p} |\Phi'''(v)|_{\sigma' p} \\
\leqslant &c(T)\{|\Delta u|_{rp} |\Phi''(u) - \Phi''(v)|_\infty + |\Delta u - \Delta v|_{rp} |\Phi''(v)|_\infty \\
&+ |\nabla u|^2_{2\sigma p} |\Phi'''(u) - \Phi'''(v)|_\infty + \| \nabla u|^2 - |\nabla v|^2|_{\sigma p} |\Phi'''(v)|_\infty\},
\end{aligned}$$

where $c(\delta) := \max\{\delta^{1/r'p}, \delta^{1/\sigma' p}\}$. On the other hand, we have

$$\nabla w \in H_p^{3\theta_1(1+\alpha_2)/4}([0,\delta]; H_p^{3(1-\theta_1)}(\Omega)) \hookrightarrow L_{2\sigma p}([0,\delta] \times \Omega),$$

and

$$\Delta w \in H_p^{\theta_2(1+\alpha_2)/2}([0,\delta]; H_p^{2(1-\theta_2)}(\Omega)) \hookrightarrow L_{rp}([0,\delta] \times \Omega),$$

for some $\theta_1, \theta_2 \in (0,1)$ and for all $w \in B_R(\phi^*)$, provided that $r, \sigma > 1$ are chosen close enough to 1. Therefore, we have

$$|\Delta\Phi'(u) - \Delta\Phi'(v)|_p \leqslant c(\delta)(R + |\phi^*|_{Z_2^{1+\alpha_2}(\delta)}) |u - v|_{Z_2^{1+\alpha_2}(\delta)},$$

which yields (4.2.32). By using the arguments above, it follows that (4.2.33) holds too. Therefore, we obtain (4.2.30)-(4.2.31) by setting $u = w + \phi^*$ and $v = z + \phi^*$. Furthermore, note that $|\phi^*|_{Z_2^{1+\alpha_2}(\delta)} \to 0$ as $\delta \to 0$, since ϕ^* is a fixed function. Hence, the contraction mapping principle yields a unique fixed point $\phi_* \in B_R(0)$ of $\mathcal{T}$. □

The lemma above delivers a unique fixed point $\phi_* \in {}_0Z_2^{1+\alpha_2}(\delta)$, for each function $\phi^* \in Z_2^{1+\alpha_2}$. If we choose (u^*, ϕ^*) as the unique solution of (4.2.21)-(4.2.22) and u_* as in (4.2.27), then $(u, \phi) \in Z_1^{1+\alpha_1}(\delta) \times Z_2^{1+\alpha_2}(\delta)$ is the unique solution of (4.2.23)-(4.2.24), where $u := u_* + u^*$ and $\phi := \phi_* + \phi^*$. In addition, if $\alpha_2 > 1/p$ then the embedding

$$Z_2^{1+\alpha_2}(\delta) \hookrightarrow C^1([0,\delta]; \widetilde{Y}_p^2(1+\alpha_2)) \cap C([0,\delta]; Y_p^2(1+\alpha_2))$$

is valid. This fact allows us to continue the solution ϕ. Indeed, let $\eta > 0$ and consider the space

$$\mathcal{M}_{\phi_*} := \{\psi \in {}_0Z_2^{1+\alpha_2}(\delta + \eta) \; : \; \psi|_{[0,\delta]} = \phi_*\}.$$

The set $\mathcal{M}_{\phi_*}$ is not empty and with the metric induced by $Z_2^{1+\alpha_2}(\delta+\eta)$, we have that $(\mathcal{M}_{\phi_*}, d)$ is a complete metric space, where

$$d(f,g) := |f - g|_{Z_2^{1+\alpha_2}(\delta+\eta)}, \; f, g \in \mathcal{M}_{\phi_*}.$$

We can now apply the contraction mapping principle to $\mathcal{M}_{\phi_*}$. From (4.2.30) and (4.2.31), it is easy to show that $\mathcal{T}$ has a unique fixed point $\overline{\phi}_* \in \mathcal{M}_{\phi_*}$ for some $\delta_1 \in (\delta, \delta+\eta)$, provided $\eta > 0$ is chosen sufficiently small. Hence, the par $(\overline{u}_*, \overline{\phi}_*)$, where $\overline{u}_*$ is obtained by (4.2.27), allows us to define the functions $\phi := \overline{\phi}_* + \phi^*$ and $u := \overline{u}_* + u^*$, which are the unique solution of (4.2.23)-(4.2.24) in $Z_1^{1+\alpha_1}(\delta_1) \times Z_2^{1+\alpha_2}(\delta_1)$. A successive application of this argument yields a solution ϕ on a maximal time interval $[0, t_{max})$, which is characterized by the two equivalent conditions

$$\begin{cases} \lim\limits_{\delta \to t_{max}} |\, \phi(\delta)\, |_{Y_p^2(1+\alpha_2)} \text{ does not exist, or} \\ \lim\limits_{\delta \to t_{max}} |\, \phi_t(\delta)\, |_{\widetilde{Y}_p^2(1+\alpha_2)} \text{ does not exist, if } \alpha_2 > 1/p, \end{cases}$$

and

$$|\, \phi\, |_{Z_2^{1+\alpha_2}(t_{max})} = \infty.$$

Actually, we have proved the following result, which corresponds to the main result of this section.

Theorem 4.2.5. *Let* $p \in (1, \infty)$. *Assume that* $\gamma = 0$ *in* (4.2.5)*, that the kernels* a_i *fulfill the condition (P0), and that* $\Phi \in C^{4-}(\mathbb{R})$. *Furthermore, suppose that* $p > n/4 + 1/(1+\alpha_2)$ *and* $\alpha_i \neq 1/p$ *for* $i = 1, 2$. *Then for some* $0 < \delta < t_{max}$*, the system* (4.2.1)-(4.2.4) *has a unique solution* $(u, \phi) \in Z_1^{1+\alpha_1}(\delta) \times Z_2^{1+\alpha_2}(\delta)$ *if the data satisfy the following conditions:*

(i) $f_i \in H_p^{\alpha_i}([0, \delta]; X)$ *for* $i = 1, 2$*;*

(ii) $u_0 \in Y_p^1(1+\alpha_1)$ *and* $\phi_0 \in Y_p^2(1+\alpha_2)$*;*

(iii) $u_t(0) \in \widetilde{Y}_p^1(1+\alpha_1)$ *and* $\phi_t(0) \in \widetilde{Y}_p^2(1+\alpha_2)$ *if* $\alpha_i > \frac{1}{p}$ *for* $i = 1, 2$.

This result remains true if $\gamma > 0$; in this case condition (iii) has to be modified as in Remark 4.2.2.

The approach used in the sections above can be also applied to the classical Cahn-Hilliard equation with temperature, that is,

$$u_t + \phi_t = \Delta u, \qquad \text{in } J \times \Omega; \tag{4.2.34}$$

$$\phi_t = -\Delta[\Delta\phi - \Phi'(\phi) + u], \qquad \text{in } J \times \Omega; \tag{4.2.35}$$

$$\partial_n u = \partial_n \phi = \partial_n(\Delta\phi) = 0, \qquad \text{on } J \times \partial\Omega; \tag{4.2.36}$$

$$u(0,x) = u_0(x),\ \phi(0,x) = \phi_0(x), \qquad \text{in } \Omega. \tag{4.2.37}$$

Indeed, set $b_i(t) = 1$ for $i = 1, 2$, $f_1(t) = f_2(t) = 0$ in (4.2.1)-(4.2.4) then (4.2.1)-(4.2.4) reduces to the integral version of (4.2.34)-(4.2.37). With this setting, it is not difficult to prove the following result.

Theorem 4.2.6. *Let* $p > n/4+1$. *Assume that* $\Phi \in C^{4-}(\mathbb{R})$. *Then for some* $0 < \delta < t_{max}$, *the system* (4.2.34)-(4.2.37) *has a unique solution* $(u, \phi) \in Z_1^1(\delta) \times Z_2^1(\delta)$ *if the data satisfy the following conditions:*

$$u_0 \in W_p^{2-2/p}(\Omega) \text{ and } \phi_0 \in W_p^{4-4/p}(\Omega).$$

4.3 Global well-posedness

In this section we want to solve the nonlinear system

$$u_t + \phi_t = \gamma \Delta u + \int_{-\infty}^{t} a_1(t-s)\Delta u(s)ds, \qquad \text{in } J \times \Omega; \tag{4.3.1}$$

$$\phi_t = -\int_{-\infty}^{t} a_2(t-s)\Delta\left[\Delta\phi - \Phi'(\phi) + u\right](s)ds, \qquad \text{in } J \times \Omega; \tag{4.3.2}$$

$$\partial_n u = \partial_n \phi = \partial_n(\Delta\phi) = 0, \qquad \text{on } J \times \partial\Omega; \tag{4.3.3}$$

$$u(0,x) = u_0(x),\ \ \phi(0,x) = \phi_0(x), \qquad \text{in } \Omega, \tag{4.3.4}$$

globally in time in the setting considered in the previous sections. We restrict ourselves to the case where the system has trivial history, i.e.

$$u(t,x) = \phi(t,x) = 0, \quad (t,x) \in (-\infty, 0) \times \Omega. \tag{4.3.5}$$

In order to obtain an a-priori estimate for our system, we assume, for instance, that the kernel a_2 is 1-regular and θ_{a_2}-sectorial with $\theta_{a_2} < \pi/2$. Let B be the operator from Theorem 1.1.8 associated with the kernel a_2, and assume that condition (4.3.5) holds. In this way, the system (4.3.1)-(4.3.4) can be written as

$$u_t + \phi_t = \gamma\Delta u + a_1 * \Delta u, \qquad \text{in } J \times \Omega; \tag{4.3.6}$$

$$B\phi_t = -\Delta[\Delta\phi - \Phi'(\phi) + u], \qquad \text{in } J \times \Omega; \tag{4.3.7}$$

$$\partial_n u = \partial_n \phi = \partial_n(\Delta\phi) = 0, \qquad \text{on } J \times \partial\Omega; \tag{4.3.8}$$

$$u(0,x) = u_0\ ,\quad \phi(0,x) = \phi_0, \qquad \text{in } \Omega. \tag{4.3.9}$$

On the other hand, since $\int_\Omega u(t,x)dx$ and $\int_\Omega \phi(t,x)dx$ are conserved quantities, i.e.

$$\int_\Omega u(t,x)dx = \int_\Omega u_0(x)dx = \text{const. and } \int_\Omega \phi(t,x)dx = \int_\Omega \phi_0(x)dx = \text{const.},$$

we may, without loss of generality, assume that

$$\int_\Omega \phi_0(x)dx = \int_\Omega u_0(x)dx = 0.$$

In fact, it suffices to replace the solution ϕ by $\phi - c$ with $c = \frac{1}{|\Omega|}\int_\Omega \phi_0(x)dx$, and to replace Φ' by $\Phi_1'(s) := \Phi'(s+c)$.

Denoting by $v = -\Delta_N^{-1} f$ the unique solution of the problem

$$\begin{aligned} -\Delta v &= f \text{ in } \Omega, \\ \partial_n v &= 0, \text{ on } \partial\Omega, \int_\Omega v dx = 0, \end{aligned}$$

where $f \in L_2(\Omega)$ and $\int_\Omega f dx = 0$, we have that

$$-\Delta_N^{-1} B\phi_t = \Delta\phi - \Phi'(\phi) + u \tag{4.3.10}$$

is equivalent to (4.3.7). On the other hand, multiplying (4.3.6) by u and (4.3.10) by ϕ_t, adding and integration by parts yields

$$\begin{aligned} &\frac{d}{dt}\{\int_\Omega (\frac{1}{2}[| u |^2 + \gamma | \nabla u |^2] + \frac{1}{2} | \nabla\phi |^2 + \Phi(\phi))dx\} \\ &\quad + \langle a_1 * \nabla u, \nabla u\rangle + \left\langle B(-\Delta_N^{-1/2}\phi_t), -\Delta_N^{-1/2}\phi_t \right\rangle = 0. \end{aligned} \tag{4.3.11}$$

Here $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $L_2(\Omega)$. Observe that the presence of the terms $\langle a_1 * \cdot, \cdot\rangle$ and $\langle B\cdot, \cdot\rangle$ in the energy equality (4.3.11) forces us to impose extra conditions on the kernels a_1 and a_2 in order to obtain an a-priori estimate. Assume that a_1 satisfies condition (P1), and a_2 the condition (P2). Integrating (4.3.11) over $[0, \delta]$ with $\delta < t_{max}$ and using the condition (4.1.2) yields

$$\begin{aligned} &\frac{1}{2}\left(| u |_2^2 + \gamma | \nabla u |_2^2\right) + \frac{1}{2}\left(| \nabla\phi |_2^2 - m_1 | \phi |_2^2\right) \\ &\quad + \int_0^\delta \langle a_1 * \nabla u, \nabla u\rangle \, ds + \int_0^\delta \left\langle B(-\Delta_N^{-1/2}\phi_t), -\Delta_N^{-1/2}\phi_t \right\rangle ds \\ &\quad \leqslant \int_\Omega | \Phi(\phi_0) | \, dx + \frac{1}{2} | u_0 |_2^2 + \frac{1}{2} | \nabla\phi_0 |_2^2 + m_2 | \Omega | \, . \end{aligned} \tag{4.3.12}$$

On the other hand, note that from the growth condition (4.1.3) it follows that

$$| \Phi''(s) | \leqslant C(1 + | s |^{\beta+1}), \tag{4.3.13}$$

$$| \Phi'(s) | \leqslant C(1 + | s |^{\beta+2}), \tag{4.3.14}$$

$$| \Phi(s) | \leqslant C(1 + | s |^{\beta+3}). \tag{4.3.15}$$

Hence, the left-hand side of (4.3.12) remains bounded from above, provided that $\phi_0 \in L_{\beta+3}(\Omega)$, which can be ensured in case $\beta < 3$ and $n \leqslant 3$ by the embedding $W_2^1 \hookrightarrow L_6$. Further, from Poincaré's inequality it follows that (4.3.12) is also bounded from below by 0. Now, we estimate the term $\Delta\Phi'(\phi)$ by using the Gagliardo-Nirenberg inequality. Observe that $\Delta\Phi'(\phi) = \Phi''(\phi)\Delta\phi + \Phi'''(\phi) \mid \nabla\phi \mid^2$. From (4.3.13), Hölder's inequality, and the Gagliardo-Nirenberg inequality, we obtain

$$\mid \Phi''(\phi)\Delta\phi \mid_p \leqslant K[1_{2p} + \mid \phi \mid_{W_p^4}^{\rho_1(\beta+1)} \mid \phi \mid_{L_6}^{(1-\rho_1)(\beta+1)}] \mid \phi \mid_{W_p^4}^{\rho_2} \mid \phi \mid_{L_6}^{1-\rho_2}, \tag{4.3.16}$$

provided that

$$(\rho_2 + (\beta+1)\rho_1)(4 - \frac{n}{p} + \frac{n}{6}) = 2 - \frac{n}{p} + \frac{n}{6}(\beta+2) \tag{4.3.17}$$

is satisfied for some $\rho_1, \rho_2 \in (0,1)$. Similarly, we estimate the term $\mid \nabla\phi \mid^2 \Phi'''(\phi)$ obtaining

$$\mid \Phi'''(\phi) \mid \nabla\phi \mid^2 \mid_p \leqslant K[1_{2p} + \mid \phi \mid_{W_p^4}^{\rho_1\beta} \mid \phi \mid_{L_6}^{(1-\rho_1)\beta}] \mid \phi \mid_{W_p^4}^{2\rho_2} \mid \phi \mid_{L_6}^{2-2\rho_2}, \tag{4.3.18}$$

provided the condition

$$(2\rho_2 + \beta\rho_1)(4 - \frac{n}{p} + \frac{n}{6}) = 2 - \frac{n}{p} + \frac{n}{6}(\beta+2) \tag{4.3.19}$$

holds for some $\rho_1, \rho_2 \in (0,1)$. Observe that the conditions (4.3.17) and (4.3.19) are satisfied if, e.g., $\rho_2 + (\beta+1)\rho_1 < 1$, and $2\rho_2 + \beta\rho_1 < 1$, and $p \geqslant 2$. Furthermore, if we choose ρ_i, for $i = 1,2$ such that $\rho, \varepsilon \in (1/p, 1)$, where $\rho := \rho_2 + (\beta+1)\rho_1$ and $\varepsilon := 2\rho_2 + \beta\rho_1$, then from (4.3.16)-(4.3.18) and Hölder's inequality we have

$$\begin{aligned} \mid \Delta\Phi'(\phi) \mid_{p,p} \leqslant & K_0[\mid \phi \mid_{L_p(W_p^4)}^{\rho_2} \mid \phi \mid_{L_\infty(L_6)}^{1-\rho_2} + \mid \phi \mid_{L_p(W_p^4)}^{2\rho_2} \mid \phi \mid_{L_\infty(L_6)}^{2-2\rho_2} \\ & + \mid \phi \mid_{L_p(W_p^4)}^{\rho} \mid \phi \mid_{L_\infty(L_6)}^{\beta+1-\rho} + \mid \phi \mid_{L_p(W_p^4)}^{\varepsilon} \mid \phi \mid_{L_\infty(L_6)}^{\beta+2-\varepsilon}] \\ \leqslant & K_1[\mid \phi \mid_{Z_2^{1+\alpha_2}(\delta)}^{\rho_2} + \mid \phi \mid_{Z_2^{1+\alpha_2}(\delta)}^{2\rho_2} + \mid \phi \mid_{Z_2^{1+\alpha_2}(\delta)}^{\rho} \\ & + \mid \phi \mid_{Z_2^{1+\alpha_2}(\delta)}^{\varepsilon}], \end{aligned} \tag{4.3.20}$$

where K_0 and K_1 are positive constants, which depend only on Ω. On the other hand, by maximal L_p-regularity, there is a constant $M := M(T) > 0$, such that

$$\mid u \mid_{Z_1^{1+\alpha_1}(\delta)} + \mid \phi \mid_{Z_2^{1+\alpha_2}(\delta)} \leqslant M\left(1 + \mid \Delta\Phi'(\phi) \mid_{p,p}\right). \tag{4.3.21}$$

Hence, from (4.3.20) and (4.3.21) it follows that

$$\mid \phi \mid_{Z_2^{1+\alpha_2}(\delta)} \leqslant M_0,$$

where the constant M_0 is independent of $\delta < t_{max}$. Therefore

$$\mid \phi \mid_{Z_2^{1+\alpha_2}(t_{max})} < \infty.$$

This in turn yields the boundedness of $u \in Z_1^{1+\alpha_1}(t_{max})$. Hence the global existence for (4.3.6)-(4.3.7) follows.

We can now state our main result of this chapter.

Theorem 4.3.1. *Let* $p \geqslant 2$*,* $n \leqslant 3$*, and* $\gamma = 0$*. Assume that* a_i *satisfies the condition (P0) for* $i = 1, 2$*, and that the potential* Φ *fulfils the conditions* (4.1.1)-(4.1.3)*. Further, suppose that the conditions (P1) and (P2) hold. Then the system* (4.3.6)-(4.3.7) *has a unique global solution* $(u, \phi) \in Z_1^{1+\alpha_1} \times Z_2^{1+\alpha_2}$ *if the following conditions hold:*

$$u_0 \in Y_p^1(1 + \alpha_1) \text{ and } \phi_0 \in Y_p^2(1 + \alpha_2).$$

In case that $\gamma > 0$, the result remains true if we set $\alpha_1 = 0$.

The arguments used above can be applied also to the classical Cahn-Hilliard equation (4.2.34)-(4.2.35) to obtain a global solution.

Theorem 4.3.2. *Let* $p \geqslant 2$ *and* $n \leqslant 3$*. Assume that the potential* Φ *fulfils the conditions* (4.1.1)-(4.1.3)*. Then the system* (4.2.34)-(4.2.35) *has a unique global solution* $(u, \phi) \in Z_1^1 \times Z_2^1$ *if the following conditions are satisfied:*

$$u_0 \in W_p^{2-2/p}(\Omega) \text{ and } \phi_0 \in W_p^{4-4/p}(\Omega).$$

Chapter 5

Convergence to steady state

In this chapter we study the asymptotic behavior of global bounded solutions of the semilinear evolutionary equation with memory

$$\dot{v}(t) + \int_0^t a(t-s)\mathcal{E}'(v(s))ds = f(t), \quad t \geqslant 0, \tag{5.0.1}$$

on a real Hilbert space H. We suppose that the nonlinear term $\mathcal{E}'$ is the Fréchet derivative of a functional $\mathcal{E} \in C^1(V)$, where V is another Hilbert space which injects continuously and densely into H. In order to prove the convergence to steady state of equation (5.0.1), we assume that $\mathcal{E}$ satisfies the so-called Łojasiewicz-Simon inequality. Examples of functionals $\mathcal{E}$, which satisfy the Łojasiewicz-Simon inequality can be found in Haraux and Jendoubi [HJ99], Haraux, Jendoubi, and Kavian [HJK03], and Chill [Chi03].

Under suitable conditions on the scalar kernel a and the function f, we show that the equation (5.0.1) is dissipative and gradient-like in the sense that for every global bounded solution v with relative compact range in V the ω-limit set is contained in the set of steady states of (5.0.1). For this we adopt ideas from Vergara and Zacher [VZ06], where a Lyapunov function was constructed in the finite dimensional case and employed to prove convergence to steady state in the framework of the Łojasiewicz inequality.

Using the ideas of this approach we prove also that any global bounded solution of a conserved phase field model with memory converges to a steady state.

5.1 Preliminaries and main assumptions

Let V and H be real Hilbert spaces (with inner product $\langle \cdot, \cdot \rangle_V$ resp. $\langle \cdot, \cdot \rangle_H$) such that V is densely and continuously embedded into H. We shall identify H with its dual H', that is, we have

$$V \hookrightarrow H \approx H' \hookrightarrow V'.$$

The operator $\mathcal{E}'$ is nonlinear and continuous from V into V', and it is the Fréchet derivative of a functional $\mathcal{E} \in C^1(V)$.

Definition 5.1.1. *We say that the function $\mathcal{E}$ satisfies the* Łojasiewicz-Simon inequality *near some point $\vartheta \in V$, if there exist constants $\theta \in (0, 1/2]$, $C > 0$, and $\sigma > 0$ such that for all $v \in V$ with $| v - \vartheta |_V \leqslant \sigma$ there holds*

$$| \mathcal{E}(v) - \mathcal{E}(\vartheta) |^{1-\theta} \leqslant C \, | \mathcal{E}'(v) |_{V'} \, .$$

The number θ will be called the Łojasiewicz exponent. This exponent plays an important role with regard to the rate of convergence to a stationary point.

We will assume that the kernel a is nonnegative and satisfies the following assumptions:

(A1) There is a nonnegative nonincreasing kernel $k \in L^1_{loc}(\mathbb{R}_+)$ such that

$$\int_0^t k(s) a(t-s) ds = 1, \;\; t > 0.$$

(A2) There is a constant $\gamma > 0$ such that the solution e of

$$e(t) + \gamma \int_0^t e(s) ds = k(t), \;\; t > 0, \tag{5.1.1}$$

is nonnegative.

Remark 5.1.1. For each $\gamma > 0$ the unique solution of (5.1.1) is given by

$$e_\gamma(t) := k(t) - \gamma(e^{-\gamma \cdot} * k)(t), \;\; t > 0.$$

Hence, if condition (A2) holds, by decreasing γ, we may assume that e is strictly positive and strictly decreasing on $(0, \infty)$. Furthermore, $e \in L_1(\mathbb{R}_+)$ and $\lim_{t\to\infty} e(t) = 0$, therefore $k_\infty := \lim_{t\to\infty} k(t) = \gamma \int_0^\infty e(s) ds > 0$.

Remark 5.1.2. The conditions (A1)-(A2) imply that $a \in L_1(\mathbb{R}_+)$.

For the function $f \in L_1(\mathbb{R}_+; H)$ we assume that $\frac{d}{dt}(k * f)(t) =: g(t)$ satisfies the condition

(B1) $g \in L_1(\mathbb{R}_+; H) \cap L_2(\mathbb{R}_+; H)$ is such that

$$\int_0^\infty \left(\int_s^\infty | g(\tau) |_H^2 \, d\tau \right)^{\frac{1}{2}} ds < \infty.$$

Lemma 5.1.2. *Let* H *be a real Hilbert space. Let* $k \in L_{1,loc}(\mathbb{R}_+)$ *be a nonnegative and nonincreasing kernel. Assume that there is a nonnegative kernel* $a \in L_{1,loc}(\mathbb{R}_+)$ *such that* $k * a = 1$*. Let* $v \in L_{2,loc}(\mathbb{R}_+; H)$ *and suppose that* $k * v \in {}_0H^1_{2,loc}(\mathbb{R}_+; H)$*. Then*

(i) $2\int_s^t \langle \frac{d}{d\tau}(k * v)(\tau), v(\tau)\rangle_H \, d\tau \geqslant \left(k * |v|_H^2\right)(t) - \left(k * |v|_H^2\right)(s) + \int_s^t k(\tau)\, |v(\tau)|_H^2 \, d\tau,$ *for all* $t > 0$ *and a.a.* $s \in (0, t)$*.*

(ii) *If in addition* $k * |v|_H^2 \in {}_0H^1_{1,loc}(\mathbb{R}_+)$ *then*

$$2\left\langle \frac{d}{dt}(k * v)(t), v(t)\right\rangle_H \geqslant \frac{d}{dt}(k * |v|_H^2)(t) + k(t)\, |v(t)|_H^2,$$

for a.a. $t > 0$*.*

Proof. Firstly, let $T > 0$ be arbitrarily fixed and let $k_\mu \in H^1_1([0, T])$ be a nonnegative and nonincreasing kernel. A simple computation yields the identity

$$2\left\langle \frac{d}{dt}(k_\mu * v)(t), v(t)\right\rangle_H = \frac{d}{dt}\left(k_\mu * |v|_H^2\right)(t) + k_\mu(t)\, |v(t)|_H^2 + \int_0^t (-\dot{k}_\mu(\tau))\, |v(t) - v(t-\tau)|_H^2 \, d\tau.$$

Hence, since $-\dot{k}_\mu(t) \geqslant 0$, $t > 0$, we obtain that

$$2\left\langle \frac{d}{dt}(k_\mu * v)(t), v(t)\right\rangle_H \geqslant \frac{d}{dt}\left(k_\mu * |v|_H^2\right)(t) + k_\mu(t)\, |v(t)|_H^2, \; t > 0; \tag{5.1.2}$$

as well as its integral version, that is

$$2\int_s^t \left\langle \frac{d}{d\tau}(k_\mu * v)(\tau), v(\tau)\right\rangle_H d\tau \geqslant \left(k_\mu * |v|_H^2\right)(t) - \left(k_\mu * |v|_H^2\right)(s) + \int_s^t k_\mu(\tau)\, |v(\tau)|_H^2 \, d\tau, \tag{5.1.3}$$

for $0 < s < t \leqslant T$.

Next, we proceed by an approximation argument. Let B_i be the operator defined in Theorem 1.4.7 associated with the kernel k, that is

$$B_i v = \frac{d}{dt} k * v, \; i = 1, 2,$$

with domain

$$D(B_1) = \left\{v \in L_1([0, T]) : \; k * v \in {}_0H^1_1([0, T])\right\};$$
$$D(B_2) = \left\{v \in L_2([0, T]; H) : \; k * v \in {}_0H^1_2([0, T]; H)\right\},$$

respectively. These operators are m-accretive on $\mathbb{R}$ respectively H. The Yosida approximation $B_{i,\mu}$ of B_i is defined by

$$B_{i,\mu} = B_i(1 + \mu B_i)^{-1}, \; \mu > 0.$$

Denote by s_μ the solution of the 1-dimensional Volterra equation

$$s(t) + \mu(a * s)(t) = 1, \; t > 0. \tag{5.1.4}$$

Since by assumption a is a completely positive kernel, it follows from [Prü93, Prop. 4.5] that the solution s_μ of (5.1.4) is positive and nonincreasing in $(0, \infty)$, for every $\mu > 0$. In addition, it is not difficult to see by differentiating (5.1.4) that $s_\mu \in H_1^1([0,T])$. Define then a sequence of kernels $k_\mu \in H_1^1([0,T])$ by

$$k_\mu(t) = \frac{1}{\mu} s_{\frac{1}{\mu}}(t), \; t > 0, \; \mu = \frac{1}{n}, \; n \in \mathbb{N}.$$

On the other hand, since $k * a = 1$ we obtain that the Yosida approximation is given by

$$B_{i,\mu} v = \frac{d}{dt}(k_\mu * v), \; v \in D(B_{i,\mu}), \; i = 1, 2.$$

Therefore, since $1 \in D(B_1)$ and by assumption $v \in D(B_2)$, we have that

$$k_\mu * v \to k * v \text{ in } {}_0H_2^1([0,T]; H), \text{ as } \mu \to 0; \tag{5.1.5}$$

$$k_\mu \to k \quad \text{in} \quad L_1([0,T]), \text{ as } \mu \to 0. \tag{5.1.6}$$

In particular $d/dt(k_\mu * v) \to d/dt(k * v)$ in $L_2([0,T]; H)$, as well as

$$\left\langle \frac{d}{dt}(k_\mu * v), v \right\rangle_H \to \left\langle \frac{d}{dt}(k * v), v \right\rangle_H \text{ in } L_1([0,T]); \tag{5.1.7}$$

$$k_\mu * |\, v\,|_H^2 \to k * |\, v\,|_H^2 \qquad \text{in } L_1([0,T]). \tag{5.1.8}$$

Hence, from (5.1.6) there is a subsequence $\mu_n \to 0$ as $n \to \infty$ such that $k_{\mu_n} \to k$ for a.e. $(0,T)$, as well as from (5.1.8), we obtain that $k_{\mu_n} * |\, v\,|_H^2\,(s) \to k * |\, v\,|_H^2\,(s)$ a.a. $s \in (0,T)$. Now, let $t \in (0,T)$ be arbitrary fixed and choose $s \in (0,t)$ such that

$$\lim_{n \to \infty} k_{\mu_n} * |\, v\,|_H^2\,(s) = k * |\, v\,|_H^2\,(s). \tag{5.1.9}$$

On the other hand, from (5.1.7) and by convergence dominated theorem we obtain that

$$\lim_{n \to \infty} \int_s^t \left\langle \frac{d}{dt}(k_{\mu_n} * v), v \right\rangle_H (\tau) d\tau = \int_s^t \left\langle \frac{d}{dt}(k * v), v \right\rangle_H (\tau) d\tau. \tag{5.1.10}$$

Therefore, from Fatou's lemma, (5.1.6), and (5.1.8)-(5.1.10) in (5.1.3) yield then (i).

Now, we assume that $k * |\, v\,|_H^2 \in {}_0H_1^1([0,T])$; hence $|\, v\,|_H^2 \in D(B_1)$. Therefore, by Yosida's approximation we obtain that

$$\frac{d}{dt} k_\mu * |\, v\,|_H^2 \to \frac{d}{dt} k * |\, v\,|_H^2 \text{ in } L_1([0,T]), \text{ as } \mu \to 0. \tag{5.1.11}$$

Hence, the desired inequality in (ii) follows from (5.1.2) by passing to limit for a.e. $t > 0$. □

The following proposition provides sufficient conditions on the kernel a and the function v, such that all assumptions in Lemma 5.1.2 hold.

Proposition 5.1.3. *Let* Y *be a Banach space of class* $\mathcal{HT}$. *Let* $a \in \mathcal{K}^1(\alpha, \theta)$ *with* $\alpha \in (0, 1)$ *and* $\theta < \pi$. *Suppose that there exists a kernel* $k \in L_{1,loc}(\mathbb{R}_+)$ *positive and nonincreasing in* $(0, \infty)$ *such that* $a * k = 1$ *holds. If* $v \in {}_0H_2^\alpha([0, T]; Y)$ *then*

$$k * v \in {}_0H_2^1([0, T]; Y) \text{ and } k* \mid v \mid_Y^2 \in {}_0H_1^1([0, T]).$$

Proof. Let B be the operator defined in Theorem 1.4.7 associated with the kernel k with domain

$$D(B) = \{v \in L_2([0, T]; Y) : k * v \in {}_0H_2^1([0, T]; Y)\}.$$

Since $a \in \mathcal{K}^1(\alpha, \theta)$ we obtain from Corollary 1.4.5 that $D(B) = {}_0H_2^\alpha([0, T]; Y)$, hence $k * v \in {}_0H_2^1([0, T]; Y)$.

Next, let $p \in (1, \min\{2, 1/(1 - \alpha)\})$, from the characterization of H_p^α via differences (see [Tri92]), it follows that there exits a constant $C(J) > 0$ such that

$$| \mid v \mid_Y^2 |_{H_p^\alpha(J)} \leqslant C(J) \mid v \mid_{H_2^\alpha(J;Y)}^2$$

holds. Therefore, $\mid v \mid_Y^2 \in {}_0H_p^\alpha(J)$, hence $k* \mid v \mid_Y^2 \in {}_0H_p^1([0, T]) \hookrightarrow {}_0H_1^1([0, T])$. □

5.2 The model equation

Let $f \in C(J; V')$ and $\mathcal{E} \in C^1(V)$ be as above, with $J := [0, T]$, $T > 0$. We consider the model equation

$$\dot{v} + a * \mathcal{E}'(v) = f, \ t \in J, \tag{5.2.1}$$

where the scalar kernel a is locally integrable on $\mathbb{R}_+$.

Remark 5.2.1. There is no existence result for solutions of the equation (5.2.1) under the general hypotheses given above. In some concrete examples, however, existence of solutions is known. Indeed, set $H = L_2(\Omega)$ and $V = H_2^1(\Omega)$. Assume that the energy functional $\mathcal{E}$ is of the form

$$\mathcal{E}(v) = \frac{1}{2}\alpha(v, v) + \int_\Omega \Phi(v)dx, \tag{5.2.2}$$

where $\alpha : V \times V \to \mathbb{R}$ is a bounded coercive bilinear form on V and Φ is a nonlinear term. Then (5.2.1) can be written as a semilinear Volterra equation of variational type, that is

$$\langle w, v(t)\rangle_V + \int_0^t b(t - s)\alpha(w, v(s))ds = \langle w, F(v, t)\rangle_{V,V'}, \ t \in J, \ w \in V, \tag{5.2.3}$$

where $b = 1 * a$ and $\langle w, F(v, t)\rangle_{V,V'} = \langle w, 1 * f\rangle_{V,V'} - 1 * a * \langle w, \Phi'(v)\rangle_{V,V'}$. By [Prü93, Thm. 7.3] (in its scalar version) we obtain that the linearized problem of equation (5.2.3) is well-posedness. This together with the contraction mapping principle and a Lipschitz condition on F yields the local well-posedness of (5.2.3). Global well-posedness is obtained by using the coercivity of the form α and assuming certain growth conditions on the nonlinear term F.

Now, assuming the condition (A1) we rewrite equation (5.2.1) as

$$\frac{d}{dt}(k * \dot{v}) + \mathcal{E}'(v) = g, \tag{5.2.4}$$

where $g(t) := \frac{d}{dt}(k * f)(t)$. In addition, from the condition (A2) equation (5.2.4) can be written as

$$\frac{d}{dt}(e * \dot{v}) + \gamma(e * \dot{v}) + \mathcal{E}'(v) = g. \tag{5.2.5}$$

The following definition gives a notion of solution of (5.2.5).

Definition 5.2.1. *A function* $v \in C(J; V)$ *is called*

(a) *a* *weak solution* *of* (5.2.5) *if* $v \in H^1_2(J; V') \cap C(J; V)$ *and* $e * \dot{v} \in {}_0H^1_2(J; V')$, *and* (5.2.5) *holds a.e. on* J;

(b) *a* *mild solution* *of* (5.2.5) *if* $v \in H^1_2(J; H) \cap C(J; V)$ *and* $e * \dot{v} \in {}_0H^1_2(J; H)$, *and* (5.2.5) *holds a.e. on* J;

(c) *a* *global bounded weak (mild) solution* *of* (5.2.5) *if* v *is a weak (mild) solution on each interval* $J = [0, T]$, $T > 0$, *and* $v \in L_\infty(\mathbb{R}_+; V)$.

In the sequel we will assume that v is a mild solution of (5.2.5).

Now, we will derive energy estimates. Let v be a mild solution of (5.2.5) and let $g \in L_2(\mathbb{R}_+; H)$. We multiply equation (5.2.5) by $\dot{v}$ to obtain

$$\left\langle \frac{d}{dt}(e * \dot{v}), \dot{v} \right\rangle_H + \gamma \langle e * \dot{v}, \dot{v} \rangle_H + \frac{d}{dt}\mathcal{E}(v) = \langle g, \dot{v} \rangle_H .$$

Assuming that $e* \mid \dot{v} \mid_H^2 \in H^1_1(J)$, then from Lemma 5.1.2 (ii) and Young's inequality, it follows that

$$\frac{d}{dt}\left\{\frac{1}{2}(e* \mid \dot{v} \mid_H^2) + \mathcal{E}(v) + \frac{1}{2\epsilon}\int_t^\infty \mid g(s) \mid_H^2 ds\right\} \leqslant -\frac{1}{2}\left\{(k(t) - \epsilon) \mid \dot{v} \mid_H^2 + \gamma e* \mid \dot{v} \mid_H^2\right\}.$$

Hence, for any global solution v of (5.2.4) the function $\Psi : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\Psi(t) := \frac{1}{2}(e* \mid \dot{v} \mid_H^2)(t) + \mathcal{E}(v(t)) + \frac{1}{2\epsilon}\int_t^\infty \mid g(s) \mid_H^2 ds \tag{5.2.6}$$

is differentiable almost everywhere and decreasing on $\mathbb{R}_+$, provided that $k_\infty > \epsilon$.

Actually, we have proved the following result.

Proposition 5.2.2. *Let* v *be a mild solution of equation* (5.2.5) *such that* $e * |\dot{v}|_H^2 \in H_1^1(J)$. *Assume (A1)-(A2) and* $g \in L_2(\mathbb{R}_+; H)$. *Then the function* $\Psi : J \to \mathbb{R}$ *defined by* (5.2.6) *is absolutely continuous and decreasing on* J.

Remark 5.2.2. By using Lemma 5.1.2 (i) one can also show without the aid of the assumption $e * |\dot{v}|_H^2 \in H_1^1(J)$, that the function Ψ is decreasing on $\mathbb{R}_+$. Hence, from [HS65, Thm. 17.12], we have that Ψ has a finite derivative a.e. on J, where $J \subset \mathbb{R}_+$ is a compact interval.

For the next result we recall the notion of ω-limit set. For every bounded solution v of (5.2.5) the ω-limit set is defined by

$$\omega(v) = \{\vartheta \in V : \text{ there exists } (t_n) \nearrow \infty \text{ s.t. } v(t_n) \to \vartheta \text{ in } V\}.$$

Proposition 5.2.3. *Let* v *be a global bounded mild solution of equation* (5.2.5) *such that* $e * |\dot{v}|_H^2 \in H_{1,loc}^1(\mathbb{R}_+)$. *Assume (A1)-(A2) and (B1), and that the set* $\{v(t) : t \geqslant 0\}$ *is relatively compact in* V. *Then*

(i) $\dot{v} \in L_2(\mathbb{R}_+; H)$.

(ii) *The potential* $\mathcal{E}$ *is constant on* $\omega(v)$ *and* $\lim_{t\to\infty} \mathcal{E}(v(t))$ *exists.*

(iii) *For every* $\vartheta \in \omega(v)$ *one has* $\mathcal{E}'(\vartheta) = 0$.

Proof. Choose $\epsilon > 0$ small enough such that $k_\infty > \epsilon$, and define $\tilde{k}_\infty := k_\infty - \epsilon > 0$. Then the function Ψ defined by (5.2.6) is such that

$$-\frac{d}{dt}\Psi(t) \geqslant \frac{1}{2}\tilde{k}_\infty |\dot{v}(t)|_H^2 + \frac{\gamma}{2}\int_0^t e(s) |\dot{v}(t-s)|_H^2 \, ds, \; t > 0,$$

holds. Therefore, $\dot{v} \in L_2(\mathbb{R}_+; H)$ and $e * |\dot{v}|_H^2 \in L_1(\mathbb{R}_+)$. Since the solution v has relatively compact range in V, it follows that the $\omega(v)$ is nonempty, compact and connected. Let $\vartheta \in \omega(v)$ and choose $t_n \nearrow \infty$ such that $v(t_n) \to \vartheta$ in V. Since $\dot{v} \in L_2(\mathbb{R}_+; H)$ we obtain

$$v(t_n + s) = v(t_n) + \int_{t_n}^{t_n+s} \dot{v}(\tau) d\tau \to \vartheta \;\text{ in }\; H, \;\; \text{for every } \; s \in [0, 1].$$

This, together with the relative compactness of the trajectory, implies that $v(t_n + s) \to \vartheta$ in V for every $s \in [0, 1]$. Therefore, $\lim_{n\to\infty} \mathcal{E}(v(t_n + s)) = \mathcal{E}(\vartheta)$ for every $s \in [0, 1]$, and thus, by the dominated convergence theorem,

$$\mathcal{E}(\vartheta) = \lim_{n\to\infty} \int_0^1 \mathcal{E}(v(t_n + s)) ds.$$

In addition, integrating $\Psi(t_n + \cdot)$ defined in (5.2.6) over $[0, 1]$, we obtain

$$\mathcal{E}(\vartheta) + \lim_{n\to\infty} \int_{t_n}^{t_n+1} \left[\frac{1}{2} e * |\dot{v}|_H^2 (s) + \frac{1}{2\epsilon}\int_s^\infty |g(\tau)|_H^2 \, d\tau\right] ds = \lim_{n\to\infty} \int_0^1 \Psi(t_n + s) ds = \Psi_\infty.$$

From assumption (B1), it follows that $\lim_{n\to\infty}\int_{t_n}^{t_n+1}\int_s^\infty |\, g(\tau)\,|_H^2\, d\tau ds = 0$, and since $e * |\,\dot{v}\,|_H^2 \in L_1(\mathbb{R}_+)$, we obtain that $\mathcal{E}(\vartheta) = \Psi_\infty$, that is $\mathcal{E}$ is constant on $\omega(v)$. Further, as a consequence of the above, we obtain that $(e* |\,\dot{v}\,|_H^2)(t) \to 0$ as $t \to \infty$. Indeed, if the contrary was true then there would be $\epsilon > 0$ and a sequence $t_n \to \infty$ as $n \to \infty$ such that $(e* |\,\dot{v}\,|_H^2)(t_n) \geqslant \epsilon$ for all $n \in \mathbb{N}$. By compactness, there exists a subsequence t_{n_k} such that $\mathcal{E}(v(t_{n_k})) \to \Psi_\infty$ as $k \to \infty$, hence $(e* |\,\dot{v}\,|_H^2)(t_{n_k}) \to 0$ as $k \to \infty$, a contradiction. Hence $(e* |\,\dot{v}\,|_H^2)(t) \to 0$ as $t \to \infty$. Moreover, we see that $\lim_{t\to\infty} \mathcal{E}(v(t)) = \Psi_\infty$. Hence the claim (ii) is proved.

Next, since $\mathcal{E} \in C^1(V)$, we have that $\mathcal{E}'(v(t_n + s)) \to \mathcal{E}'(\vartheta)$ in V' for every $s \in [0, 1]$. Further, using the dominated convergence theorem and equation (5.2.4) we obtain that

$$\begin{aligned}\mathcal{E}'(\vartheta) &= \lim_{n\to\infty}\int_0^1 \mathcal{E}'(v(t_n+s))ds \\ &= \lim_{n\to\infty}\left[-\int_0^1 \frac{d}{dt}(k*\dot{v})(t_n+s)ds + \int_{t_n}^{t_n+1} g(s)ds\right] \\ &= -\lim_{n\to\infty}\{(k*\dot{v})(t_n+1) - (k*\dot{v})(t_n)\} = 0.\end{aligned} \tag{5.2.7}$$

Indeed, since $e \in L_1(\mathbb{R}_+)$ and $e * |\,\dot{v}\,|_H^2\,(t) \to 0$ as $t \to \infty$, it follows from Jensen's inequality that $(e * \dot{v})(t) \to 0$ as $t \to \infty$ in H. Furthermore, since $e * |\,\dot{v}\,|_H^2 \in L_1(\mathbb{R}_+)$, it follows that $e * \dot{v} \in L_2(\mathbb{R}_+; H)$. Hence, using the definition of k in (5.2.7), we obtain that

$$\mathcal{E}'(\vartheta) = -\gamma \lim_{n\to\infty}\int_{t_n}^{t_n+1}(e*\dot{v})(s)ds = 0.$$

With this, our claims are proved. □

In order to state our main result, we firstly define $\langle\cdot,\cdot\rangle_{V'}$ by

$$\langle v, u\rangle_{V'} = (R^{-1}v, u)_{V,V'},\ u, v \in V',$$

where $R : V \to V'$ stands for the Riesz map and $R^{-1} : V' \to V$ its inverse, we will denote this in the sequel as K.

Theorem 5.2.4. *Let v be a global mild solution of equation* (5.2.5) *such that* $e* |\,\dot{v}\,|_H^2 \in H^1_{1,loc}(\mathbb{R}_+)$. *Suppose that*

(i) *the set* $\{v(t) : t \geqslant 0\}$ *is relatively compact in* V*;*

(ii) *(A1)-(A2) and (B1) hold;*

(iii) $\mathcal{E} \in C^2(V)$*;*

(iv) *for every* $v \in V$ *the operator* $K \circ \mathcal{E}''(v) \in B(V)$ *extends to an element of* $B(H)$*, and the mapping* $K \circ \mathcal{E}''$ *is continuous from* V *into* $B(H)$ *equipped with the strong operator topology;*

(v) $\mathcal{E}$ *satisfies the Łojasiewicz-Simon inequality near each point* $\vartheta \in \omega(v) \subset V$.

Then $\lim_{t\to\infty} v(t) = \vartheta$ *in* V, *and* ϑ *is a stationary solution, i.e.* $\mathcal{E}'(\vartheta) = 0$.

Proof. From the assumptions (iii), (iv) and equation (5.2.5) we obtain

$$\begin{aligned}\frac{d}{dt}\left\langle \mathcal{E}'(v), e * \dot{v}\right\rangle_{V'} &= \frac{d}{dt}\left\langle K\mathcal{E}'(v), e * \dot{v}\right\rangle_{V,V'} = \frac{d}{dt}\left\langle K\mathcal{E}'(v), e * \dot{v}\right\rangle_{H} \\ &= \left\langle K \circ \mathcal{E}''(v)\dot{v}, e * \dot{v}\right\rangle_{H} + \left\langle \mathcal{E}'(v), \frac{d}{dt}(e * \dot{v})\right\rangle_{V'} \\ &= \left(K \circ \mathcal{E}''(v)\dot{v}, e * \dot{v}\right)_{H} - |\, \mathcal{E}'(v)\,|_{V'}^2 + \left\langle \mathcal{E}'(v), g - \gamma e * \dot{v}\right\rangle_{V'}.\end{aligned}$$

Since, $K\circ\mathcal{E}''(v(t))$ is uniformly bounded for $t > 0$ in H, it follows from the uniform boundedness principle and Jensen's inequality we have

$$\left(K \circ \mathcal{E}''(v)\dot{v}, e * \dot{v}\right)_{H} \leqslant M\,|\,\dot{v}\,|_H|\, e * \dot{v}\,|_H \leqslant \frac{M}{2}\,|\,\dot{v}\,|_H^2 + \frac{M\,|\,e\,|_1}{2} e * |\,\dot{v}\,|_H^2 .$$

In addition,

$$\left\langle \mathcal{E}'(v), g - \gamma e * \dot{v}\right\rangle_{V'} \leqslant \frac{1}{2}\,|\,\mathcal{E}'(v)\,|_{V'}^2 + \gamma^2\,|\,e\,|_1\, e * |\,\dot{v}\,|_H^2 + |\,g\,|_H^2 .$$

Hence,

$$-\frac{d}{dt}\left\langle \mathcal{E}'(v), e * \dot{v}\right\rangle_{V'} \geqslant -\frac{M}{2}\,|\,\dot{v}\,|_H^2 - \frac{M\,|\,e\,|_1}{2} e * |\,\dot{v}\,|_H^2 - \gamma^2\,|\,e\,|_1\, e * |\,\dot{v}\,|_H^2 - |\,g\,|_H^2 + \frac{1}{2}\,|\,\mathcal{E}'(v)\,|_{V'}^2 .$$

Next, let $\vartheta \in \omega(v)$, and define a new energy function $\Upsilon : \mathbb{R}_+ \to \mathbb{R}$ by

$$\Upsilon(t) := \Psi(t) - \mathcal{E}(\vartheta) + \delta\left\{\left\langle \mathcal{E}'(v(t)), (e * \dot{v})(t)\right\rangle_{V'} + \int_t^\infty |\,g(s)\,|_H^2\, ds\right\},\ t > 0, \tag{5.2.8}$$

for some $\delta > 0$ fixed, where the function Ψ is defined as in (5.2.6). Then the Υ is differentiable, and its derivative satisfies the estimate

$$\begin{aligned}-\frac{d}{dt}\Upsilon(t) &\geqslant \frac{1}{2}\tilde{k}_\infty\,|\,\dot{v}(t)\,|_H^2 + \frac{\gamma}{2} e * |\,\dot{v}\,|_H^2 \\ &\quad + \delta\left\{-\frac{M}{2}\,|\,\dot{v}\,|_H^2 - \frac{M\,|\,e\,|_1}{2} e * |\,\dot{v}\,|_H^2 + \frac{1}{2}\,|\,\mathcal{E}'(v)\,|_{V'}^2 - \gamma^2\,|\,e\,|_1\, e * |\,\dot{v}\,|_H^2\right\} \\ &\geqslant \frac{1}{2}(\tilde{k}_\infty - \delta M)\,|\,\dot{v}\,|_H^2 + \frac{1}{2}[\gamma - \delta\,|\,e\,|_1\,(M + \gamma^2)] e * |\,\dot{v}\,|_H^2 + \frac{\delta}{2}\,|\,\mathcal{E}'(v)\,|_{V'}^2 .\end{aligned}$$

Hence, if we choose $\delta > 0$ small enough, then there is a constant $C_0 > 0$ such that

$$-\frac{d}{dt}\Upsilon(t) \geqslant C_0\left\{|\,\dot{v}\,|_H^2 + e * |\,\dot{v}\,|_H^2 + |\,\mathcal{E}'(v)\,|_{V'}^2\right\}. \tag{5.2.9}$$

Therefore, the function $\Upsilon(t)$ is decreasing, and by the proof of Proposition 5.2.3

$$\lim_{t\to\infty} \Upsilon(t) = 0.$$

In addition, we can assume that $\Upsilon(t) > 0$ for all $t > 0$. Since, if there is a $t > 0$ such that $\Upsilon(t) = 0$ then $\Upsilon(s) = 0$ for all $s \geqslant t$, and in this case, from (5.2.9), it follows that $\dot{v}(s) = \mathcal{E}'(v(s)) = 0$ for all $s \geqslant t$, hence $v(t)$ is a steady state.

Now, we will use our main assumption (v). Let $\vartheta \in \omega(v)$, since $\mathcal{E}$ is constant on $\omega(v)$, it follows from assumption (v) and compactness of the ω-limit set that there is a open set $U \subset V$ such that $\omega(v) \subset U$, and there are constants $\theta \in (0, 1/2]$ and $C > 0$ such that

$$| \mathcal{E}(v(t)) - \mathcal{E}(\vartheta) |^{1-\theta} \leqslant C \mid \mathcal{E}'(v(t)) |_{V'} \tag{5.2.10}$$

holds for every $v(t) \in U$. Further, since $\lim_{t\to\infty} \operatorname{dist}(v(t), \omega(v)) = 0$ we have that there is a $t^* \geqslant 0$ such that $v(t) \in U$ for all $t \geqslant t^*$ and (5.2.10) holds. Next, we compute and estimate the time derivative of $\Upsilon(t)^{1-\theta}$. By (5.2.8) we obtain

$$\begin{aligned}\Upsilon(t)^{1-\theta} \leqslant & C_1 \Big\{ | \mathcal{E}(v(t)) - \mathcal{E}(\vartheta) |^{1-\theta} + (e* \mid \dot{v} |_H^2)^{1-\theta} + | \mathcal{E}'(v(t)) |_{V'}^{1-\theta} | e * \dot{v} |_{V'}^{1-\theta} \\ & + \left(\int_t^\infty | g(s) |_H^2 \, ds \right)^{1-\theta} \Big\} \\ \leqslant & C_2 \left\{ | \mathcal{E}'(v(t)) |_{V'} + (e* \mid \dot{v} |_H^2)^{\frac{1}{2}2(1-\theta)} + | e * \dot{v} |_{V'}^{\frac{1-\theta}{\theta}} + \left(\int_t^\infty | g(s) |_H^2 \, ds \right)^{\frac{1}{2}2(1-\theta)} \right\}.\end{aligned}$$

Since, $2(1-\theta) \geqslant 1$ and $(1-\theta)/\theta \geqslant 1$ for all $\theta \in (0, 1/2]$, it follows that

$$\begin{aligned}\Upsilon(t)^{1-\theta} & \leqslant C_3 \left\{ | \mathcal{E}'(v(t)) |_{V'} + (e* \mid \dot{v} |_H^2)^{\frac{1}{2}} + | e * \dot{v} |_{V'} + \left(\int_t^\infty | g(s) |_H^2 \, ds \right)^{\frac{1}{2}} \right\} \\ & \leqslant C_4 \left\{ | \mathcal{E}'(v(t)) |_{V'} + (e* \mid \dot{v} |_H^2)^{\frac{1}{2}} + \left(\int_t^\infty | g(s) |_H^2 \, ds \right)^{\frac{1}{2}} \right\}.\end{aligned} \tag{5.2.11}$$

Therefore, from (5.2.9) and (5.2.11) it follows that

$$\begin{aligned}-\frac{d}{dt}[\Upsilon(t)^\theta] &= -\theta \Upsilon(t)^{\theta-1} \frac{d}{dt} \Upsilon(t) \\ &\geqslant \frac{\theta C_0 \left\{ | \dot{v} |_H^2 + e* \mid \dot{v} |_H^2 + | \mathcal{E}'(v) |_{V'}^2 \right\}}{C_4 \left\{ | \mathcal{E}'(v(t)) |_{V'} + (e* \mid \dot{v} |_H^2)^{\frac{1}{2}} + \left(\int_t^\infty | g(s) |_H^2 \, ds \right)^{\frac{1}{2}} \right\}} \\ &\geqslant C_5 \left\{ | \dot{v} |_H^2 + e* \mid \dot{v} |_H^2 + | \mathcal{E}'(v) |_{V'}^2 \right\}^{1/2} - \tilde{C}_5 \left(\int_t^\infty | g(s) |_H^2 \, ds \right)^{\frac{1}{2}} \\ &\geqslant C_6 \left\{ | \dot{v} |_H + | e * \dot{v} |_H + | \mathcal{E}'(v) |_{V'} \right\} - \tilde{C}_5 \left(\int_t^\infty | g(s) |_H^2 \, ds \right)^{\frac{1}{2}}.\end{aligned} \tag{5.2.12}$$

This in turn implies that $\dot{v} \in L_1([t^*, \infty), H)$. Therefore $\lim_{t\to\infty} v(t)$ exist in H, hence from the relative compactness of $v(t)$ in V, it follows our claim. □

Remark 5.2.3. Theorem 5.2.4 remains true if the assumption $e * \mid \dot{v} |_H^2 \in H^1_{1,loc}(\mathbb{R}_+)$ is dropped. In fact, by Remark 5.2.2 the function Υ in (5.2.8) is still nonincreasing and thus differentiable a.e. on $\mathbb{R}_+$. In order, to deduce $\dot{v} \in L_1([t^*, \infty); H)$ from (5.2.12) we apply [HS65, Thm. 18.14] to the function $-\Upsilon^\theta$.

5.3 Long-time behaviour for a phase field model

Let $J = [0, T]$ with $T > 0$ be an interval, and let Ω be a smooth bounded domain in $\mathbb{R}^3$. We consider the system

$$\dot{u} + \dot{\phi} = a_1 * \Delta u, \qquad \text{in } J \times \Omega; \tag{5.3.1}$$

$$\mu = \mathcal{E}'(\phi) - u, \qquad \text{in } J \times \Omega; \tag{5.3.2}$$

$$\dot{\phi} = a_2 * \Delta \mu, \qquad \text{in } J \times \Omega; \tag{5.3.3}$$

$$\partial_n u = \partial_n \phi = \partial_n \mu = 0, \qquad \text{on } J \times \partial\Omega \tag{5.3.4}$$

$$u(0, x) = u_0(x), \;\; \phi(0, x) = \phi_0(x), \qquad \text{in } \Omega, \tag{5.3.5}$$

where the kernels a_i are 1-regular and θ_i-sectorial with $\theta_i < \pi/2$, $i = 1, 2$. The nonlinear term $\mathcal{E}'$ is defined by

$$\mathcal{E}'(\phi) := -\Delta\phi + \Phi'(\phi),$$

with Φ satisfying the following growth conditions

(B2) $\Phi \in C^{4-}(\mathbb{R})$ such that

$$| \Phi'''(s) | \leqslant C \left(1 + | s |^{\beta}\right), \;\; s \in \mathbb{R},$$

for some constants C, and some $\beta \in (0, 3)$;

(B3) there are constants $m_1, m_2 \in \mathbb{R}$ such that

$$\Phi(s) \geqslant -\frac{m_1}{2} s^2 - m_2, \;\; \text{for each } s \in \mathbb{R}, \;\; \text{and } \lambda_1 > m_1,$$

where $\lambda_1 > 0$ is the smallest nontrivial eigenvalue of the negative Laplacian on Ω with homogeneous Neumann boundary conditions.

The system (5.3.1)-(5.3.5) is a conserved phase field model with memory and relaxed chemical potential, which has been studied in Chapter 4, where the global well-posedness was obtained (cf. Theorem 4.3.1).

We will assume in the sequel that we regard a global solution of (5.3.1)-(5.3.5) enjoys the following regularity

$$(\phi, u) \in H_2^1(J; L_2(\Omega) \times L_2(\Omega)) \cap L_2(J; H_2^4(\Omega) \times H_2^2(\Omega)).$$

On the other hand, since the solution u and ϕ of (5.3.1)-(5.3.5) are conserved quantities, we can w.l.o.g. assume that

$$\int_\Omega u(t, x) dx = \int_\Omega \phi(t, x) dx = 0,$$

for all $t \geqslant 0$. In fact, it suffice to replace u by $u-\overline{u}$, ϕ by $\phi-\overline{\phi}$, and $\Phi(\cdot)$ by $\Phi(\cdot+\overline{\phi})$, where the bar means $\bar{v} := \frac{1}{|\Omega|}\int_\Omega v dx$. In addition, the long-time behaviour is not affected by this normalization. By means of this normalization we can rewrite the system (5.3.1)-(5.3.5) in an abstract form in the L_p-settings as follows

$$\dot{u} + \dot{\phi} = - a_1 * Au, \qquad \text{in } J \times \Omega; \tag{5.3.6}$$

$$\mu = \mathcal{E}'(\phi) - u, \qquad \text{in } J \times \Omega; \tag{5.3.7}$$

$$\dot{\phi} = - a_2 * AP\mu, \qquad \text{in } J \times \Omega; \tag{5.3.8}$$

$$u(0,x) = u_0(x), \;\; \phi(0,x) = \phi_0(x), \qquad \text{in } \Omega, \tag{5.3.9}$$

where $A := -\Delta$ with domain

$$D(A) = \left\{w \in H^2_p(\Omega) : \; \partial_n w = 0 \text{ on } \partial\Omega\right\} \cap X, \text{ with } X := \left\{w \in L_p(\Omega) : \; \int_\Omega w(x)dx = 0\right\},$$

and P is the projection onto $R(A)$ in $L_p(\Omega)$ defined by $Pv := v - \bar{v}$.

We set $H = L_2(\Omega) \cap X$ and $V = D(A^{1/2}) = H^1_2(\Omega) \cap X$, hence the operator A is self-adjoint, invertible, positive definite, and coercive i.e.

$$\langle Aw, w\rangle \geqslant \lambda_1 \mid w \mid_2^2, \text{ for each } w \in V,$$

where $\lambda_1 > 0$ is the smallest nontrivial eigenvalue of the negative Laplacian on Ω with homogeneous Neumann boundary conditions.

Next, assume that the kernels a_1 and a_2 satisfy the condition (A1). Multiplying (5.3.6) by u we obtain

$$\frac{1}{2}\frac{d}{dt} \mid u \mid_2^2 + \left\langle \dot{\phi}, u \right\rangle + \left\langle \frac{d}{dt}(k_1 * v_1), v_1 \right\rangle = 0, \tag{5.3.10}$$

where $v_1 := a_1 * A^{1/2} u = -A^{-1/2}(\dot{u} + \dot{\phi})$. As to equation (5.3.8), we multiply by $P\mu$, this yields

$$\left\langle \dot{\phi}, P\mu \right\rangle + \left\langle \frac{d}{dt}(k_2 * v_2), v_2 \right\rangle = 0, \tag{5.3.11}$$

where $v_2 := a_2 * A^{1/2} P\mu = -A^{-1/2}\dot{\phi}$. Using the definition of μ and adding equation (5.3.10) to (5.3.11) we obtain

$$\frac{d}{dt}\left\{\frac{1}{2} \mid u \mid_2^2 + \mathcal{E}(\phi)\right\} + \left\langle \frac{d}{dt}(k_1 * v_1), v_1 \right\rangle + \left\langle \frac{d}{dt}(k_2 * v_2), v_2 \right\rangle = 0, \tag{5.3.12}$$

where $\mathcal{E}(\phi) := \int_\Omega \frac{1}{2} \mid \nabla\phi \mid^2 + \Phi(\phi)dx$. Observe that, since the kernels a_i are of positive type, it follows from equation (5.3.12) that

$$\frac{1}{2} \mid u(t) \mid_2^2 + \mathcal{E}(\phi(t)) \leqslant \frac{1}{2} \mid u_0 \mid_2^2 + \mathcal{E}(\phi_0).$$

In view of condition (B3), we then deduce, using the Poincaré-Wirtinger inequality, that

$$|\, u(t)\,|_{L_2} + |\, \phi(t)\,|_{H_2^1} \leqslant C(u_0, \phi_0),\ \ t > 0. \qquad (5.3.13)$$

Moreover, since Ω is a bounded domain in $\mathbb{R}^3$, this shows that $\phi \in L_\infty(\mathbb{R}_+; L_6(\Omega))$, by Sobolev embedding.

Next, we will discuss the integrability of the operator family $\{A^\kappa a * S(t)\}_{t\geqslant 0}$, $\kappa \in [0,1)$, where $S(t)$ denotes the resolvent family associated to the Volterra equation

$$\dot z + a * Az = f,\ z(0) = z_0. \qquad (5.3.14)$$

Observe that the mild solution of (5.3.14) can be written by means of the variation of parameters formula as

$$z(t) = S(t)z_0 + (S * f)(t),\ t > 0.$$

Lemma 5.3.1. *Let* Y *be a Banach space,* $A \in S(Y)$ *be an invertible sectorial operator in* Y *with spectral angle* $\varphi_A < \pi$. *Assume that the kernel* $a \in L_1(\mathbb{R}_+)$ *in* (5.3.14) *satisfies the following assumptions:*

(i) a *is 2-regular and* θ_a*-sectorial such that* $\varphi_A + \theta_a < \pi/2$ *holds;*

(ii) $\lim_{\lambda\to 0} \hat a(\lambda) \neq 0$ *and* $\left(\frac{1}{\hat a(i\cdot)}\right)' \in L_1(-1,1)$.

Then there exists a uniform integrable resolvent family S *for equation* (5.3.14)*, that is* $S \in L_1(\mathbb{R}_+; B(Y))$. *Moreover, for each* $\kappa \in [0,1)$, $A^\kappa a * S \in L_1(\mathbb{R}_+; B(Y))$.

Proof. Firstly, the existence of a resolvent family $S \in C((0,\infty); B(Y))$ follows from assumption (i) (see Remark 1.4.1). The uniform integrability of the resolvent follows from [Prü93, Thm. 10.2 and Lem. 10.2].

Note that the uniform integrability of the resolvent together with the assumption $a \in L_1(\mathbb{R}_+)$ implies $a*S \in L_1(\mathbb{R}_+; B(H))$, that is the last statement of lemma holds in case $\kappa = 0$. Let now $\kappa \in (0,1)$. Observe that the Laplace transform of $A^\kappa a * S$ is given by

$$\widehat{A^\kappa a * S}(\lambda) = A^\kappa \left(\frac{\lambda}{\hat a(\lambda)} + A\right)^{-1} =: T(\lambda),\ \ \mathrm{Re}\,\lambda > 0.$$

From the sectoriality of the operator A, the parabolicity condition $\varphi_A + \theta_a < \pi/2$, and the 1-regularity of a, we can see that there is a constant $M > 0$ such that

$$|\, T(\lambda)\,|_{B(Y)} \leqslant \frac{M}{(1 + |\, \frac{\lambda}{\hat a(\lambda)}\,|)^{1-\kappa}} \leqslant \frac{M}{(1 + |\, \lambda\,|)^{1-\kappa}},\ \ \mathrm{Re}\,\lambda \geqslant 0$$

holds. Moreover, from the 2-regularity of a we obtain

$$|T'(\lambda)|_{B(Y)} \leqslant \frac{M}{(1+|\lambda|)^{2-\kappa}}, \ \mathrm{Re}\lambda \geqslant 0; \tag{5.3.15}$$

$$|T''(\lambda)|_{B(Y)} \leqslant \frac{1}{|\lambda|} \frac{M}{(1+|\lambda|)^{3-\kappa}}, \ \mathrm{Re}\lambda \geqslant 0, \ \lambda \neq 0. \tag{5.3.16}$$

Now, we define the inverse Fourier transformation of T in the distributional sense as the operator $R : \mathbb{R}_+ \to B(Y)$ given by

$$(R|\chi) = (T| \frac{1}{2\pi} \lim_{N\to\infty} \int_{-N}^{N} e^{i\rho t}\chi(\rho)d\rho), \ \text{ for all } \chi \in C_0^\infty(0,\infty). \tag{5.3.17}$$

Next, we choose a $C_0^\infty(\mathbb{R})$-function $\tilde{\varphi}(\rho)$ such that $\tilde{\varphi}(\rho) = 1$ for $|\rho| \leqslant M+1$, $\tilde{\varphi}(\rho) = 0$ for $|\rho| \geqslant M+2$, $0 \leqslant \tilde{\varphi} \leqslant 1$ elsewhere. Then for $M > 0$ arbitrarily fixed, after two integrations by parts (5.3.17) becomes

$$\begin{aligned}(R|\chi) = &\int_0^\infty \left(\frac{1}{2\pi} \lim_{N\to\infty} \int_{-N}^{N} e^{i\rho t}\tilde{\varphi}(\rho)T(i\rho)d\rho \right) \chi(t)dt \\ &- \int_0^\infty \left(\frac{1}{2\pi t^2} \lim_{N\to\infty} \int_{-N}^{N} e^{i\rho t}[(1-\tilde{\varphi}(\rho))T(i\rho)]''d\rho \right) \chi(t)dt.\end{aligned}$$

Since the integrands

$$\frac{1}{2\pi} \lim_{N\to\infty} \int_{-N}^{N} e^{i\rho t}\tilde{\varphi}(\rho)T(i\rho)d\rho \text{ and } \frac{1}{2\pi t^2} \lim_{N\to\infty} \int_{-N}^{N} e^{i\rho t}[(1-\tilde{\varphi}(\rho))T(i\rho)]''d\rho$$

above are locally integrable functions on $(0,\infty)$, hence from the estimate (5.3.16) we can then represent the operator R by means of the formula

$$\begin{aligned}R(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\rho t}\tilde{\varphi}(\rho)T(i\rho)d\rho - \frac{1}{2\pi t^2} \int_{-\infty}^{\infty} e^{i\rho t}[(1-\tilde{\varphi}(\rho))T(i\rho)]''d\rho, \text{ a.a } t > 0 \\ &= R_1(t) + R_2(t).\end{aligned}$$

In order to show that R_i belongs to $L_1(\mathbb{R}_+; B(Y))$, for $i = 1, 2$, we proceed exactly as in the proof of [Prü93, Thm. 10.1-10.2] with R_i in place of S_i. We will not repeat it here. □

The following result states the global boundedness of the solution (ϕ, u) of the system (5.3.6)-(5.3.9).

Theorem 5.3.2. *Let $a_i \in L_1(\mathbb{R}_+)$. Assume that the kernels a_i for $i = 1, 2$, satisfy the assumptions of Lemma 5.3.1. Further, suppose that the conditions (A1)-(A2) and (B2)-(B3) hold. Assume that $(\phi_0, u_0) \in D(A^2) \times D(A)$. Then the solution (ϕ, u) of (5.3.6)-(5.3.9) is globally bounded, that is $(\phi, u) \in L_\infty(\mathbb{R}_+ \times \Omega)^2$, moreover this has relative compact range in $W := V \times H$.*

Proof. Consider the Volterra equations

$$\dot{z}_i(t) + \int_0^t a_i(s)A^i z_i(t-s)ds = f_i(t),\ t > 0,\ z_i(0) = z_{0i},\ i = 1,2. \tag{5.3.18}$$

From Lemma 5.3.1 we have that there exists resolvent families $S_i \in L_1(\mathbb{R}_+; B(Y))$ for $i = 1, 2$. By means of the variation of parameters, the solutions of (5.3.18) are given by

$$z_i(t) = S_i(t)z_{0i} + \int_0^t S_i(s)f_i(t-s)ds,\ t > 0.$$

So, for $q \in [1, \infty]$, it is easy to see that if $z_{0i} \in Y$ and $f_i \in L_q(\mathbb{R}_+; Y)$ then $z_i \in L_q(\mathbb{R}_+; Y)$ (see [Prü93, p. 257]).

Next, we rewrite the system (5.3.6)-(5.3.9) as follows:

$$\dot{e}(t) + \int_0^t a_1(t-s)Ae(s)ds = \int_0^t a_1(t-s)A\phi(s)ds,\ t > 0,\ \text{ with } e = u + \phi; \tag{5.3.19}$$

$$\dot{\phi}(t) + \int_0^t a_2(t-s)A^2\phi(s)ds = -\int_0^t a_2(t-s)A[\Phi'(\phi(s)) + \phi(s) - e(s)]ds,\ t > 0; \tag{5.3.20}$$

$$e(0) = e_0 = u_0 + \phi_0,\ \phi(0) = \phi_0. \tag{5.3.21}$$

The variation of parameters formula then yields the integral equations

$$e(t) = S_1(t)e_0 + (A^{1/2}a_1 * S_1 * A^{1/2}\phi)(t), \tag{5.3.22}$$

$$\phi(t) = S_2(t)\phi_0 - (Aa_2 * S_2 * [\Phi'(\phi) + \phi - e])(t),\ t > 0. \tag{5.3.23}$$

By assumptions (B2)-(B3) and the solution properties of the linear problems, this system of integral equations can be solved locally by means of the contraction mapping principle, say for any $p \geqslant 2$. Therefore, from the energy estimation (5.3.13), we have that there is precisely one global bounded mild solution

$$(\phi, u) \in C(\mathbb{R}_+; W),$$

which depends continuously on the data.

Now, to prove that $(\phi, u) \in L_\infty(\mathbb{R}_+ \times \Omega)$ we proceed with a bootstrap argument. Set $p_0 = 6$ and $r_0 = 2$. Suppose that we already know

$$\phi \in L_\infty(\mathbb{R}_+; H^1_{r_n}(\Omega)) \hookrightarrow L_\infty(\mathbb{R}_+; L_{p_n}(\Omega)),\ \text{ with } \frac{1}{p_n} = \frac{1}{r_n} - \frac{1}{3}.$$

Then from condition (B2) we obtain $\Phi'(\phi) \in L_\infty(\mathbb{R}_+; L_{p_n/(\beta+2)}(\Omega))$. On the other hand, since $A^{1-\delta/2}a_1 * S_1 \in L_1(\mathbb{R}_+; B(L_{r_n}(\Omega))$, for each $\delta \in (0, 1)$, by Lemma 5.3.1, it follows from (5.3.22) that

$$e \in L_\infty(\mathbb{R}_+; H^{1-\delta}_{r_n}(\Omega)) \hookrightarrow L_\infty(\mathbb{R}_+; L_{t_n}(\Omega)),\ \text{ with } \frac{1}{t_n} = \frac{1}{p_n} + \frac{\delta}{3}.$$

Note that

$$\Phi'(\phi) + \phi - e \in L_\infty(\mathbb{R}_+; L_{p_n/(\beta+2)}(\Omega)) \hookrightarrow L_\infty(\mathbb{R}_+; L_{s_n}(\Omega)) \text{ with } \frac{1}{s_n} = \frac{\beta+2}{p_n}.$$

Moreover, since $(A^2)^{(1-\delta/2)} a_2 * S_2 \in L_1(\mathbb{R}_+; B(L_{s_n}(\Omega))$ by Lemma 5.3.1, it follows that

$$\phi \in L_\infty(\mathbb{R}_+; H^{2-\delta}_{s_n}(\Omega)) \hookrightarrow L_\infty(\mathbb{R}_+; H^1_{r_{n+1}}(\Omega)), \text{ with } \frac{1}{s_n} = \frac{1}{r_{n+1}} + \frac{1-\delta}{3}.$$

Hence,

$$\frac{\beta+2}{p_n} = \frac{1}{s_n} = \frac{1}{p_{n+1}} + \frac{2-\delta}{3}.$$

Inductively this yields

$$\frac{1}{p_n} = (\beta+2)^n \left[\frac{1}{p_0} - \frac{2-\delta}{3(\beta+1)}\right] + \frac{2-\delta}{3(\beta+1)}.$$

Since by assumption $\beta < 3$, we may choose $0 < \delta < (3-\beta)/2$ to get the bracket negative. Then the iteration ends after finitely many steps. As a consequence we obtain

$$\phi \in L_\infty(\mathbb{R}_+ \times \Omega).$$

Moreover, since $H^{2-\delta}_{s_n}(\Omega)$ and $H^{1-\delta}_{r_n}(\Omega)$ are compactly embedded in $H^1_2(\Omega)$, and $L_2(\Omega)$ respectively, it follows that $\{(\phi(t), u(t)); t \geqslant 0\}$ is relative compact in W. □

Define a functional Ξ on W by

$$\Xi(\phi, u) := \int_\Omega \frac{1}{2} \mid \nabla\phi \mid^2 + \Phi(\phi) + \frac{1}{2} \mid u \mid^2 dx = \mathcal{E}(\phi) + \frac{1}{2} \mid u \mid_2^2, \tag{5.3.24}$$

and define the ω-limit set of the solution (ϕ, u) of (5.3.1)-(5.3.3) by

$$\omega(\phi, u) = \{(\zeta, \vartheta) \in W: \text{ there exits } (t_n) \nearrow \infty \text{ s.t., } (\phi, u)(t_n) \to (\zeta, \vartheta) \text{ in } W \text{ as } n \to \infty\}.$$

Our main results of this section read as follows.

Theorem 5.3.3. *Let* (ϕ, u) *be a global bounded solution of* (5.3.1)-(5.3.3). *Assume that the assumptions of Theorem 5.3.2 hold and that the kernels* $a_i \in \mathcal{K}^1(\alpha_i, \theta_i)$ *with* $\alpha_i \in (0,1)$ *and* $\theta_i \in (0, \pi/2)$. *Further, suppose that the functional* Ξ *defined in* (5.3.24) *satisfies the Łojasiewicz-Simon inequality near some point* (ζ, ϑ) *of* $\omega(\phi, u)$. *Then* $\lim_{t\to\infty}(\phi(t), u(t)) = (\zeta, \vartheta)$ *in* W, *and* (ζ, ϑ) *is a stationary solution, i.e.* $\Xi'(\zeta, \vartheta) = 0$.

Proof. We begin computing the Frechét derivative of the functional $\Xi : W \to \mathbb{R}$ on W, it is given by

$$(\Xi'(\phi, u), (h, k))_{W', W} = (\mathcal{E}'(\phi), h)_{V', V} + \langle u, k \rangle_H \,,$$

hence

$$|\,\Xi'(\phi,u)\,|_{W'}^2 = |\,A^{-1/2}P\mathcal{E}'(\phi)\,|_2^2 + |\,u\,|_2^2\,.$$

Next, assume the condition (A2) for k_i with $i=1,2$ and define a Lyapunov function $\widetilde{\Psi} : \mathbb{R}_+ \to \mathbb{R}$ by

$$\widetilde{\Psi}(t) := \Xi(\phi,u) + \frac{1}{2}\left(e_1 * |\,v_1\,|_2^2 + e_2 * |\,v_2\,|_2^2\right). \tag{5.3.25}$$

Since $a_i \in \mathcal{K}^1(\alpha_i,\theta_i)$ we obtain from Corollary 1.4.5 that the functions v_i defined in (5.3.10) and (5.3.11) for $i=1,2$, respectively satisfy the assumptions of Lemma 5.1.2. Hence, by applying the inequality of Lemma 5.1.2 (ii) in (5.3.12) a simple computation shows that

$$-\frac{d}{dt}\widetilde{\Psi}(t) \geqslant \frac{1}{2}\left(\gamma_1 e_1 * |\,v_1\,|_2^2 + k_1^\infty\,|\,v_1\,|_2^2 + \gamma_2 e_2 * |\,v_2\,|_2^2 + k_2^\infty\,|\,v_2\,|_2^2\right), \tag{5.3.26}$$

where $k_i^\infty := \lim_{t\to\infty} k_i(t) > 0$ for $i=1,2$, which exist by Remark 5.1.1. Hence,

$$v_i \in L_2(\mathbb{R}_+ \times \Omega) \text{ and } e_i * |\,v_i\,|_2^2 \in L_1(\mathbb{R}_+) \cap C_0(1,\infty), \text{ for } i=1,2.$$

By applying the arguments in the proof of Proposition 5.2.3, to the function $\widetilde{\Psi}$, it is not hard to check that the functional $\Xi(\phi,u)$ is constant on $\omega(\phi,u)$ and that for all $w \in \omega(\phi,u)$ one has that $\Xi'(w) = 0$. Indeed, since $v_i \in L_2(\mathbb{R}_+ \times \Omega)$, it follows that $\dot{\phi}, \dot{u} \in L_2(\mathbb{R}_+; V')$, hence $\phi(t_n+s) \to \zeta$ in V' for all $s \in [0,1]$ and the same holds for $u(t_n+s)$. The compactness of the range of (ϕ,u) in W yields the convergence in W for all $s \in [0,1]$. The rest of the arguments follow of the same way as the proof of Proposition 5.2.3.

As in the proof of Theorem 5.2.4, we must modify the Lyapunov function $\widetilde{\Psi}$ to prove convergence in the desired space. Define $\widetilde{\Upsilon} : \mathbb{R}_+ \to \mathbb{R}$ by

$$\widetilde{\Upsilon}(t) := \widetilde{\Psi}(t) - \Xi(\zeta,\vartheta) - \delta_1\left\langle A^{-1/2}u(t), (e_1 * v_1)(t)\right\rangle - \delta_2\left\langle A^{-1}P\mathcal{E}'(\phi(t)), A^{-1/2}(e_2 * v_2)(t)\right\rangle, \tag{5.3.27}$$

for some fixed $\delta_i > 0$, for $i=1,2$.

Next, we will check that $\widetilde{\Upsilon}(t)$ is a Lyapunov function. To this end, we begin with some estimates.

From the definition of v_1 in (5.3.10), v_2 in (5.3.11), and the condition (A2) for k_1, it follows that

$$u = A^{-1/2}\frac{d}{dt}(k_1 * v_1),$$

and

$$\begin{aligned}\frac{d}{dt}\left\langle e_1 * v_1, A^{-1/2}u\right\rangle &= \left\langle \frac{d}{dt}(k_1 * v_1) - \gamma_1 e_1 * v_1, A^{-1/2}u\right\rangle + \left\langle e_1 * v_1, A^{-1/2}\dot{u}\right\rangle \\ &= |\,u\,|_2^2 - \gamma_1\left\langle e_1 * v_1, A^{-1/2}u\right\rangle - \langle e_1 * v_1, v_1 - v_2\rangle\end{aligned}$$

holds. Hence, by Young's inequality, it follows that

$$\frac{d}{dt}\left\langle e_1 * v_1, A^{-1/2}u\right\rangle \geqslant \frac{1}{2}\,|\,u\,|_2^2 - c_1\left[|\,v_1\,|_2^2 + |\,v_2\,|_2^2 + e_1 * |\,v_1\,|_2^2\right], \tag{5.3.28}$$

with some constant $c_1 > 0$. In a similar way we have

$$\frac{d}{dt}\left\langle A^{-1}P\mathcal{E}'(\phi), A^{-1/2}e_2 * \nu_2\right\rangle = \left\langle -\nu_2 + A^{-3/2}P\Phi''(\phi)\dot{\phi}, e_2 * \nu_2\right\rangle$$
$$+ \left\langle A^{-1}P\mathcal{E}'(\phi), P\mu - A^{-1/2}\gamma_2 e_2 * \nu_2\right\rangle$$
$$= -\left\langle \nu_2 + A^{-3/2}P\Phi''(\phi)A^{1/2}\nu_2, e_2 * \nu_2\right\rangle$$
$$+ | A^{-1/2}P\mathcal{E}'(\phi) |_2^2 - \left\langle A^{-1}P\mathcal{E}'(\phi), u + A^{-1/2}\gamma_2 e_2 * \nu_2\right\rangle .$$

Hence, using the L_∞-bound for ϕ, and Young's inequality yield

$$\frac{d}{dt}\left\langle A^{-1}P\mathcal{E}'(\phi), A^{-1/2}e_2 * \nu_2\right\rangle \geqslant \frac{1}{2} | A^{-1/2}P\mathcal{E}'(\phi) |_2^2 - c_2\left[| \nu_2 |_2^2 + e_2 * | \nu_2 |_2^2 + | u |_2^2\right], \tag{5.3.29}$$

for some constant $c_2 > 0$.

Choosing $\delta_1 > 0$ small and then $\delta_2 > 0$ even smaller in (5.3.27), then we obtain from the estimates (5.3.26), and (5.3.28)-(5.3.29) that

$$-\frac{d}{dt}\widetilde{\Upsilon}(t) \geqslant c\left[| \nu_1 |_2^2 + e_1 * | \nu_1 |_2^2 + | \nu_2 |_2^2 + e_2 * | \nu_2 |_2^2 + | u |_2^2 + | A^{-1/2}P\mathcal{E}'(\phi) |_2^2\right]$$
$$= c\left[| \nu_1 |_2^2 + e_1 * | \nu_1 |_2^2 + | \nu_2 |_2^2 + e_2 * | \nu_2 |_2^2 + | \Xi'(\phi, u) |_{W'}^2\right], \tag{5.3.30}$$

for some constant $c > 0$. Therefore, $\widetilde{\Upsilon}(t)$ is positive, decreasing, and since $e_i * | \nu_i |_2^2 \to 0$, we also have $| e_i * \nu_i |_2 \to 0$ as $t \to \infty$ for $i = 1, 2$, hence

$$\lim_{t\to\infty} \widetilde{\Upsilon}(t) = \lim_{t\to\infty} \Xi(\phi(t), u(t)) - \Xi(\zeta, \vartheta) = 0.$$

In a similar way as in the previous section we estimate $\widetilde{\Upsilon}(t)^{1-\theta}$ obtaining

$$\widetilde{\Upsilon}(t)^{1-\theta} \leqslant C\left\{| \Xi'(\phi(t), u(t)) |_{W'} + (e_1 * | \nu_1 |_H^2)^{\frac{1}{2}} + (e_2 * | \nu_2 |_H^2)^{\frac{1}{2}}\right\}. \tag{5.3.31}$$

as well as

$$-\frac{d}{dt}[\widetilde{\Upsilon}(t)^\theta] = -\theta\widetilde{\Upsilon}(t)^{\theta-1}\frac{d}{dt}\widetilde{\Upsilon}(t)$$
$$\geqslant C_0\frac{| \nu_1 |_2^2 + e_1 * | \nu_1 |_2^2 + | \nu_2 |_2^2 + e_2 * | \nu_2 |_2^2 + | \Xi'(\phi, u) |_{W'}^2}{| \Xi'(\phi(t), u(t)) |_{W'} + (e_1 * | \nu_1 |_2^2)^{\frac{1}{2}} + (e_2 * | \nu_2 |_2^2)^{\frac{1}{2}}}$$
$$\geqslant C_1\left\{| \nu_1 |_2 + | \nu_2 |_2 + | \Xi'(\phi(t), u(t)) |_{W'}\right\}. \tag{5.3.32}$$

Therefore, we obtain that

$$A^{-1/2}\dot{\phi},\ A^{-1/2}\dot{u} \in L_1(\mathbb{R}_+; L_2(\Omega)),$$

hence $\lim_{t\to\infty} A^{-1/2}\phi$ and $\lim_{t\to\infty} A^{-1/2}u$ exist in $L_2(\Omega)$. In addition, since $(\phi(t), u(t))$ has range relative compact in W, it follows that

$$\lim_{t\to\infty}(\phi(t), u(t)) = (\zeta, \vartheta) \text{ in } W.$$

□

Remark 5.3.1. Following the lines of the proof of [CFP06, Proposition 6.6] it holds that the functional Ξ defined in (5.3.24) satisfies the Łojasiewicz-Simon inequality near some point (ζ, ϑ) of $\omega(\phi, u)$ in Theorem 5.3.3 is satisfied if for e.g. the potential Φ is real analytic in a neighborhood of the ω-limit set of the solution (ϕ, u) and if the nonlinearity Φ satisfies the growth condition (B2).

Combining the results of Chapter 4 and 5 we obtain

Theorem 5.3.4. *Let* $a_j \in L_1(\mathbb{R}_+)$ *be scalar kernels for* $j = 1, 2$. *Assume that* $a_j \in \mathcal{K}^1(\alpha_j, \theta_{a_j})$, $\alpha_j \in (0,1)$, $\alpha_2 \geqslant \alpha_1$, *for* $j = 1, 2$. *In addition, we suppose that the subsequent conditions hold:*

(i) *the nonlinearity* Φ *satisfies (B2)-(B3);*

(ii) a_1 *is of positive type, that is*

$$Re \int_0^T [a_1 * \psi](t)\overline{\psi(t)}dt \geqslant 0 \quad \textit{for all } \psi \in L_2((0,T);\mathbb{C}), \textit{ and } T > 0;$$

(iii) $\mathrm{Im}\hat{a}_1(i\rho) \cdot \mathrm{Im}\hat{a}_2(i\rho) \geqslant 0$, $\rho \in \mathbb{R} \setminus \{0\}$;

(iv) a_j *satisfies the conditions (A1)-(A2), for* $j = 1, 2$.

Then for each $p \geqslant 2$ *the system* (5.3.1)-(5.3.5) *is globally well-posedness and the solution* (ϕ, u) *enjoy the following regularity*

$$(\phi, u) \in H_p^{1+\alpha_2}(J; L_p(\Omega)) \cap L_p(J; D(A^2)) \times H_p^{1+\alpha_1}(J; L_p(\Omega)) \cap L_p(J; D(A)),$$

provided the initial condition $(\phi_0, u_0) \in D(A^2) \times D(A)$. *If in addition we assume that*

(v) Φ *is real analytic in a neighborhood of the* ω*-limit set of the solution* ϕ*;*

(vi) a_j *is 2-regular,* $\lim_{\lambda \to 0} \hat{a}_j(\lambda) \neq 0$, *and* $\left(\frac{1}{\hat{a}_j(i\cdot)}\right)' \in L_1(-1,1)$ *for* $j = 1, 2$.

Then $\lim_{t\to\infty}(\phi(t), u(t)) = (\zeta, \vartheta)$ *exists in* $H_2^1(\Omega) \times L_2(\Omega)$ *and* (ζ, ϑ) *is a stationary solution of the system* (5.3.1)-(5.3.5), *that is*

$$\begin{aligned} \vartheta &= \textit{const.}, \; x \in \Omega; \\ -\Delta\zeta + \Phi'(\zeta) - \vartheta &= \textit{const.}, \; x \in \Omega; \\ \partial_n \zeta &= 0, \qquad x \in \partial\Omega. \end{aligned}$$

5.4 Rate of convergence

In this section we show that the Łojasiewicz exponent θ in the Łojasiewicz-Simon inequality determines the decay rate of the solution to the steady state.

First observe that for $t > 0$ large enough, from (5.3.30) and (5.3.31), we have that there is a constant $c > 0$ such that

$$-\frac{d}{dt}\widetilde{\Upsilon}(t) \geqslant c\widetilde{\Upsilon}(t)^{2(1-\theta)}$$

holds. Since $\widetilde{\Upsilon}(t) > 0$ for all $t > t_0$ with $t_0 > 0$ large enough, we obtain from this inequality that

$$\frac{-1}{1-2\theta}\frac{d}{dt}\widetilde{\Upsilon}(t)^{-(1-2\theta)} \leqslant -c \text{ if } \theta \in \left(0, \frac{1}{2}\right),$$

and

$$\frac{d}{dt}(\log \widetilde{\Upsilon}(t)) \leqslant -c \text{ if } \theta = \frac{1}{2}.$$

Hence, integrating these differential inequalities, we obtain that there is a constant $C > 0$ such that for large $t > 0$,

$$\widetilde{\Upsilon}(t) \leqslant \begin{cases} C(1+t)^{-\frac{1}{1-2\theta}} & \text{if } \theta \in \left(0, \frac{1}{2}\right) \\ Ce^{-ct} & \text{if } \theta = \frac{1}{2}. \end{cases}$$

In addition, since

$$-\frac{d}{dt}\left[\widetilde{\Upsilon}(t)^{\theta}\right] \geqslant C_1 \mid v_i \mid_2, \text{ for } i = 1, 2$$

we obtain

$$\mid \phi(t) - \zeta \mid_{V'} \leqslant \int_t^{\infty} \mid v_2(s) \mid_2 ds \leqslant C_2\widetilde{\Upsilon}(t)^{\theta},$$

the same holds for the solution u, that is

$$\mid u(t) - \vartheta \mid_{V'} \leqslant \int_t^{\infty} \mid v_1(s) \mid_2 ds \leqslant C_2\widetilde{\Upsilon}(t)^{\theta}.$$

The same argument can be applied to obtain rate of convergence to steady state for the abstract model (5.0.1) in case that $f = 0$. Actually, the argument in this section was first used in [HJ01], and [HJK03] in case without kernel and [CF05] for the equation (0.0.3). However, note that no convergence rate are obtained in the energy space $W = V \times H$.

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig und ohne fremde Hilfe verfasst, keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

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