

# Portfolio optimization with bounded shortfall risks

## Dissertation

zur Erlangung des akademischen Grades  
doctor rerum naturalium (Dr. rer. nat.)

vorgelegt der  
Mathematisch-Naturwissenschaftlich-Technischen Fakultät  
der Martin-Luther-Universität Halle-Wittenberg

von Herrn Dipl-Math Abdelali Gabih  
geboren am 04. April 1974 in El Jadida

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eingereicht am: 14.07.2005

Tag der Verteidigung: 14.10.2005

**urn:nbn:de:gbv:3-000009218**

[<http://nbn-resolving.de/urn/resolver.pl?urn=nbn%3Ade%3Agbv%3A3-000009218>]

*Dedicated to the memory of my friends Hamsi Said and Semkaoui Said*

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# Contents

1	Introduction . . . . .	1
<b>I</b>	<b>The Portfolio optimization problem: A general overview</b>	<b>14</b>
2	Financial markets . . . . .	14
3	Utility function of an investor . . . . .	16
4	The portfolio optimization problem . . . . .	19
5	Risk measures . . . . .	29
<b>II</b>	<b>Portfolio optimization with deterministic benchmark</b>	<b>32</b>
6	Optimization under Value at Risk constraint . . . . .	33
7	Optimization under Expected Loss constraint . . . . .	36
8	Optimization under Expected Utility Loss constraint . . . . .	40
9	Numerical results . . . . .	44
<b>III</b>	<b>Portfolio optimization with stochastic benchmark</b>	<b>51</b>
10	The aim of this chapter . . . . .	51
11	Benchmarking the stock market . . . . .	52
12	Properties of the optimal portfolio . . . . .	56
13	Numerical results . . . . .	57
<b>IV</b>	<b>The case of partial information</b>	<b>66</b>
14	An HMM for the stock return . . . . .	66
15	The optimization problem under partial information . . . . .	68
16	HMM filtering results . . . . .	71
17	Malliavin derivative . . . . .	72
18	Optimal Trading Strategies . . . . .	75
<b>V</b>	<b>Appendix</b>	<b>81</b>

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A	Proof of Lemma 7.2 . . . . .	81
B	Proof of Lemma 8.2 . . . . .	82
C	Computation of the current terminal wealth in Proposition 8.4 . . . . .	84
D	Proof of Lemma 11.2 . . . . .	86
E	Evaluation of the derivative $\mathbf{F}_z(\mathbf{z}, \mathbf{t})$ in Eq. (11.3) . . . . .	87
F	Proof of Proposition 11.5 . . . . .	88
G	Malliavin derivative . . . . .	90
<b>Bibliography</b>		<b>93</b>

# 1 Introduction

The first revolution in finance began with Harry Markowitz who published in 1952 in his doctoral dissertation a portfolio selection principle based on identifying the *best stock* for an investor and quantified the trade-offs between risk and return inherent in an entire portfolio of stocks. Later William Sharpe used Markowitz's ideas to develop the concept of determining covariances not between every possible pair of stocks, but between each stock and the *market*. Then one can address the *optimization* problem of choosing the portfolio with the largest mean return, subject to keeping the risk below a specified acceptable threshold. For purpose of this optimization problem each stock could then be characterized by its rate of return and its correlation with the market. For the pioneering work, Markowitz and Sharpe shared with Merton Miller the 1990 Nobel Prize in economics, the first ever awarded for work in finance. The portfolio selection work of Markowitz and Sharpe introduced mathematics to the investment management; and thanks to Robert Merton and Paul Samuelson, one-period models were replaced by continuous-time, Brownian motion-driven models, and the quadratic utility function implicit in mean-variance optimization was replaced by more general increasing, concave utility functions.

The second revolution in finance is connected with the explosion in the market for *derivative securities*. The fundamental work here was done by Fisher Black, Robert Merton, and Myron Scholes in the early 1970s. Black, Merton, and Scholes were seeking to understand the value of the option to buy one share of stock at a future date and price specified in advance. This so-called *European call-option* derives its value from that of the underlying stock, hence the name *derivative security*. Based on the simple principle called *absence of arbitrage*, Black and Scholes [9] derived in 1973 the now famous formula for the value of the European call-option, which bears their name, and which was extended by Merton 1973 [58] in a variety of very significant ways. For this fundamental work, Robert Merton and Myron Scholes were awarded the 1997 Nobel Prize in economics.

## History

The modern portfolio theory started with the famous works of Markowitz (see [54], [55]), who conceived the idea of trading off the mean return of a portfolio against its variance. Merton introduced in two works (see [56], [57]) the concept of Itô calculus with methods of continuous-time stochastic optimal control to solve the problem of portfolio optimization. In a model with constant coefficients Merton [56] solved the relevant Hamilton-Jacobi-Bellman equation and produced solutions to both finite and infinite-horizon models when the utility function is a power function or the logarithm. The modern mathematical approach to portfolio management in complete markets, built around the ideas of equivalent martingale measures and the creation of portfolios from martingale representation theorems, began with Harrison and Kreps [33] and was further developed by Harrison and Pliska (see [34], [35]) in the context of the option pricing. Pliska [63], Cox and Huang [11], [12], and Karatzas, Lehoczky and,

Shreve [38] adapted the martingale ideas to problems of utility maximization. Much of this development appears in [41].

The stochastic duality theory of Bismut [8] was first applied to study the portfolio optimization problems in the doctoral dissertation of Xu [70] who formulated a dual problem whose solution could be shown to exist and to be useful in constructing and characterizing the solution to the original optimization problem. The methodology of Xu was applied to deal with incomplete market models by Karatzas, Lehoczky, Shreve, and Xu [39], where they combined the martingale method with duality methods to maximize expected utility in incomplete market.

The techniques of Malliavin calculus to compute optimal portfolios, were introduced by Ocone and Karatzas [61]. The mixed control/filtering problem was studied by Kuwana (see [50], [51]) and by Lakner [52], see also Karatzas [37], Duffie and Richardson [20], and Schweizer [67]. Sass and Haussmann [65] consider the case of a hidden Markov model for the drift. Pham and Quenez [62] consider a financial market where price process of risky assets follows a stochastic volatility model. Basak and Shapiro [4]; Basak, Shapiro, and Tepla [5] embed risk management into portfolio optimization problem and analyse the impact of different risk constraints to the portfolio optimization. Emmer, Klüppelberg, and Korn [24] studied the utility maximization problem under the constraint of an upper bound for the Capital-at-Risk (CaR) of a portfolio which is defined as the difference between the mean of the profit-loss distribution and the VaR. Thereby, the portfolio strategy is restricted to constants. Dmitrasinovic-Vidovic, Lari-Lavassani, and Li [16] extended the method used in [24] to the continuous setting to investigate the portfolio optimization problem under Conditional Capital-at-Risk (CCaR) which is proved to be a coherent risk measure if the random variable describing the risk is continuously distributed.

In this thesis we deal with optimal strategies for portfolios consisting of  $n$  risky stocks and one risk-free bond. Giving a finite planning horizon  $[0, T]$  and starting with some initial endowment, the aim is to maximize the expected utility of the terminal wealth of the portfolio by optimal selection of the proportions of the portfolio wealth invested in stocks and bond, respectively. Assuming a continuous-time market allowing for permanent trading and rebalancing the portfolio, these proportions have to be found for every time  $t$  up to  $T$ .

The utility maximization problem admits a simple solution in the context of the Black-Scholes model of a complete financial market; this solution has been derived by Karatzas, Lehoczky, and Shreve [38] and also by Cox and Huang [11], [12]. The case of an incomplete market has been treated by Karatzas, Lehoczky, Shreve, and Xu [39]. Here the portfolio can contain shares of a risk-free bond and of stocks whose prices follow a geometric Brownian motion.

Following the optimal portfolio strategy leads (by definition) to the maximum expected utility of the terminal wealth. Nevertheless, the terminal wealth is a random variable with a distribution which is often extremely skew and shows considerable probability in regions of small values of the terminal wealth. This means that the optimal terminal wealth may exhibit large so-called shortfall risks. By the term shortfall risk we denote the event, that the terminal wealth falls below some threshold value.

In Germany companies offering some kind of private pension insurances (Riester-Rente) are obliged by law to pay at least the invested capital without any interest to the insured person. So the company is confronted with the risk of a terminal wealth of the portfolio (created with the deposits of the insured person) below the value of the non-interest-bearing deposits.

In order to incorporate such shortfall risks into the optimization it is necessary to quantify them by using appropriate risk measures. Let's denote the terminal wealth of the portfolio at time  $t = T$  by  $X_T$ , and let  $Q > 0$  be a shortfall level which we will specify later. Then the shortfall risk consists in the random event  $\{X_T < Q\}$  or  $\{G := X_T - Q < 0\}$ . Next we assign risk measures to the random variable (risk)  $G$  and denote them by  $\rho(G)$ . Using these measures, constraints of the type  $\rho(G) \leq \varepsilon$  for some  $\varepsilon > 0$  can be added to the formulation of the portfolio optimization problem.

A natural idea is to restrict the probability of a shortfall, i.e.,

$$\rho(G) = P(G < 0) = P(X_T < Q).$$

This approach corresponds to the widely used concept of Value at Risk (VaR) which is defined as

$$\text{VaR}_\varepsilon(G) = -\zeta_\varepsilon(G)$$

where  $\zeta_\varepsilon(G)$  denotes the  $\varepsilon$ -quantile of the random variable  $G$ . VaR can be interpreted as the threshold value for the risk  $G$  such that  $G$  falls short below this value with some given probability  $\varepsilon$ , i.e., VaR describes the loss that can occur over a given period, at a given confidence level, due to exposure to market risk. It holds

$$P(G < 0) \leq \varepsilon \Leftrightarrow \text{VaR}_\varepsilon(G) \leq 0 \Leftrightarrow \text{VaR}_\varepsilon(X_T) \leq -Q.$$

Value-at-risk describes probabilistically the market risk of a trading portfolio. This risk measure is widely used by banks, securities firms, commodity and energy merchants, and other trading organizations because it is an easily interpretable summary measure of risk (see Bodnar et al. [10]).

Another risk measure is the Expected Loss defined by

$$\text{EL}(G) = \text{E}[G^-] = \text{E}[(X_T - Q)^-].$$

In the example of the pension insurance this is a measure for the average additional capital the company is obliged to pay as compensation for the shortfall. The constraint  $\text{E}[G^-] \leq \varepsilon$  bounds this average additional capital by  $\varepsilon > 0$ .

Another interesting risk measure is the so called Expected Utility Loss which we denote by (EUL)

$$\text{EUL}(G) := \text{E}[G^-] = \text{E}[(U(X_T) - U(Q))^-], \quad (1.1)$$

this measure deals with the expected loss in utility, see Definition 4.2.

Further risk measures can be found in the class of coherent measures introduced by Artzner, Delbaen, Eber and Heath [2], and Delbaen [14]. These are measures possessing the properties of monotonicity, subadditivity, positive homogeneity and the

translation property. The above (VaR, EL, EUL) risk measures do not belong to this class, since VaR is not subadditive, and EL and EUL violate the translation invariance property.

In this thesis we analyse the effects of risk management on optimal terminal wealth choices and on optimal portfolio policies. We first focus on modeling portfolio managers as expected utility maximizers, who derive utility from wealth at some horizon and who must comply with different risk constraints imposed at that horizon, requiring that the wealth may decrease below a given floor. This analysis was initiated by the early studies on shortfall constraints (see for example Roy [64], Telser [69], Kataoka [46]) and is extended in recent studies (see for example Klüppelberg and Korn [47], Alexander and Baptista [1], Emmer [23]), where the authors conduct a mean-variance analysis to VaR-based risk management to manage the risk of portfolios. Dmitrasinovic-Vidovic, Lari-Lavassani, Li, and Ware (see [16] and [17]) extended the result obtained in [24] to deal with Conditional Capital at-Risk in continuous setting. Bäuerle and Rieder [3] used stochastic control methods to investigate the problem of maximizing the expected utility in a hidden Markov model.

At more fundamental level Basak and Shapiro [4] study a more general preference structure and merge the utility maximization and risk management into one optimization problem. The authors demonstrate that risk management practices under the VaR approach may yield unintended results. They find that a VaR-risk manager often optimally chooses a larger exposure to risky assets than non-risk managers and consequently incurs larger losses when losses occur. To overcome this shortcoming of VaR, we propose and evaluate an alternative forms of risk management that maintain limited expected losses when losses occur. In [31], [28] we embed risk management objectives into utility maximization problem using Value at Risk (VaR), Expected Loss (EL). In Gabih et al. [29] we compare the loss in utility of the portfolio along with the return of the stock market. Finally, we combine and extend in [30] the methods used in [29, 65] to study the portfolio optimization problem in the case of partial information under the Expected Utility Loss.

Basically, we consider as in Black-Scholes model [9], financial investment opportunities which are given by an instantaneously *risk-free market account* providing an interest rate  $r$  so that its price  $S^0 = (S_t^0)_{t \in [0, T]}$  is given by

$$dS_t^0 = rS_t^0 dt, \quad (1.2)$$

and  $n$  *risky stocks* whose prices  $S = (S_t)_{t \in [0, T]}$ ,  $S_t = (S_t^1, \dots, S_t^n)^\top$  evolve according to

$$dS_t^i = S_t^i \left[ \mu_i dt + \sum_{j=1}^n \sigma_{ij} dW_t^j \right], \quad S_0^i = s_0^i \in \mathbb{R}, \quad i = 1, \dots, n. \quad (1.3)$$

We analyse the effects of risk management on optimal portfolio policies within the familiar (continuous-time) complete market setting, but in two different situations:



## Full information

First we consider the standard case of *full information*, where the drift process  $\mu = (\mu_1, \dots, \mu_n)^\top$ , the volatility matrix  $\sigma$ , and the Brownian motion  $W = (W_t)_{t \in [0, T]}$ ,  $W_t = (W_t^1, \dots, W_t^n)^\top$  appearing in the stochastic differential equation (1.3) for the security price are well known to the portfolio manager. Here we adopt the approach used by Basak and Shapiro [4] and we directly embed risk management objectives into utility maximizing framework. Especially, we use three different risk constraints, Value at Risk VaR for which we give a slightly different optimal solution than the one given by Basak and Shapiro [4]. Here we find that the expected losses in the state where there are large losses, are higher than those the portfolio manager would have incurred if he had not used Value at Risk. This shortcoming we overcome by proposing an alternative form for risk management, i.g.  $E[(X_T - Q)^-]$  which we call Expected Loss and which we denote by EL. Since the objective of the portfolio manager is to maximize the expected utility from the terminal wealth, we find it interesting to deal with the portfolio optimization problem where the portfolio manager is confronted with a risk measured by a constraint of the type

$$\text{EUL}(G) := E[G^-] = E[(U(X_T) - U(Q))^-] \leq \varepsilon, \quad (1.4)$$

where  $\varepsilon$  is a given bound for the Expected Utility Loss. This risk constraint has the convenient property that it leads to more explicit calculations for the optimal strategy we are looking for. In addition it allows to the constrained static problem to be solved for a large class of utility functions. We keep the shortfall level  $Q$  to be constant. A typical choice for the shortfall level is  $Q = q$  with

$$q = x e^{(r+\delta)T}, \quad \delta \in \mathbb{R},$$

where  $x$  is the initial endowment. Here, the shortfall level is related to the result of an investment into the money market,  $x e^{rT}$  is the terminal wealth of a pure bond portfolio where the portfolio manager follows the buy-and-hold strategy. In this case shortfall means to reach not an target interest rate of  $r + \delta$ . For  $\delta = -r$  we have  $q = x$ , i.e., the shortfall level is equal to the initial capital.

The EUL risk measure (1.4), proves to be efficient once again when we model a portfolio manager who manages the relative performance, or tracking error of his portfolio along with a stochastic benchmark  $Q$  which can be the level of a portfolio, or an index, or any economic indicator. Here we consider only one stock market with initial price 1. The benchmark  $Q$  is not a constant but is a random variable and chosen to be proportional to the result of an investment in a pure stock portfolio, i.e.,

$$Q = e^{\delta T} x S_T, \quad \delta \in \mathbb{R}.$$

The real number  $\delta$  measures the over- ( $\delta > 0$ ) or under-performance ( $\delta < 0$ ) of the stock market in terms of the annual logarithmic return. We refer to [29] for more details.

We define the tracking error of the portfolio manager's horizon wealth  $X_T$  relative to the benchmark  $Q$  as

$$G = G(X_T, Q) = U(X_T) - U(Q)$$

where the random variable  $G$  can be interpreted as the utility gain of the terminal wealth relative to the benchmark. Moreover, the shortfall risk is quantified by assigning to the random variable  $G$  a real valued risk measure  $\rho(G)$  given by

$$\rho(G) = \text{EUL}(G) := \mathbb{E}[G^-] = \mathbb{E}[(U(X_T) - U(Q))^-].$$

Here, the shortfall is related to the event, that the terminal wealth  $X_T$  of the portfolio is smaller than the benchmark  $Q = e^{\delta T} x S_T$  and we measure the shortfall risk using the Expected Utility Loss. Contrary to the case of a benchmarked money market, we find here that the decision of the portfolio manager depends on the sensitivity of the benchmark  $Q$  to economic conditions. In [5] VaR is used to measure the shortfall in the case of a stochastic benchmark.

## Partial information

In the second part of this thesis we deal with the case of *partial information*. This is the situation when neither the drift process  $\mu$  nor the Brownian motion  $W$  appearing in (1.3) are observable for the portfolio manager. We assume that he can only observe the stock price  $S$ . The volatility  $\sigma$  is known and constant. In contrast to the case of *full information*, the case of partial information is more realistic since prices and interest rates are published and available to the public, but drifts and paths of Brownian motions are only mathematical tools used to create models, but certainly not observable. Therefore, the fact that portfolio managers have only a partial information will be modeled by requiring that investment decisions and all processes appearing in this market have to be adapted to  $\mathcal{F}^S = (\mathcal{F}_t^S)_{t \in [0, T]}$ , the  $P$ -augmented filtration generated by the stock price, which is smaller than the original filtration. Basically, the portfolio optimization problem with partial information can be solved only if the dynamics of the unknown drift is specified. Without other restrictions such as risk management, models with partial informations were studied by Detemple [15], Dothan and Feldman [18], Genotte [32] in a linear Gaussian setting. Later Karatzas and Xu [43] adopted Bayesian approach for the utility maximization problems, by combining filtering and martingale representation theory. Karatzas and Zhao [44] reduced the optimization problem with partial observations to the case of a drift process which is adapted to the observation process, so that the martingale method can be applied. Using the same methodology Lakner [52] [53] proposed a Gaussian drift driven by an independent Brownian motion and used Clark's formula (see Theorem 17.1) to reduce the determination of the optimal trading strategy to the calculation of the Malliavin derivative  $D_t m_s$ ,  $s \in [t, T]$ , of the *Kalman filter*  $m_t = \mathbb{E}[\mu_t | \mathcal{F}_t^S]$ , which is the best estimator for the signal  $\mu_t$  given the observation  $R_s$ ,  $s \leq t$ , where  $R$  is the return process associated with stocks

$$dR_t = (\text{Diag}(S_t))^{-1} dS_t, \quad \text{or equivalently} \quad R_t = \int_0^t \mu_s ds + \int_0^t \sigma dW_s. \quad (1.5)$$

Lakner [53] exploits the fact that the filtering equation for  $m_t$  provides a closed form solution for the linear Kalman filter and computes  $D_t m_s$  explicitly.

Sass and Haussmann [65] proposed a drift as an independent continuous-time Markov chain in a model with constant stock volatility and simple dynamics for the interest rate process. In contrast of the case of Gaussian process, it is easier to consider the unnormalized filter  $\mathcal{E}_t$  (see Chapter IV, Definition 16.1) which satisfies a linear stochastic differential equation (SDE). The Malliavin derivative  $D\mathcal{E}_t$  of the unnormalized filter is characterized by another SDE, and the optimal strategy is given in [65] in terms of the unnormalized filter and its Malliavin derivative. In [66], Sass and Haussmann extend this model to stochastic volatility and more general stochastic interest rates whose dynamics are driven by another Brownian motion which is defined with respect to an equivalent risk neutral measure.

We consider a multi-stock market model as in [65], where prices satisfy a SDE with instantaneous rates of return  $\mu = (\mu_t)_{t \in [0, T]}$ , the *drift process* appearing in Equation (1.3), is given by

$$\mu_t = BY_t, \quad t \in [0, T], \quad (1.6)$$

where  $Y = (Y_t)_{t \in [0, T]}$  is a stationary, irreducible, *continuous time Markov chain* independent of  $W$  with state space  $\{e_1, \dots, e_d\}$ , the standard unit vectors in  $\mathbb{R}^d$ , and  $B = (B_{ik})_{i=1, \dots, n; k=1, \dots, d}$  is given by  $B_{ik} = b_i^k$ . The columns of the *state matrix*  $B \in \mathbb{R}^{n \times d}$  contain the  $d$  possible states of  $\mu_t$ . Moreover, the continuous Markov chain  $Y$  is characterized by its rate matrix  $Q \in \mathbb{R}^{d \times d}$ , where  $Q_{kl}$ ,  $k \neq l$ , is the jump rate or transition rate from  $e_k$  to  $e_l$ , and  $\lambda_k = \sum_{l=1, l \neq k}^d Q_{kl}$  the rate of leaving  $e_k$ . For investment decisions only the prices of the stocks are available. Thus we have a hidden Markov model (HMM) for the stock returns. Under restrictions on the loss in utility compared to a benchmark, we embed a risk management task to the portfolio optimization problem. Combining filtering and martingale method, we obtain in Proposition 15.2 the optimal terminal wealth  $X_T^* = f(\zeta_T)$  as a function of the conditional state price density  $\zeta_T$ . The function  $f$  does not fulfill the conditions of chain rule to state that our optimal terminal wealth has a Malliavin derivative. To do so, we use approximation arguments in a way to approximate  $f$  by a sequence of functionals  $(f_n)_{n \in \mathbb{N}^*}$  for which the chain rule is applicable and its Malliavin derivative  $(f_n(\zeta_T))_{n \in \mathbb{N}^*}$  converges to a limit proved to be the Malliavin derivative of our terminal wealth  $X_T^* = f(\zeta_T)$ . To derive the optimal corresponding strategy, we use the extension of Clark's formula from  $D_{2,1}$  to  $D_{1,1}$ , because we avoid unnecessarily restrictive moment bound on our terminal wealth  $X_T^*$ . In our model it is convenient to work in  $\mathbb{D} = \bigcap_{p>1} \mathbb{D}_{p,1}$  as subspace of  $\mathbb{D}_{1,1}$ , since the corresponding integrability conditions we need to apply our chain rules follow directly from Hölder's inequality. In our main result, we provide in Proposition 18.3 an explicit representation for the optimal trading strategy in terms of observable processes and in terms of the unnormalized filter  $\mathcal{E}$  and the current wealth  $X_t^*$ . All quantities involved are adapted to the augmented filtration of the stock prices  $\mathcal{F}^S$ . Further, the filters and their derivatives can be approximated very well because of the linear structure of the equations they are satisfying. Finally, we examine the particular case of a constant drift. Here the unnormalized filter coincides with the inverse of the state price density and its Malliavin derivative can be computed explicitly so that the optimal trading

strategy coincides with the optimal trading strategy we have obtained in the case of full information under the same risk constraint. Although the two methodologies of full information and partial information are different, they lead to the same optimal trading strategy in the particular case of a Markovian model with a constant drift.

## Outline

In Chapter I we set up the background concerning the expected utility maximization in the familiar continuous-time and complete setting. Section 2 introduces in general way the Brownian-motion-driven model for our financial market which consists of one risk-free bond and  $n$  stocks, the later being driven by an  $n$ -dimensional Brownian motion. The prices of risky assets and risk-free bond are supposed to evolve according to Black-Scholes model [9]. Then we describe the different investment decisions of an investor acting in this market by introducing three different equivalent quantities modeling the trading operations in the market. These equivalent quantities are the number of shares  $\phi_t^i$  held in each stock  $S_t^i$  at time  $t$ , the amount of wealth  $\pi_t^i = \phi_t^i S_t^i$  invested in this stock, and the fraction of wealth  $\theta_t^i = \frac{\pi_t^i}{X_t}$  invested in this stock, where  $X_t$  is the wealth generated by  $\phi$ . Section 3 presents the different properties that a given utility function should have depending to the investor's taste of risk. Typical economic utility functions are presented at the end of this section. Section 4 deals with the portfolio optimization problem of a portfolio manager who receives a deterministic initial capital, which he must then invest in a complete market so as to maximize the expected utility of his wealth at a prespecified final horizon. We set up the martingale method used to solve this problem in the case of complete market. The martingale approach is introduced by Karatzas et al. [38], Cox and Huang [11] and it proceeds basically in three steps. First, on the underlying probability space we determine a new measure which discounts the growth inherent in the market. Under this measure, the expected value of the discounted final wealth attained by any reasonable portfolio is equal to the initial endowment. Second among all random variables whose expectation under the new measure is equal to the initial endowment, a most desirable one is determined. Third, it is shown that a portfolio can be constructed in a way that attains this most desirable terminal wealth. This portfolio is optimal, and its construction uses the fact that any martingale with respect to a Brownian filtration can be represented as a stochastic integral with respect to the Brownian motion, the integrand in this representation leads to the optimal portfolio. Section 5 gives a short review of risk measures used in this thesis and their properties.

In Chapter II we examine the portfolio optimization problem where the shortfall risk is concerned with a deterministic shortfall level  $Q := q$  which can be written as  $q = xe^{(r+\delta)T}$ ,  $\delta \in \mathbb{R}$ , where  $x$  is the initial capital. Basically, we associate the risk with the random variable  $G = X_T - q$  and we adopt three different risk constraints:

- the Value at Risk:  $P(G < 0) = P(X_T < q)$ ,
- the Expected Loss:  $EL(G) = E[G^-] = E[(X_T - q)^-]$ ,

- and the Expected Utility Loss  $EUL(G) = E[G^-] = E[(U(X_T) - U(q))^-]$ .

In order to analyse the impact of the different risk constraints to the behavior of the portfolio manager, we formulate the dynamic optimization problem of maximizing the expected utility from terminal wealth with additional risk constraints bounding the different forms of loss with a given  $\varepsilon$ .

To solve the dynamic optimization problem associated with risk constraints, we adapt the martingale representation approach [Karatzas et al. [38], Cox and Huang [11]], which allows the problem to be restated as a static variational problem used to obtain the optimal candidate maximizing the expected utility among the set of all admissible terminal wealth. By exploiting the market completeness, it turns out that there exists a strategy whose terminal wealth coincides with the optimal solution given by the static problem. From the other hand the static problem is solved by adapting the common duality approach, thereby we define the convex conjugate of the utility function to which we add additional terms capturing the different risk constraints.

First in Section 6 the shortfall probability or equivalently the Value at Risk is bounded and added in a form of risk constraint to the optimization. We follow the paper of Basak and Shapiro [4], but give slightly different solutions. In Section 7 the Expected Loss is bounded and added to the optimization. This case is not considered explicitly in [4] and we give the detailed solution for the case of a CRRA utility function. Section 8 deals with the Expected Utility Loss as a modification of the Expected Loss constraint considered in Section 7. Finally, Section 9 illustrates the findings of the preceding sections with an example.

In Chapter III we investigate the impact of adding a Utility Expected Loss constraint to the problem of portfolio manager who aims to beat the return of a given portfolio. More precisely, we deal with a portfolio manager who manages the relative performance, or tracking error of his portfolio along with other objectives. For a given benchmark  $Q$  representing the performance of a portfolio or an index or any economic indicator. The task in this chapter is to examine the behavior of a portfolio manager benchmarking the stock price, this situation leads to a random benchmark which is proportional to the result of an investment in a pure stock portfolio. The portfolio optimization problem is formulated with a risk constraint bounding the Expected Utility Loss with a given  $\varepsilon$ . Contrary to the case of a benchmarked money market studied in Chapter II, we find here that the decision of the portfolio manager depends on the sensitivity of the benchmark to economic conditions.

Section 11 describes the economy and presents the different facts related to the benchmarking of the stock market. Subsection 11.1 solves the portfolio optimization problem under the Expected Utility Loss constraint. In particular, we characterize the optimal terminal wealth and its associated optimal strategy. Section 12 presents some properties of the optimal portfolio and its asymptotic behavior when the time  $t$  approaches the horizon time  $T$ . Finally Section 13 illustrates the findings of the previous sections with numerical examples.

In Chapter IV we investigate the portfolio optimization problem for a model of the financial market with partial information by choosing portfolio strategies based only on information about the asset-prices. We embed this problem to a risk management

by requiring that the loss in utility compared to a benchmark has to be bounded. We specify the dynamics of the drift process as a stationary, irreducible, *continuous time Markov chain* independent of  $W$ . Then we combine the results of filtering obtained by Sass and Haussmann in [65] with the results of convex-duality approach we have obtained in Chapter II to get the optimal strategy (Proposition 18.3) which is expressed in terms of the unnormalized filter  $\mathcal{E}$ , its Malliavin derivative  $D\mathcal{E}$ , and the parameters of the model. These quantities are all  $\mathcal{F}^S$ -adapted. In Section 14 we introduce the market model, where we consider one risk-free asset whose price process is assumed for simplicity to be equal to 1 at each date. Similar to the case of full information studied in Chapter II, Section 15 deals with the portfolio optimization problem under partial information which we solve using the martingale duality approach. The basic difference is that instead of the state price density

$$Z_t = \exp\left(-\int_0^t (\sigma^{-1}BY_s)^\top dW_s - \frac{1}{2}\int_0^t \|\sigma^{-1}BY_s\|^2 ds\right),$$

we use  $\zeta_t = E[Z_t|\mathcal{F}_t^S]$ , the  $\mathcal{F}^S$ -conditional state price density as driving process for the economy. In Section 16 we present the results of HMM filtering. In Section 17 we use approximation arguments to prove that the optimal terminal wealth we have obtained in Section 15, has a Malliavin derivative. Finally, in Section 18 we derive our main result Theorem 18.1, which provides the optimal trading strategy in terms of the unnormalized filter, its Malliavin derivative, the return process, and the parameters of the model. These processes are all adapted to  $\mathcal{F}^S$ , and are observable or can be estimated. The particular logarithmic and power utility allows us to compute in Proposition 18.3 the optimal trading strategy in terms of the current unnormalized filter, the current terminal wealth, and the return process. Finally we prove that in the case of a constant drift, our optimal trading strategy obtained in Proposition 18.3 coincides with the optimal trading strategy we have obtained in the case of full information (Chapter II) under the same Expected Utility Loss. The Appendix contains proofs of propositions and lemmas related to our portfolio optimization problems. In Appendix G we recall results about Malliavin calculus and the chain rules we need to apply Clark's formula.

## Acknowledgements

I would like to say how I am grateful to everyone who has taught me about mathematics from the first class of school until this moment where this work is achieved.

It was for me a very nice destiny to meet you Professor Wilfried Grecksch. I am really unable to find the words to express my gratitude to you Professor. I am deeply grateful to you for supervising me and giving me the opportunity to work in your group at the institute of optimization und stochastic at Martin-Luther university Halle-Wittenberg. In spite of all your various duties, you used to get time to discuss with me mathematics and in a very nice way. I would like to tell you how grateful I am.

I feel unable to find the words to thank you Professor Ralf Wunderlich. I am very grateful to you Professor for arising my interest to risk management and for supervising me together with Professor Wilfried Grecksch. I like to discuss with you problems in mathematics. Thanks to you and to your family.

I will be still very grateful to you Professor Abdelaziz Rhandi for directing my attention to financial mathematics about five years ago, thank you very much Professor for your help, for your encouragement and stimulation.

I would like to thank Professor Rüdiger Frey for attracting my interest to financial markets with incomplete information, Doctor Jörn Saß for stimulations and fruitful decisive discussions on portfolio optimization under partial information, and Professor Youssef Ouknin who has taught me so much about mathematics and let me appreciate stochastic analysis.

I would like to thank all my friends, staff of the institute of optimization und stochastic at Martin-Luther university for their help and nice friendship.

The financial support of this work by the Deutsche Akademische Austauschdienst (DAAD) and Martin-Luther University (MLU) is also gratefully appreciated.

I am very grateful to my close family, particularly my father (Ba M'hammed), my mother (Mi Fatima), my brother Ibrahim, my sisters Chaiibia, Fatiha and to you Khadija 'Zina'. I say thanks for your patience over the years, you helped me, you encouraged me, you supported me. I would never be this far if you were not here. A special thanks goes to El Haj Fetnassi, Es Sadik and Rachida. To my little 'sisters' Nidal, Bassma, Ikram, Salma, Fatimzahra, and Diaa I say I love you all and very much.

To the man with whom I used to get inspiration in life, to the man who learn me to inspire and appreciate the beauty, the passion and the nice sense of life, to my forever

friend and brother Moulay Ahmed Bouchti.

I would like to thank my friends Professor El Kahoui M'hmmmed and Professor Maniar Lahcen for their encouragement and for their nice friendship.

To my forever friends Mofadal Attaib, Mostafa Kalmoun, Mohammed Barahhou, Otmane Souhar, Sabour Ibrahim, Es Saghir Abdelhadi, and Hassan Essaky. I would like to say thank you for your nice friendship and fraternity.

To my friends Abdelhadi Warii, Mustafa Mabkhout, Hicham Alex, Younes el Maghrawi and to you Samir Lemjihdi I say thanks friends for everything.

*Wer in seiner Leidenschaft entschläft, ist besser als wer seine Leidenschaft verliert.*

*Celui qui se perd dans sa passion a moins perdu que celui qui perd sa passion.*

Abdelali Gabih.



## General notation

$\mathbb{N}, \mathbb{R}$	set of integer, real numbers
$v^\top$	transposition of the vector $v$
$\mathbf{X} = (X_{ij})_{i,j=1,\dots,n}$	matrix
$\ \mathbf{x}\  = (\mathbf{x}, \mathbf{x})^{\frac{1}{2}}$	Euclidean norm
$\text{Diag}(v)$	diagonal matrix with diagonal $v$
$\mathbf{1}_n$	$n$ -dimensional vector whose entries are all equal 1
$\mathbf{1}_{n \times d}$	a matrix whose entries are all equal 1
$\mathbf{C}^k(\mathcal{D}), k \in \mathbb{N}^*$	set of $k$ -times continuously differentiable functions on $\mathcal{D}$
$\mathbf{C}_c^k(\mathcal{D}),$	functions of $\mathbf{C}^k(\mathcal{D})$ with a compact support
$\mathbf{C}_b^\infty(\mathcal{D}),$	functions of $\mathbf{C}^\infty(\mathcal{D})$ which are bounded and have bounded derivatives of all orders
$\mathbf{L}_p(\mathcal{D}), p \in \mathbb{N}$	set of functions on $\mathcal{D}$ with $\int_{\mathcal{D}} \ \mathbf{f}(\mathbf{x})\ ^p \mathbf{d}\mathbf{x} < \infty$
$\mathbf{1}_M(x)$	indicator function
$(\Omega, \mathcal{H}, \mathbf{P})$	complete probability space
$\omega$	element of $\Omega$
$\mathbf{E}[F]$	expectation of the random variable $F$
$\mathbf{E}[F \mathcal{F}]$	conditional expectation of the random variable $F$ given the $\sigma$ -algebra $\mathcal{F}$
$\mathcal{F}^X = (\mathcal{F}_t^X)_{t \in [0, T]}$	the augmented $\sigma$ -algebra generated by an $\mathcal{F}$ -adapted process $X$
$\mathcal{S}$	class of smooth random variables. See Appendix G
$DF$	Malliavin derivative of the smooth random variable $F$
$\ F\ _{p,1}$	norm, see the definition in Appendix G
$\mathbf{D}_{p,1}, p \geq 1$	the closure of space $\mathcal{S}$ with respect to the norm $\ F\ _{p,1}$

# Chapter I

## The Portfolio optimization problem: A general overview

### 2 Financial markets

We consider a continuous-time economy with finite horizon  $[0, T]$  which is built on a complete probability space  $(\Omega, \mathcal{H}, P)$ , on which is defined an  $n$ -dimensional Brownian motion  $W = (W_t)_{t \in [0, T]}$ ,  $W_t = (W_t^1, \dots, W_t^n)^\top$ . We shall denote by  $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]}$  the  $P$ -augmentation of the natural filtration and assume that  $\mathcal{H} = \mathcal{F}_T$ . Further, we assume that all stochastic processes are adapted to  $\mathcal{F}$ . It is assumed through this thesis that all inequalities as well as equalities hold  $P$ -almost surely. Moreover, it is assumed that all stated processes are well defined without giving any regularity conditions ensuring this. As in the Black-Scholes model [9], financial investment opportunities are given by an instantaneously *risk-free market account* providing an interest rate  $r$  so that its price  $S^0 = (S_t^0)_{t \in [0, T]}$  is given by

$$dS_t^0 = rS_t^0 dt, \quad (2.1)$$

and  $n$  *risky stocks* whose prices  $S = (S_t)_{t \in [0, T]}$ ,  $S_t = (S_t^1, \dots, S_t^n)^\top$  evolve according to

$$dS_t^i = S_t^i [\mu_i dt + \sum_{j=1}^n \sigma_{ij} dW_t^j], \quad S_0^i = s_0^i \in \mathbb{R}, \quad i = 1, \dots, n, \quad (2.2)$$

where the interest rate  $r$ , the stock instantaneous mean return  $\mu = (\mu_1, \dots, \mu_n)^\top$  and the volatility matrix  $\sigma = (\sigma_{ij})_{i,j=1,\dots,n}$  may be time-dependent, but defined in a way to insure the integrability conditions. Trading in the this economy requires from an investor which we shall call *portfolio manager*

- an initial capital  $x > 0$ ,
- a portfolio process  $\theta = (\theta_t)_{t \in [0, T]}$ ,  $\theta_t = (\theta_t^1, \dots, \theta_t^n)^\top$  where  $\theta_t^i$  indicates the fraction of wealth invested in stock  $i$  at time  $t$ ,

- a consumption process  $c = (c_t)_{t \in [0, T]}$  where  $c_t$  indicates the rate with which the wealth generated by the portfolio  $\theta$  is consumed at time  $t$ .

Depending on the financial conditions of the investor one can impose to the portfolio and the consumption processes appropriate conditions to determine the set of admissible strategies. A classical example would be that strategies taking values only in the interval  $[0, 1]$ , i.e., there is no short-selling.

At any time  $t$ , a portfolio manager acts in this market by choosing a trading strategy  $\psi = (\psi_t)_{t \in [0, T]}$ ,  $\psi_t = (\psi_t^0, \dots, \psi_t^n)^\top$  where  $\psi_t^i$  is the number of shares held by the portfolio manager in the asset  $i$ . The  $\mathbb{R}^{n+1}$ -valued process  $\psi$  is assumed to be  $\mathcal{F}$ -measurable such that

$$\sum_{i=0}^n \int_0^T (\psi_t^i S_t^i)^2 dt < \infty.$$

The wealth process  $X_t$  of the portfolio manager is defined at time  $t$  in terms of the trading strategy by

$$X_t = \sum_{i=0}^n \psi_t^i S_t^i.$$

Moreover, we consider that the trading strategy is self-financing in the sense that no other money is going in or out the market except the money generated by the trading strategy, see [49]. Under this assumption and if the wealth  $X_t > 0$ ,  $P$ -a.s., the portfolio manager can act in the market using the associated portfolio process  $\theta = (\theta_t)_{t \in [0, T]}$ ,  $\theta_t = (\theta_t^1, \dots, \theta_t^n)^\top$  defined as the fractions of wealth invested in each risky stock, i.e.,

$$\theta_t^i = \frac{\psi_t^i S_t^i}{X_t^\theta}, \quad i = 1, \dots, n,$$

with  $\theta_t^0 = 1 - \sum_{i=1}^n \theta_t^i$  is the fraction of wealth invested in the risk-free bond. As a consequence, the wealth process can be formulated in terms of the portfolio process as a linear stochastic differential equation given by

$$dX_t^\theta = X_t^\theta \left( r \left( 1 - \sum_{i=1}^n \theta_t^i \right) + \sum_{i=1}^n \mu_i \theta_t^i \right) dt + X_t^\theta \left( \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} \theta_t^i dW_t^j \right).$$

Using matrix notation, the dynamics of the wealth process can be formulated as

$$dX_t^\theta = X_t^\theta [r + \theta_t^\top (\mu - r \mathbf{1}_n)] dt + X_t^\theta \theta_t^\top \sigma dW_t, \quad X_0^\theta = x. \quad (2.3)$$

**Remark 2.1.** In Chapter IV, we shall use another equivalent quantity for the investment decision which is the amount of wealth invested in risky stocks. More precisely, we define the amount of wealth invested in the risky stocks by  $\pi = (\pi_t)_{t \in [0, T]}$ ,  $\pi_t = (\pi_t^1, \dots, \pi_t^n)^\top$ , where  $\pi_t^i = \psi_t^i S_t^i$  indicates the amount of wealth invested in stock  $i$  at time  $t$ . Further,  $\pi_t^0 = X_t - \sum_{i=1}^n \pi_t^i$  is the amount of wealth invested in the risk-free bond. As consequence the wealth process can be reformulated in terms of the process  $\pi$  as

$$dX_t^\pi = \pi_t^\top (\mu dt + \sigma dW_t) + (X_t^\pi - \mathbf{1}_n^\top \pi_t) r dt, \quad X_0^\pi = x. \quad (2.4)$$

### 3 Utility function of an investor

In his exposition about the theory on the measurement of risk Bernoulli [6] proposed that the value of an item should not be determined by the price somebody has to pay for it, but by the utility that this item has for the owner. A classical example would be that a glass of water has a much higher utility for somebody who is lost in the desert than somebody in the civilization. Although the glass of water might be exactly the same and therefore its price, the two persons in the mentioned situation will perceive its value differently.

Following partially the structure given in [22] Chapter 10, we discuss different properties that a given utility function should have and we look at some typical economic utility functions. Intuitively everybody prefers more wealth  $X$  than less wealth and this is the first property we are interested to get for the utility function. Economists call this the non-satiation attribute and it expresses the fact that an investment with higher return has always a higher utility than an investment with a lower return assuming that both options equally likely.

From this it seems to be important to focus on a utility function whose first derivative has to be positive. Hence, the first requirement placed on a utility function  $U$  from a wealth parameter  $X$  is therefore

$$U'(X) > 0.$$

The second property of a utility function is an assumption about an investor's taste of risk. Three assumptions are possible: The investor is averse to risk, the investor is neutral toward risk and the investor seeks risk. A simple example illustrating the three different risks is the so called fair gamble which is an investment with expected value equal to its cost.

Invest		Do Not Invest	
Outcome	Probability	Outcome	Probability
2	1/2	1	1
0	1/2		

The option "invest" has an expected value of  $\frac{1}{2}2 + \frac{1}{2}0 = 1$  Euro. Assume that the investor would have to pay 1 Euro to undertake this investment and obtain these outcomes. If the investor prefers to not invest, then 1 Euro is kept. In this case the expected utility of not investing must be higher than the expected utility of investing

$$U(1) > \frac{1}{2}U(2) + \frac{1}{2}U(0),$$

which implies that

$$U(1) - U(0) > U(2) - U(1),$$

and this inequality expresses that the change in utility when wealth changes from 0 to 1 is more valuable than the change in utility when wealth changes from 1 to 2 and hence the utility is modeled by a function such that an additional unit increase is

less valuable than the last unit increase. This property is fulfilled for a function with negative second derivative.

Risk neutrality means that an investor is indifferent to whether or not a fair gamble option is undertaken. When the investor is indifferent between investing and not investing, the expected utility of investing, or not investing, must be the same

$$U(1) = \frac{1}{2}U(2) + \frac{1}{2}U(0),$$

and this yields after rearranging

$$U(1) - U(0) = U(2) - U(1).$$

This expression implies that the change in utility of wealth is independent of the changes in wealth. Such a characteristic is fulfilled for functions that exhibit a zero second derivative, therefore indifference to a fair gamble implies a utility function that should have a zero second derivative.

Risk seeking means that the investor would select a fair gamble and hence the expected utility of investment must be higher than the expected utility of not investing

$$U(1) < \frac{1}{2}U(2) + \frac{1}{2}U(0),$$

and this yields after rearranging

$$U(1) - U(0) < U(2) - U(1).$$

This inequality expresses that the utility of one-unit change from 2 to 1 is greater than the utility of one-unit change from 0 to 1. Functions with positive second derivative are functions that exhibit the property of greater change in value for larger unit changes in the argument.

Another property of utility functions can be derived from the example of an investor who is deriving some utility from the wealth obtained by investing in risky assets and risk-free bond. Here, the property is an assumption about how the size of the wealth invested in risky assets changes when the size of the wealth has changed. In other words, if the investor's wealth increases, will be more or less of that wealth invested in risky stock? Here three kinds of investor's behavior are possible:

- Decreasing absolute risk aversion: The investor increases the amount invested in risky assets when the wealth increases.
- Constant absolute risk aversion: The investor keeps the amount invested in risky assets when the wealth increases.
- Increasing absolute risk aversion: The investor decreases the amount invested in risky assets when the wealth increases.

As it is pointed out in [22], the absolute risk aversion can be measured by

$$A(X) = -\frac{U''(X)}{U'(X)},$$

and as a consequence the type of the investor can be determined according to  $A'(X)$ :

- $A'(X) > 0$  : Increasing absolute risk aversion
- $A'(X) = 0$  : Constant absolute risk aversion
- $A'(X) < 0$  : Decreasing absolute risk aversion.

It is also possible to use the change of the percentage of wealth invested in risky assets as wealth changes. This is evaluated by

$$R(X) = -\frac{XU''(X)}{U'(X)} = XA(X),$$

and has the following interpretation

- $R'(X) > 0$  : Increasing relative risk aversion
- $R'(X) = 0$  : Constant relative risk aversion
- $R'(X) < 0$  : Decreasing relative risk aversion.

The most frequently used utility function is the power utility function

$$U(z) = \begin{cases} \frac{z^{1-\gamma}}{1-\gamma}, & \gamma \in (0, \infty) \setminus \{1\}, \\ \ln z, & \gamma = 1. \end{cases} \quad (3.5)$$

With positive first derivative and negative second derivative, the power utility function (3.5) meets the requirement of risk averse investor who prefers more than less wealth. Moreover, this function exhibits decreasing absolute risk aversion and constant relative risk aversion. The parameter  $\gamma$  of the power utility function can be interpreted as relative risk aversion since in this case  $R(X) = \gamma$ .

**Remark 3.1.** If we choose a logarithmic utility function  $U(z) = \ln z$ , i.e., we set  $\gamma = 1$ , then the utility  $U(X)$  of the terminal wealth  $X$  is equivalent to the annual logarithmic return. The annual logarithmic return is defined as

$$L(X) := \frac{1}{T} \ln \frac{X}{x},$$

where  $x$  is the initial capital, i.e.,  $X_0 = x$ .

## 4 The portfolio optimization problem

### 4.1 Pointwise maximization

In this section we examine the particular case of portfolio optimization when the investor maximizes the expected logarithmic utility of the terminal wealth of one stock with a constant stock return  $\mu$  and a constant volatility  $\sigma$ , and a bond with a constant interest rate  $r$ . The portfolio manager, starts with initial capital  $x > 0$  and follows a trading strategy  $\theta = (\theta_t)_{t \in [0, T]}$  which leads to the wealth  $X^\theta = (X_t^\theta)_{t \in [0, T]}$  defined by

$$dX_t^\theta = X_t^\theta[(r + \theta_t(\mu - r))dt + \theta_t\sigma dW_t], \quad X_0^\theta = x,$$

which can be expressed as

$$X_t^\theta = x \exp\left(\int_0^t \left(r + \theta_s(\mu - r) - \frac{1}{2}(\theta_s\sigma)^2\right) ds + \int_0^t \theta_s\sigma dW_s\right).$$

The strategy  $\theta$  is required to be such that the stochastic integral  $(\int_0^t \theta_s\sigma dW_s)_{t \in [0, T]}$  is a martingale, which is the case when the inequality  $\mathbb{E}\left[\int_0^t \theta_s^2 ds\right] < \infty$  is fulfilled, or when  $\theta$  is assumed to be a bounded and deterministic. Moreover, the particular structure of the logarithmic utility allows to get

$$\begin{aligned} \mathbb{E}[\ln(X_T^\theta)] &= \ln(x) + \mathbb{E}\left[\int_0^t \left(r + \theta_s(\mu - r) - \frac{1}{2}(\theta_s\sigma)^2\right) ds + \int_0^t \theta_s\sigma dW_s\right] \\ &= \ln(x) + \mathbb{E}\left[\int_0^t \left(r + \theta_s(\mu - r) - \frac{1}{2}(\theta_s\sigma)^2\right) ds\right]. \end{aligned}$$

The portfolio maximization is now equivalent to the following pointwise maximization problem

$$\text{find a strategy } \theta^* \text{ which maximizes } \left(r + \theta_t(\mu - r) - \frac{1}{2}(\theta_t\sigma)^2\right).$$

Here, the optimal solution is given by a constant strategy

$$\theta_t^* = \theta^*, \quad \text{for all } 0 \leq t \leq T \quad \text{with} \quad \theta^* = \frac{\mu - r}{\sigma^2}.$$

This strategy is bounded and hence satisfies the required condition for admissibility. Moreover, if the coefficients of the model are such that  $\mu > r$  and  $\mu - r < \sigma^2$ , this strategy takes values in the interval  $(0, 1)$ .

### 4.2 Stochastic control method

The stochastic control method is adequate for Markov models. This method consists in deriving the so called Hamilton-Jacobi-Bellman equation which allows the analysis of the model. Merton [56] was the first who applied stochastic control method to

a dynamic optimization problem in a standard Black-Scholes model with constant coefficients.

At time  $t$  the portfolio manager is assumed to consume its wealth at consumption rate  $c_t$ , where by definition a consumption rate is a non-negative  $\mathcal{F}_t$ -adapted process  $c = (c_t)_{t \in [0, T]}$  such that

$$\int_0^T c_t dt < \infty.$$

With the additional consumption rate, the wealth process is formulated as controlled stochastic process satisfying

$$dX_t^{\theta, c} = [rX_t^{\theta, c} - c_t]dt + X_t^{\theta, c}\theta_t[(\mu - r)dt + \sigma dW_t], \quad X_0^{\theta, c} = x,$$

and the dynamic optimization problem reads as follows

$$\text{find a pair } (\theta^*, c^*) \text{ which maximizes } \mathbb{E} \left[ \int_0^T e^{-\alpha t} U_1(c_t) dt + U_2(X_T^{\theta, c}) \right],$$

for a given  $\alpha \in (0, 1)$  and a utility functions  $U_1$  and  $U_2$ . Let us define the value function of the maximization problem as

$$V(t, x) = \sup_{(\theta_s, c_s)_{t \leq s \leq T}} \mathbb{E} \left[ \int_t^T e^{-\alpha s} U_1(c_s) ds + U_2(X_T^{\theta, c}) \right].$$

The value function expresses the evaluation of the maximal value of the portfolio manager's costs as function of the initial capital  $x$  at the starting time  $t$ . Moreover, Bellman principle implies that

$$V(t, x) = \sup_{(\theta_s, c_s)_{t \leq s \leq u}} \mathbb{E} \left[ \int_t^u e^{-\alpha s} U_1(c_s) ds + V(u, X_u^{\theta, c}) \right],$$

for  $u \leq T$ . Itô's lemma applied to the function  $V(t, x)$  leads to the so called Hamilton-Jacobi-Bellman equation

$$\begin{aligned} \sup_{\alpha_1 \leq \theta \leq \alpha_2, c \geq 0} \left[ \frac{1}{2} (x\sigma\theta)^2 V_{xx}(t, x) + \left( (1 - \theta)xr + \theta x\mu - c \right) V_x(t, x) \right. \\ \left. + V_t(t, x) - rV(t, x) + U_1(c) \right] = 0, \\ \text{where } V(T, x) = U_2(x), \quad V(t, 0) = U_2(0), \end{aligned}$$

where  $\alpha_1, \alpha_2$  are real numbers. Solving this HJB equation leads to the optimal value function  $V(t, x)$  as a solution of a partial differential equation obtained by substituting the optimal strategy and the consumption processes in the HJB-equation. These optimal processes are obtained here independently of the derivatives  $V_t$ ,  $V_x$  and  $V_{xx}$ .



### 4.3 Martingale method

In this section we address the portfolio optimization problem of a portfolio manager who wishes to maximize the expected utility from terminal wealth without any other restrictions such as risk management. This problem is solved by [Karatzas et al. [38], Cox and Huang [11]] using the so called *martingale approach* which we present in this section. As in Section 2, we consider a risk-free money market account with a constant interest rate  $r$  so that its price at time  $t$ , is  $S_t^0 = e^{-rt}$ . Further we consider  $n$  risky stocks whose prices  $S = (S_t)_{t \in [0, T]}$ ,  $S_t = (S_t^1, \dots, S_t^n)^\top$  evolve according to

$$dS_t^i = S_t^i [\mu_i dt + \sum_{j=1}^n \sigma_{ij} dW_t^j], \quad S_0^i = s_0 \in \mathbb{R}, \quad i = 1, \dots, n, \quad (4.6)$$

with stock instantaneous mean returns  $\mu = (\mu_1, \dots, \mu_n)^\top$  and the volatility matrix  $\sigma = (\sigma_{ij})_{i,j=1,\dots,n}$  are assumed to be constants. Moreover,  $\sigma$  is assumed to satisfy the non-degeneracy condition

$$x^\top \sigma \sigma^\top x \geq \delta x^\top x \quad \text{for all } x \in \mathbb{R}^n,$$

where  $\delta > 0$  is a given constant.

The dynamic market completeness implies the existence of a unique state price density process  $H = (H_t)_{t \in [0, T]}$ , given by

$$dH_t = -H_t(r dt + \kappa^\top dW_t), \quad H_0 = 1, \quad (4.7)$$

where  $\kappa := \sigma^{-1}(\mu - r\mathbf{1}_n)$  is the market price of risk in the economy and  $\mathbf{1}_n$  is the  $n$ -dimensional vector whose all components are one.

The market price of risk  $\kappa$  or equivalently the state price density process  $H$  can be regarded as the driving economic parameter in a portfolio managers dynamic investment problem.

As it is outlined in Section 2, the portfolio manager is acting in this market using the portfolio process  $\theta = (\theta_t)_{t \in [0, T]}$ ,  $\theta_t = (\theta_t^1, \dots, \theta_t^n)^\top$  defined as the fractions of wealth invested in each risky stock, i.e.,

$$\theta_t^i = \frac{\psi_t^i S_t^i}{X_t^\theta}, \quad i = 1, \dots, n,$$

with  $\theta_t^0 = 1 - \sum_{i=1}^n \theta_t^i$  is the fraction of wealth invested in the risk-free bond. As a consequence, the wealth process can be formulated in terms of the portfolio process as a linear stochastic differential equation given by

$$dX_t^\theta = X_t^\theta [r + \theta_t^\top (\mu - r\mathbf{1}_n)] dt + X_t^\theta \theta_t^\top \sigma dW_t, \quad X_0^\theta = x. \quad (4.8)$$

At time  $t = T$  the portfolio manager reaches the terminal wealth  $X_T^\theta$ . Thereby, the portfolio process  $\theta$  is assumed to be admissible in the following sense.

**Definition 4.1.**

Given  $x > 0$ , we say that a portfolio process  $\theta$  is admissible at  $x$ , if the wealth process  $X^\theta = (X_t^\theta)_{t \in [0, T]}$  starting at  $X_0^\theta = x$  satisfies  $P(X_t^\theta > 0, \text{ for all } t \in [0, T]) = 1$ . We shall denote by  $\mathcal{A}(x)$  the set of all admissible strategies.

**Definition 4.2.**

We call  $U : [0, \infty) \mapsto \mathbb{R} \cup \{-\infty\}$ , a utility function if  $U$  is strictly increasing, strictly concave, twice continuously differentiable on  $(0, \infty)$ , and its derivative satisfies  $\lim_{x \rightarrow 0^+} U'(x) = \infty$  and  $\lim_{x \rightarrow \infty} U'(x) = 0$ . Further we denote by  $I : (0, \infty) \mapsto (0, \infty)$  the inverse function of  $U'$ .

In this economy, the portfolio manager is assumed to derive from the terminal wealth  $X_T^\theta$  a utility  $U(X_T^\theta)$  and his objective is to maximize the expected utility by choosing an optimal strategy  $\theta^*$  from the set of admissible strategies  $\mathcal{A}(x)$ .

**The dynamic problem**

Find a strategy  $\theta^*$  in  $\mathcal{A}(x)$  that solves

$$\max_{\theta \in \mathcal{A}(x)} \mathbb{E}[U(X_T^\theta)]. \quad (4.9)$$

With no additional restrictions such as risk management, the maximization problem (4.9) was solved in the case of a complete market, by Cox and Huang [11] and independently by Karatzas, Lehocky and Shreve [38] using martingale and duality approaches.

In fact, Itô's Formula implies that the process  $(H_t X_t^\theta)_{t \in [0, T]}$  is a supermartingale which implies that the so called *budget constraint*

$$\mathbb{E}[H_T X_T^\theta] \leq x \quad (4.10)$$

is satisfied for every  $\theta \in \mathcal{A}(x)$ . This means that the expected discounted terminal wealth can not exceed the initial wealth.

In the present case of a complete market, the following theorem is a basic tool in martingale method, see [41].

**Theorem 4.3.**

Let  $x > 0$  be given and let  $\xi$  be a nonnegative,  $\mathcal{F}_T$ -measurable random variable such that

$$\mathbb{E}[H_T \xi] = x.$$

Then there exists a portfolio process  $\theta$  in  $\mathcal{A}(x)$  such that  $\xi = X_T^\theta$ .

Define

$$\mathcal{B}(x) := \{\xi > 0 : \xi \text{ is } \mathcal{F}_T\text{-measurable and } \mathbb{E}[H_T \xi] \leq x\}.$$

In contrast to the dynamic problem, where the portfolio manager is required to maximize the expected utility from terminal wealth over a set of processes, in a first

step the static problem is considered. Here the portfolio manager has the advantage to maximize only over a set of random variables which are the admissible terminal wealth.

### The static problem

Find a random variable  $\xi^*$  in  $\mathcal{B}(x)$  that solves

$$\max_{\xi \in \mathcal{B}(x)} \mathbb{E}[U(\xi)]. \quad (4.11)$$

### Assumption

$$\mathbb{E}[H_T I(yH_T)] < \infty \quad \text{for all } y \in (0, \infty).$$

Under the last assumption the function  $\chi : (0, \infty) \rightarrow (0, \infty)$  defined by  $\chi(y) := \mathbb{E}[H_T I(yH_T)]$  inherits the property of being a continuous, strictly decreasing mapping of  $(0, \infty)$  onto itself, and so  $\chi$  has a (continuous, strictly decreasing) inverse  $\chi^{-1}$  from  $(0, \infty)$  onto itself. So we define

$$\xi^* := I(\chi^{-1}(x)H_T),$$

where  $x$  is the initial endowment of the portfolio manager.

**Proposition 4.4.** The random variable  $\xi^* := I(\chi^{-1}(x)H_T)$  satisfies

$$\mathbb{E}[H_T \xi^*] = x, \quad (4.12)$$

and for every portfolio  $\theta \in \mathcal{A}(x)$ , we have

$$\mathbb{E}[U(X_T^\theta)] \leq \mathbb{E}[U(\xi^*)]. \quad (4.13)$$

**Proof.** Equation (4.12) follows directly from the definition of  $\xi^*$  and  $\chi^{-1}$ . The inequality (4.13) is a consequence of (4.12), (4.10) and of the following property of the utility function  $U$  with the inverse of its derivative  $I$ , see [39]:

For  $0 < x, y < \infty$ , we have

$$U(I(y)) \geq U(x) + y(I(y) - x). \quad \blacksquare$$

Theorem 4.5 which is proved in [41] solves the static optimization problem (4.11).

### Theorem 4.5.

Consider the portfolio problem (4.9). Let  $x > 0$  and set  $y := \chi^{-1}(x)$ , i.e.,  $y$  solves  $x = \mathbb{E}[H_T I(yH_T)]$ . Then there exists for  $\xi^* = \xi^N := I(yH_T)$ , a self-financing portfolio process  $\theta^N = (\theta_t^N)_{t \in [0, T]}$ , such that

$$\theta^N \in \mathcal{A}(x), \quad X_T^{\theta^N} = \xi^N,$$

and the portfolio process solves the dynamic problem (4.9).

So far we have not used the assumption of market completeness, this assumption is used only insure the existence of the portfolio  $\theta^N$  which finances  $\xi^N$ . However, when specialized to the case of logarithmic utility or to power utility defined in (3.5), one can directly use the Markov property of solutions of stochastic differential equations to express the current optimal wealth process  $X_t^{\theta^N}$  before the horizon  $T$ , as a function of  $H_t$  for which we apply Itô's Formula. The optimal portfolio can be derived by equating coefficients with the wealth process given in (4.8), a task that we investigate in the following example.

**Example 4.6.** The problem of the so called non-risk portfolio manager was studied by Cox and Huang [12], [11] where the manager has a power utility  $U$  given by (3.5) with a constant relative risk aversion  $\gamma$ . According to Theorem 4.5, the static problem (4.11) has the optimal solution

$$\xi^N = I(yH_T),$$

with  $I(x) = x^{-\frac{1}{\gamma}}$  is the inverse function of the derivative of the utility function  $U$  and  $y^N := \frac{1}{x^\gamma} e^{(1-\gamma)(r + \frac{\|\kappa\|^2}{2\gamma})T}$ .

Let  $X_t^N$  be the optimal wealth before the horizon. Itô's lemma applied to Equations (4.7) and (4.8) implies that the process  $(H_t X_t^N)_{t \in [0, T]}$  is  $\mathcal{F}$ -martingale, i.e.,  $X_t^N = \mathbb{E}[\frac{H_T}{H_t} X_T^N | \mathcal{F}_t]$ .

Here the optimal terminal wealth  $X_T^N$  is given by Theorem 4.5 as  $X_T^N := \xi^N = I(y^N H_T)$ . Moreover, Markov's property of the solution  $(H_t)_{t \in [0, T]}$  of Equation (4.7) leads to the computation of this conditional expectation using the fact that  $\ln H_T$  is normally distributed with mean  $\ln H_t - (r + \frac{\|\kappa\|^2}{2})(T - t)$  and variance  $\|\kappa\|^2(T - t)$ . We get for the optimal terminal wealth before the horizon the following form

$$X_t^N = \frac{e^{\Gamma(t)}}{(y^N H_t)^{\frac{1}{\gamma}}} \quad \text{with} \quad \Gamma(t) := \frac{1 - \gamma}{\gamma} \left( r + \frac{\|\kappa\|^2}{2\gamma} \right) (T - t).$$

The optimal strategy is obtained by a representation approach. In this case, we have  $X_t^N = F(H_t, t)$  with  $F(z, t) = \frac{e^{\Gamma(t)}}{(y^N z)^{\frac{1}{\gamma}}}$  for which we apply Itô's lemma to get

$$dX_t^N = \left[ F_t(H_t, t) - rF_z(H_t, t)H_t + \frac{\|\kappa\|^2}{2} F_{zz}(H_t, t)H_t^2 \right] dt - F_z(H_t, t)H_t \kappa^\top dW_t,$$

where  $F_z, F_{zz}$  and  $F_t$  denote the partial derivatives of  $F(z, t)$  w.r.t.  $z$  and  $t$ , respectively. If we equate the volatility coefficient of this equation with the volatility coefficient of Equation (4.8), we derive the following constant optimal strategy

$$\theta_t^N \equiv \theta^N = \frac{1}{\gamma} \sigma^{-1} \kappa = \frac{1}{\gamma} (\sigma \sigma^\top)^{-1} (\mu - r \mathbf{1}_n) = \text{const}$$

for the optimization in the absence of a risk-constraint which we call normal strategy.

## 4.4 The case of stochastic volatility

In [27] we consider an extended Merton's problem of optimal consumption and investment in continuous-time with stochastic volatility. We approximate the linear wealth process by a particular weak Itô-Taylor approximation called *Euler scheme*, and then we show that the optimal control of the value function generated by the Euler scheme is an  $\varepsilon$ -optimal control of the original problem of maximizing total expected discounted utility from consumption. More precisely, we consider a financial market with a risk-free bond

$$dS_t^0 = rS_t^0 dt, \quad (4.14)$$

and one *risky stock* whose prices are given according to

$$dS_t = S_t[\mu dt + \sigma(\Lambda_t)dW_t^1], \quad S = s_0 \in \mathbb{R}. \quad (4.15)$$

The process  $\Lambda_t$  represents an economic factor with mean reversion property and can be observed by the investor. Models of stochastic volatility are of great interest in finance since they have the property to capture the stock return variability.

Following Fleming and Hernandez [25], we assume that the dynamics of  $\Lambda_t$  is given by

$$d\Lambda_t = g(\Lambda_t)dt + \beta[\rho dW_t^1 + (1 - \rho^2)^{\frac{1}{2}}dW_t^2], \quad \Lambda_0 = \lambda \in \mathbb{R} \quad (4.16)$$

with  $\mu > r$  and  $-1 \leq \rho \leq 1$ .  $W^1$  and  $W^2$  are two independent Brownian motions. The functions  $\sigma$  and  $g$  are assumed to be in  $C^1(\mathbb{R})$  and satisfy

- $\sigma_z = \frac{d}{dz}\sigma$  is bounded and  $\sigma_l \leq \sigma(\cdot) \leq \sigma_u$  for a given constants  $0 < \sigma_l < \sigma_u$ ;
- $g_z = \frac{d}{dz}g$  is bounded and there exists  $k > 0$  such that  $g_z \leq -k$ .

We denote by  $\theta_t$  the trading portfolio corresponding to the fraction of wealth invested in the risky asset at time  $t$ , and let  $\tilde{c}_t = \frac{c_t}{X_t^\theta}$  be the fraction of wealth which is consumed. Then the wealth process satisfies

$$dX_t^\theta = X_t^\theta[(r + (\mu - r)\theta_t - \tilde{c}_t)dt + \theta_t\sigma(\Lambda_t)dW_t^1], \quad X_0^\theta = x > 0, \quad (4.17)$$

where the processes  $\theta = (\theta_t)_{t \in [0, T]}$  and  $\tilde{c} = (\tilde{c}_t)_{t \in [0, T]}$  are supposed to be admissible in the sense that they are  $\mathcal{F}_t$ -progressively measurable such that  $\mathbf{P}[\forall t > 0 : |\theta_t| \leq A_1, 0 \leq \tilde{c}_t \leq A_2] = 1$ , where  $A_1$  and  $A_2$  are given constants. Let  $\mathcal{A}(x)$  denote the set of admissible strategies.

The portfolio manager's objective is to find an investment portfolio and a consumption rate so as to maximize over the set of admissible strategies the following expected total discounted utility from consumption

$$J(x, \Lambda; \theta, \tilde{c}) = \mathbf{E} \left[ \int_0^\infty \frac{1}{1 - \gamma} e^{-\alpha t} (\tilde{c}_t X_t^\theta)^{1 - \gamma} dt \right], \quad \alpha > 0 \text{ and } 0 < \gamma < 1. \quad (4.18)$$

This is well defined since it is shown in [25] that the value function

$$V(x, \Lambda) = \sup_{\theta, \tilde{c} \in \mathcal{A}(x)} \mathbb{E} \left[ \int_0^\infty \frac{1}{1-\gamma} e^{-\alpha t} (\tilde{c}_t X_t^\theta)^{1-\gamma} dt \right], \quad \alpha > 0 \text{ and } 0 < \gamma < 1,$$

is bounded. This problem is solved by Flemming and Hernandez in [25], where the authors write  $V(x, \Lambda)$  in the form  $V(x, \Lambda) = \frac{x^\gamma}{\gamma} \tilde{V}(\Lambda)$  for some function  $\tilde{V}$ , then, by a change of probability measure argument the portfolio optimization problem was reduced to a stochastic control problem for which the dynamic programming equation is a non linear differential equation with state  $\Lambda_t$  given by equation (4.16), and they show that the value function  $\tilde{V}(\Lambda)$  is the unique positive classical solution to the dynamic programming equation associated with this stochastic problem. In particular, they obtained via analytic and stochastic control arguments optimal investment and consumption policies as feedback policies of the current wealth  $X^\theta$  which we approximate by a particular stochastic Itô-Taylor approximation called Euler scheme. Let us restrict our study to the case of finite time horizon  $T > 0$  and let

$$V(x, \Lambda; T) = \sup_{\theta, \tilde{c} \in \mathcal{A}(x)} J(x, \Lambda; \theta, \tilde{c}, T) \quad (4.19)$$

be the associated optimal costs function.

The Euler approximation of the wealth process (4.17) is defined for a given discretization  $0 = t_0 < t_1 < \dots < t_N = T$  of the time interval  $[0, T]$  by

$$Y_N^\theta(t_{i+1}) = Y_N^\theta(t_i) + (r + \theta_{t_i}(\mu - r) - \tilde{c}_{t_i}) Y_N^\theta(t_i)(t_{i+1} - t_i) + \sigma(\Lambda(t_i)) \theta_{t_i} Y_N^\theta(t_i)(W_{t_{i+1}} - W_{t_i}),$$

for  $i = 0, 1, 2, \dots, N - 1$  with initial value  $Y_0^\theta = x$ . We shall use the interpolated Euler scheme which is defined by

$$\begin{aligned} Y_N^\theta(t) = Y_N^\theta(t_{n_t}) &+ (r + \theta_{t_{n_t}}(\mu - r) - \tilde{c}_{t_{n_t}}) Y_N^\theta(t_{n_t})(t - t_{n_t}) \\ &+ \sigma(\Lambda(t_{n_t})) \theta_{t_{n_t}} Y_N^\theta(t_{n_t})(W_t - W_{t_{n_t}}). \end{aligned} \quad (4.20)$$

with  $n_t := \max\{0 \leq i \leq N, t_i \leq t\}$ . This approximation is a particular case of the truncated Itô-Taylor expansion which has the important property that it allows a sufficiently smooth function of an Itô process to be expanded as the sum of a finite number of terms represented by multiple Itô integrals with constant integrands and a remainder consisting of a finite number of other multiple Itô integrals with non constant integrands. We refer to [45] for more details. In [27] we first prove that the functional

$$J_N(x, \Lambda; \theta, \tilde{c}, T) = \mathbb{E} \left[ \int_0^T \frac{1}{1-\gamma} e^{-\alpha t} (\tilde{c}_t Y_N^\theta(t))^{1-\gamma} dt \right], \quad \alpha > 0 \text{ and } 0 < \gamma < 1 \quad (4.21)$$

is continuous with respect to the controls  $\theta, \tilde{c} \in \mathcal{A}(x)$ , i.e., if  $u^m = (\theta^m, \tilde{c}^m)$  is a sequence of admissible controls converging uniformly in  $L^\infty(\Omega)$  to a given admissible control  $u = (\theta, \tilde{c})$ , i.e.,

$$\sup_{0 \leq t \leq T} \|\theta_t^m - \theta_t\|_{L^\infty(\Omega)}; \quad \sup_{0 \leq t \leq T} \|\tilde{c}_t^m - \tilde{c}_t\|_{L^\infty(\Omega)} \longrightarrow 0 \text{ as } m \rightarrow \infty.$$

Then, we have

$$|J_N(x, \Lambda; u^m, T) - J_N(x, \Lambda; u, T)| \longrightarrow 0 \text{ as } m \rightarrow \infty.$$

Further we have proved in [27] that the interpolated Euler approximation process defined in (4.20) converges strongly to the wealth process given by (4.17), i.e.,

$$\mathbb{E}[|Y_N^\theta(T) - X_T^\theta|^2] \longrightarrow 0 \text{ as } N \rightarrow \infty.$$

From the other hand, the conditions of (Theorem 14.5.1 [45]) are satisfied in our model so that we have the convergence of the Euler scheme in the weak sense and as consequence we prove in [27] the following main result.

**Theorem 4.7.**

For  $T > 0$ ,  $\theta, c \in \mathcal{A}(x)$  let

$$J(x, \Lambda; \theta, \tilde{c}, T) = \mathbb{E} \left[ \int_0^T \frac{1}{1-\gamma} e^{-\alpha t} (\tilde{c}_t X_t^\theta)^{1-\gamma} dt \right], \quad \alpha > 0 \text{ and } 0 < \gamma < 1$$

be the total discounted utility from consumption, where  $X^\theta$  is the wealth process given in (4.17). Let  $Y_N(t)$  be its interpolated Euler approximation and

$$J_N(x, \Lambda; \theta, \tilde{c}, T) = \mathbb{E} \left[ \int_0^T \frac{1}{1-\gamma} e^{-\alpha t} (\tilde{c}_t Y_N^\theta(t))^{1-\gamma} dt \right], \quad \alpha > 0 \text{ and } 0 < \gamma < 1$$

its associated total discounted utility of consumption. Let  $u_t^* = (\theta_t^*, c_t^*)$  be an optimal control of optimal costs function

$$V(x, \Lambda; T) = \sup_{\theta, \tilde{c} \in \mathcal{A}(x)} J(x, \Lambda; \theta, \tilde{c}, T), \quad (4.22)$$

and let  $u_N^*(t) = (\theta_N^*(t), \tilde{c}_N^*(t))$  be an optimal control of the value function

$$V_N(x, \Lambda; T) = \sup_{\theta, \tilde{c} \in \mathcal{A}(x)} J_N(x, \Lambda; \theta, \tilde{c}, T). \quad (4.23)$$

Then, we have

$\forall \epsilon > 0$  there exists a partition  $(t_i)_{0 \leq i \leq N_\epsilon}$  such that

$$|J(x, \Lambda; u_t^*, T) - J(x, \Lambda; u_N^*(t), T)| < \epsilon$$

for every  $N \geq N_\epsilon$ .

**Proof.** Let  $T > 0$ ,  $u_t^* = (\theta_t^*, \tilde{c}_t^*)$  be an optimal policy of the original problem

$$V(x, \Lambda, T) = \sup_{\theta, \tilde{c} \in \mathcal{A}(x)} \mathbb{E} \left[ \int_0^T \frac{1}{1-\gamma} e^{-\alpha t} (\tilde{c}_t X_t^\theta)^{1-\gamma} dt \right], \quad \alpha > 0 \text{ and } 0 < \gamma < 1.$$

We denote by  $u_N^*(t) = (\theta_N^*(t), \tilde{c}_N^*(t))$  the optimal control of the objective function

$$V_N(x, \Lambda) = \sup_{\theta, \tilde{c} \in \mathcal{A}(x)} \mathbb{E} \left[ \int_0^T \frac{1}{1-\gamma} e^{-\alpha t} (\tilde{c}_t Y_N^\theta(t))^{1-\gamma} dt \right], \quad \alpha > 0 \text{ and } 0 < \gamma < 1.$$

associated with the controlled interpolated Euler scheme  $Y_N^\theta$ .

Then, we have

$$\begin{aligned} J(x, \Lambda; u_t^*, T) - J(x, \Lambda; u_N^*(t), T) &= J(x, \Lambda; u_t^*, T) - J_N(x, \Lambda; u_t^*, T) \\ &+ J_N(x, \Lambda; u_t^*, T) - J_N(x, \Lambda; u_N^*(t), T) \\ &+ J_N(x, \Lambda; u_N^*(t), T) - J(x, \Lambda; u_N^*(t), T). \end{aligned} \quad (4.24)$$

Since  $u_N^*(t)$  is the global optimal control associated with  $J_N$ , it follows that

$$J_N(x, \Lambda; u_t^*, T) - J_N(x, \Lambda; u_N^*(t), T) \leq 0.$$

From the one hand, we have

$$\begin{aligned} |J(x, \Lambda; u_t^*, T) - J_N(x, \Lambda; u_t^*, T)| &= \left| \mathbb{E} \left[ \int_0^T e^{-\alpha t} (\tilde{c}_t^*)^{1-\gamma} \left( (X_t^{\theta^*})^{1-\gamma} - (Y_N^{\theta^*}(t))^{1-\gamma} \right) dt \right] \right| \\ &\leq \frac{A_2^{1-\gamma}}{1-\gamma} \int_0^T e^{-\alpha t} \mathbb{E} \left[ |g(X_t^{\theta^*}) - g(Y_N^{\theta^*}(t))| \right] dt, \end{aligned}$$

with  $g(x) = x^{1-\gamma}$  for  $x > 0$ . The function  $g$  as well the coefficient  $a(t, x) = x[r + \pi_t(b - r) - \tilde{c}_t]$ ,  $b(t, x) = \sigma(z)\pi_t x$  of the wealth process satisfy the conditions of (Theorem 14.5.1 [45]), and therefore there exists a positive constant  $C_g$  such that

$$|J(x, \Lambda; u_t^*, T) - J_N(x, \Lambda; u_t^*, T)| \leq \frac{A_2^{1-\gamma}}{1-\gamma} \left( \int_0^T e^{-\alpha t} dt \right) C_g \sup_{1 \leq i \leq N} |t_i - t_{i-1}|$$

and

$$|J_N(x, \Lambda; u_N^*(t), T) - J(x, \Lambda; u_N^*(t), T)| \leq \frac{A_2^{1-\gamma}}{1-\gamma} \left( \int_0^T e^{-\alpha t} dt \right) C_g \sup_{1 \leq i \leq N} |t_i - t_{i-1}|.$$

On the other hand, the right hand side of (4.24) is positive since  $u^*$  is global optimal control, it follows that

$$|J(x, \Lambda; u_t^*, T) - J(x, \Lambda; u_N^*(t)^*, T)| \leq 2 \frac{A_2^{1-\gamma}}{1-\gamma} \left( \int_0^T e^{-\alpha t} dt \right) C_g \sup_{1 \leq i \leq N} |t_i - t_{i-1}|.$$

■

**Remark 4.8.** We emphasize the important fact that the main goal of a stochastic time-discrete approximation is a practical simulation of solutions of stochastic differential equations needed in situations where a good pathwise approximation is required, or in situations dealing with expectations of functionals of an Itô process which can not be determined analytically. From this fact the last result could be interesting for numerical computations.



## 5 Risk measures

In general, a risk is related to the possibility of losing wealth and assumed to be an undesirable characteristic of a random outcome of a given financial investment. Without other considerations such as risk constraints, the optimal portfolio strategy given in Example (4.6) leads (by definition) to the maximum expected utility of the terminal wealth. Nevertheless, these strategies are more risky since they lead to extreme positions, and as consequence the optimal terminal wealth does not exceed the initial investment with a high probability. This distribution is not desirable for a portfolio manager, who offers, say, a life insurance with a fixed minimum rate of return. In order to incorporate such shortfall risks into the optimization it is necessary to quantify them by using appropriate risk measures. By the term shortfall risk we denote the event, that the terminal wealth falls below some threshold value  $Q > 0$ . In Section 8 we shall introduce a shortfall level which is related to the result of an investment into the money market. A typical choice is

$$Q := q = e^{\delta T} X_T^{\theta \equiv 0} = x e^{(r+\delta)T}, \quad \delta \in \mathbb{R}.$$

Here,  $X_T^{\theta \equiv 0} = x e^{rT}$  is the terminal wealth of a pure bond portfolio where the portfolio manager follows the buy-and-hold strategy  $\theta \equiv 0$ . In this case shortfall means to reach not an target interest rate of  $r + \delta$ . For  $\delta = -r$  we have  $q = x$ , i.e., the shortfall level is equal to the initial capital.

In Section 11 we deal with a stochastic shortfall level  $Q$ , which is proportional to the result of an investment into a pure stock portfolio managed by the buy-and-hold strategy  $\theta \equiv 1$ . Especially we set for  $S_0 = 1$

$$Q = e^{\delta T} X_T^{\theta \equiv 1} = e^{\delta T} x S_T, \quad \delta \in \mathbb{R}.$$

The shortfall risk consists of the random event  $C = \{X_T < Q\}$  or  $\{G := X_T - Q < 0\}$ . Next we assign risk measures to the random variable (risk)  $G$  and denote them by  $\rho(G)$ . Using these measures, constraints of the type  $\rho(G) \leq \varepsilon$  for some  $\varepsilon > 0$  can be added to the formulation of the portfolio optimization problem.

We now present some risk measures used in this thesis. A natural idea is to restrict the probability of a shortfall, i.e.,

$$\rho_1(G) = P(G < 0) = P(X_T < Q) \leq \varepsilon.$$

Here  $\varepsilon \in (0, 1)$  is the maximum shortfall probability which is accepted by the portfolio manager. This approach corresponds to the widely used concept of Value at Risk (VaR) which is defined as

$$\text{VaR}_\varepsilon(G) = -\zeta_\varepsilon(G),$$

where  $\zeta_\varepsilon(G)$  denotes the  $\varepsilon$ -quantile of the random variable  $G$ . VaR can be interpreted as the threshold value for the risk  $G$  such that  $G$  falls short this value with some given probability  $\varepsilon$ . It holds

$$P(G < 0) \leq \varepsilon \Leftrightarrow \text{VaR}_\varepsilon(G) \leq 0 \Leftrightarrow \text{VaR}_\varepsilon(X_T) \leq -Q.$$

Value at Risk is the most common tool in risk management for banks and many financial institutions. It is defined as the worst loss for a given confidence level. For a confidence level of  $\alpha = 99\%$  one is 99% certain that at the end of a chosen risk horizon there will be no smaller wealth than the VaR. In the academic literature many works have focused on the Value at Risk as risk measure, see for example Duffie and Pan [19]. Theoretical properties of the Value at risk are discussed in Artzner [2], Cvitanic and Karatzas [13].

As it is pointed out in [28], the VaR risk measure has the shortcoming to control only the probability of loss rather than its magnitude and as consequence the expected losses in the states where there are large losses are higher than the expected losses the portfolio manager would have incurred by avoiding the use of VaR risk measure. In order to overcome this shortcoming of the VaR, the risk manager uses as alternative the so-called Expected Loss denoted by EL and defined as

$$\rho_2(G) = \text{EL}(G) := \mathbb{E}[G^-] = \mathbb{E}[(X_T - Q)^-].$$

Since the aim is to maximize the expected utility of the terminal wealth  $X_T$ , one can also compare the utilities of  $X_T$  and of a given benchmark  $Q$ .

Let  $U$  denote a utility function given by Definition 4.2. Realizations of  $X_T$  with  $U(X_T)$  below the target utility  $U(Q)$  are those of an unacceptable shortfall. Then the random event  $C$  can also be written as  $C = \{X_T < Q\} = \{U(X_T) < U(Q)\}$ . Defining the random variable  $G = G(X_T, Q) = U(X_T) - U(Q)$  we have  $C = \{G < 0\}$ . The random variable  $G$  can be interpreted as the utility gain of the terminal wealth relative to the benchmark. In order to quantify the shortfall risk we assign to the random variable  $G$  a real-valued risk measure  $\rho(G)$  given by

$$\rho_3(G) = \text{EUL}(G) := \mathbb{E}[G^-] = \mathbb{E}[(U(X_T) - U(Q))^-],$$

and call it Expected Utility Loss (EUL). Here similarly to the EL risk measure, the risk measure (VaR) can be defined in terms of the utility function in the following sense

$$\rho_1(G) = P(G < 0) = P(U(X_T) < U(Q)) = P(X_T < Q)$$

since  $U$  is strictly increasing. Further risk measures can be found in the class of coherent measures introduced by Artzner, Delbaen, Eber and Heath [13] and Delbaen [14] where the characteristics of a risk function  $\rho(X)$  have been proposed.

**Definition 5.1.** Consider a set  $V$  of real-valued random variables. A function  $\rho : V \rightarrow \mathbb{R}$  is called coherent risk measure if it is

- (i) *monotonous:*  $X \in V, X \geq 0 \implies \rho(X) \geq 0$
- (ii) *sub-additive:*  $X, Y, X + Y \in V \implies \rho(X + Y) \leq \rho(X) + \rho(Y)$
- (iii) *positively homogeneous:*  $X \in V, h > 0, X \in V \implies \rho(hX) = h\rho(X)$
- (iv) *translation invariant:*  $X \in V, a \in \mathbb{R} \implies \rho(X + a) = \rho(X) - a$ .

Delbaen [14] proved that the VaR measure is not a coherent risk measure since it does not fulfil the sub-additivity property. This property expresses the fact that a portfolio made of sub-portfolios will risk an amount which is at most the sum of the separate amounts risked by its sub-portfolios.

EUL and EL risk measures do not belong to the class of coherent risk measures, since they both violate the translation property. We refer to Basak, Shapiro et.al. [4, 5] and our papers [28, 31], where VaR-based risk measures are used as constraints of portfolio optimization problems. Constraints modeling the Expected Loss and the Expected Utility Loss are studied within a portfolio maximization problem in [28, 30, 31].

We shall discuss in the next chapters the behavior of a portfolio manager who wants to maximize its expected utility from terminal wealth in presence of different shortfall risks measured by the last discussed risk measures.

**Remark 5.2.** If we choose a logarithmic utility function  $U(z) = \ln z$ , i.e., we set  $\gamma = 1$ , then the maximization of the expected utility  $E[U(X_T^\theta)]$  of the terminal wealth of the portfolio  $\theta$ , is equivalent to the maximization of the expected annual logarithmic return of this portfolio. The annual logarithmic return is defined as

$$L(X_T^\theta) := \frac{1}{T} \ln \frac{X_T^\theta}{x}$$

where  $x$  is the initial capital, i.e.,  $X_0 = x$ . Hence, we find

$$E[L(X_T^\theta)] := \frac{1}{T} E[U(X_T^\theta) - U(x)].$$

For the Expected Utility Loss we derive

$$E[(U(X_T^\theta) - U(Q))^-] = T E[(L(X_T^\theta) - L(Q))^-].$$

It can be seen, that bounding the Expected Utility Loss by  $\varepsilon$  is equivalent to bounding the Expected Loss of the annual logarithmic return by  $\frac{\varepsilon}{T}$ .

## Chapter II

# Portfolio optimization with deterministic benchmark

In this chapter we examine the portfolio optimization problem in the presence of additional risk constraints taking into account that the terminal wealth  $X_T$  may fall short of a given deterministic shortfall level  $Q := q$ . Basically, we adopt three different risk constraints

- the shortfall probability  $P(X_T < q)$ , which describes the loss that can occur over a given period, at a given confidence level,
- the Expected Loss:  $E[(X_T - q)^-]$ ,
- the Expected Utility Loss  $E[(U(X_T) - U(q))^-]$ , which takes care of the magnitude of loss in utility.

We shall discuss the impact of the different risk constraints to the behavior of the portfolio manager.

We formulate the dynamic optimization problem of maximizing the expected utility from terminal wealth with additional risk constraints bounding the shortfall probability of the eventual loss with a given  $\varepsilon$ . The EL and the EUL risk constraints bound the Expected Loss and the Expected Utility Loss with a given  $\varepsilon$ , respectively.

Following the normal strategy  $\theta^N$  the portfolio manager reaches the terminal wealth  $X_T^N = \xi^N$  given in Theorem 4.5 and in Example 4.6 for the case of a CRRA-utility function. Let

$$\varepsilon_{\max} := E[(U(\xi^N) - U(q))^-]$$

be the corresponding Expected Utility Loss of the optimal terminal wealth  $\xi^N$  measuring the risk of the normal strategy. Obviously, for  $\varepsilon \geq \varepsilon_{\max}$  the risk constraint is not binding and the normal strategy  $\theta^N$  is optimal for the constrained problem, too. Therefore, we restrict to the case  $\varepsilon < \varepsilon_{\max}$ .

On the other hand, if the bound  $\varepsilon > 0$  is chosen too small it may happen, that it is impossible to find any strategy  $\theta_t$  which (starting with the given initial capital

$x$ ) generates a terminal wealth  $X_T$  that fulfills the risk constraint, i.e., there is no admissible solution. Depending on the chosen parameters of the financial market this case can be observed, if the deterministic benchmark  $q$  is chosen to be larger than  $xe^{rT}$ , i.e., the result of an investment in the risk-free bond. Then, there is some positive minimum (or infimum) value  $\varepsilon_{\min}$  which bounds the Expected Utility Loss from below. Choosing the bound  $\varepsilon$  for the risk constraint such that  $0 \leq \varepsilon_{\min} < \varepsilon < \varepsilon_{\max}$  provides that the risk constraint is binding and that there exist admissible solutions from which an optimal solution for the constrained optimization problem can be determined. The same remark can be derived for the Value at Risk and the Expected Loss.

## 6 Optimization under Value at Risk constraint

In this section we present the portfolio maximization problem constrained by the Value at Risk. More precisely, we consider an investor who wishes in addition to maximize his expected utility from terminal wealth, to control the probability for a shortfall. Let  $G = X_T - q$  be the random variable quantifying the shortfall risk. Given a probability  $\varepsilon \in (0, 1)$  this constraint can be written as

$$P(G < 0) = P(X_T < q) \leq \varepsilon. \quad (6.1)$$

From the definition of VaR given in Section 5 this is equivalent to

$$\text{VaR}_\varepsilon(G) \leq 0 \Leftrightarrow \text{VaR}_\varepsilon(X_T) \leq -q.$$

With constraint (6.1) the agent bounds the probability of negative values of the risk  $G = X_T - q$  by  $\varepsilon$ . We will denote this strategy as VaR-strategy and we call  $X_t^{\text{VaR}}$  the wealth generated by following this strategy.

We give a slightly different solution of the dynamic optimization problem of the VaR agent studied by Basak and Shapiro in [4]. The problem is solved using the martingale representation approach which consists of formulating the problem as the following static variational problem:

$$\begin{aligned} & \max_{\xi \in \mathcal{B}(x)} \mathbb{E}[U(\xi)] \\ & \text{subject to } P(\xi < q) \leq \varepsilon. \end{aligned}$$

The VaR constraint leads to non-concavity with which the maximization is more complicated. The optimal terminal wealth  $X_T^{\text{VaR}}$  is characterized by Basak and Shapiro [4], Proposition 1, where the authors assumed that the solution exists.

**Proposition 6.1.** Let  $Q = q > 0$  be a fixed benchmark. Moreover, let for  $y > 0$  be

defined

$$\begin{aligned} \underline{h} = \underline{h}(y) &:= \frac{1}{y}U'(q), \quad \text{and } \bar{h} \text{ such that } P(H_T > \bar{h}) = \varepsilon \quad \text{and} \\ f(z) = f(z; y) &:= \begin{cases} I(yz) & \text{if } z < \underline{h}, \\ q & \text{if } \underline{h} \leq z < \bar{h}, \\ I(yz) & \text{if } \bar{h} \leq z, \end{cases} \end{aligned}$$

and let be such that for  $z > 0$ . Finally, let the initial capital  $x > 0$  and the bound for the risk constraint  $\varepsilon$  be such that there exists strictly positive and unique solution  $y$  of the following equation

$$\mathbb{E}[H_T f(H_T; y)] = x$$

Then the VaR-optimal terminal wealth is

$$X_T^{VaR} = f(H_T) = f(H_T; y).$$

In the following proposition we present explicit expressions for the VaR agent's optimal wealth and portfolio strategies before the horizon.

**Proposition 6.2.** Let the assumptions of Proposition 6.1 be fulfilled. Moreover, let  $U$  be the utility function given in (3.5).

(i) The VaR-optimal wealth at time  $t < T$  before the horizon is given by

$$\begin{aligned} X_t^{VaR} &= F(H_t, t), \tag{6.2} \\ \text{with } F(z, t) &:= \frac{e^{\Gamma(t)}}{(yz)^{\frac{1}{\gamma}}} - \left[ \frac{e^{\Gamma(t)}}{(yz)^{\frac{1}{\gamma}}} \Phi(-d_1(\underline{h}, z, t)) - qe^{-r(T-t)} \Phi(-d_2(\underline{h}, z, t)) \right] \\ &+ \left[ \frac{e^{\Gamma(t)}}{(yz)^{\frac{1}{\gamma}}} \Phi(-d_1(\bar{h}, z, t)) - qe^{-r(T-t)} \Phi(-d_2(\bar{h}, z, t)) \right], \end{aligned}$$

for  $z > 0$ . Thereby,  $\Phi(\cdot)$  is the standard-normal distribution function,  $y$  and  $\underline{h}, \bar{h}$  are as in Proposition 6.1, and

$$\begin{aligned} \Gamma(t) &:= \frac{1-\gamma}{\gamma} \left( r + \frac{\|\kappa\|^2}{2\gamma} \right) (T-t), \\ d_1(u, z, t) &:= \frac{\ln \frac{u}{z} + \left( r - \frac{\|\kappa\|^2}{2} \right) (T-t)}{\|\kappa\| \sqrt{T-t}}, \\ d_2(u, z, t) &:= d_1(u, z, t) + \frac{1}{\gamma} \|\kappa\| \sqrt{T-t}. \end{aligned}$$

- (ii) The VaR-optimal fraction of wealth invested in stock at time  $t < T$  before the horizon is

$$\begin{aligned} \theta_t^{VaR} &= \theta^N \Theta(H_t, t), \\ \text{where } \Theta(z, t) &:= 1 - \frac{qe^{-r(T-t)}}{F(z, t)} \left[ \Phi(-d_2(\underline{h}, z, t)) - \Phi(-d_2(\bar{h}, z, t)) \right] \\ &\quad + \frac{\gamma}{\|\kappa\| \sqrt{T-t} F(z, t)} \frac{e^{\Gamma(t)}}{(yz)^{\frac{1}{\gamma}}} \left[ \varphi(d_1(\underline{h}, z, t)) - \varphi(d_1(\bar{h}, z, t)) \right] \\ &\quad - \frac{\gamma q e^{-r(T-t)}}{\|\kappa\| \sqrt{T-t} F(z, t)} \left[ \varphi(d_2(\bar{h}, z, t)) - \varphi(d_2(\underline{h}, z, t)) \right], \end{aligned}$$

for  $z > 0$ . Thereby,  $\theta^N = \frac{1}{\gamma} \sigma^{-1} \kappa = \frac{1}{\gamma} (\sigma \sigma^\top)^{-1} (\mu - r \mathbf{1}_n)$  denotes the normal strategy,  $\Theta(H_t, t)$  is the exposure to risky assets relative to the normal strategy and  $\varphi(\cdot)$  is the standard-normal probability density function.

**Proof.**

- (i) Using Equations (4.7) and (4.8), Itô's lemma implies that the process  $H X^{VaR} = (H_t X_t^{VaR})_{t \in [0, T]}$  is an  $\mathcal{F}_t$ -martingale:

$$\begin{aligned} X_t^{VaR} &= \mathbb{E} \left[ \frac{H_T}{H_t} \xi^{VaR} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E} \left[ \frac{H_T}{H_t} I(y H_t) (\mathbf{1}_{\{H_T < \underline{h}\}} + \mathbf{1}_{\{\bar{h} \leq H_T\}}) \middle| \mathcal{F}_t \right] + \mathbb{E} \left[ \frac{H_T}{H_t} q \mathbf{1}_{\{\underline{h} \leq H_T < \bar{h}\}} \middle| \mathcal{F}_t \right]. \end{aligned}$$

These conditional expectations are computed by applying Markov's property of solution of stochastic differential equation and using the fact that  $\ln H_T$  is normally distributed with mean  $\ln H_t - (r + \frac{\|\kappa\|^2}{2})(T-t)$  and variance  $\|\kappa\|^2(T-t)$ . For more explicit computations we refer to Appendix C.

- (ii) From Equality (6.2) it follows  $X_t^{VaR} = F(H_t, t)$ . The process  $H = (H_t)_{t \in [0, T]}$  satisfies the SDE (4.7). Applying Itô's lemma to the function  $F(H_t, t)$  we find that  $X^{VaR} = (X_t^{VaR})_{t \in [0, T]}$  satisfies the SDE

$$dX_t^{VaR} = \left[ F_t(H_t, t) - r F_z(H_t, t) H_t + \frac{\|\kappa\|^2}{2} F_{zz}(H_t, t) H_t^2 \right] dt - F_z(H_t, t) H_t \kappa^\top dW_t,$$

where  $F_z, F_{zz}$  and  $F_t$  denote the partial derivatives of  $F(z, t)$  w.r.t.  $z$  and  $t$ , respectively. Equating coefficients in front of  $dW_t$  in the above equation and Equation (4.8) leads to the following equality:

$$\theta_t^{VaR} = -\sigma^{-1} \kappa \frac{F_z(H_t, t) H_t}{F(H_t, t)} = -\theta^N \gamma \frac{F_z(H_t, t) H_t}{F(H_t, t)}. \quad (6.3)$$

Computing the derivative  $F_z$  we get

$$F_z(z, t) = \frac{1}{\gamma z} \left[ -F(z, t) + qe^{-r(T-t)} \left( \Phi(-d_2(\underline{h}, z)) - \Phi(-d_2(\bar{h}, z)) \right) \right] \\ - \frac{e^{\Gamma(t)}}{(yz)^{\frac{1}{\gamma}} \|\kappa\| \sqrt{T-t} z} \left[ \varphi(d_1(\underline{h}, z)) - \varphi(d_1(\bar{h}, z)) \right] \\ + \frac{qe^{-r(T-t)}}{\|\kappa\| \sqrt{T-t} z} \left[ \varphi(d_2(\bar{h}, z)) - \varphi(d_2(\underline{h}, z)) \right].$$

Substituting into (6.3), we get the final form of the optimal strategies before the horizon.  $\blacksquare$

## 7 Optimization under Expected Loss constraint

We consider in this section a portfolio manager who wishes to limit his Expected Loss. In this case he defines his strategy as one which fulfills the constraint

$$\text{EL}(G) := \mathbb{E}[G^-] = \mathbb{E}[(X_T - q)^-] \leq \varepsilon, \quad (7.4)$$

where  $G = X_T - q$  and  $\varepsilon$  is a given bound for the Expected Loss. We will denote this strategy as EL-strategy and let  $X_t^{EL}$  be the wealth corresponding to this strategy at time  $t$ . Our objective in this section is to solve the optimization problem constrained by (7.4). The dynamic optimization problem of the EL-portfolio manager can be restated as the following static variational problem

$$\max_{\xi \in \mathcal{B}} \mathbb{E}[U(\xi)] \\ \text{subject to } \mathbb{E}[(\xi - q)^-] \leq \varepsilon.$$

The following proposition characterizes the optimal terminal wealth  $X_T^{EL}$  in the presence of the EL-constraint (7.4). We prove that if a terminal wealth satisfies  $X_T^{EL} = f(H_T)$  where  $f$  is given in Proposition 7.1, then it is the optimal solution of the static variational problem.

**Proposition 7.1.** Let  $Q = q > 0$  be a fixed benchmark. Moreover, let for  $y_1, y_2 > 0$  be defined

$$\underline{h} = \underline{h}(y_1) := \frac{1}{y_1} U'(q), \\ \bar{h} = \bar{h}(y_1, y_2) := \frac{U'(q) + y_2}{y_1} \quad \text{and} \\ f(z) = f(z; y_1, y_2) := \begin{cases} I(y_1 z) & \text{if } z < \underline{h}, \\ q & \text{if } \underline{h} \leq z < \bar{h}, \\ I(y_1 z - y_2) & \text{if } \bar{h} \leq z, \end{cases}$$



for  $z > 0$ . Finally, let the initial capital  $x > 0$  and the bound for the risk constraint  $\varepsilon$  be such that there are strictly positive and solutions  $y_1, y_2$  of the following system of equations

$$\begin{aligned} \mathbb{E}[H_T f(H_T; y_1, y_2)] &= x \\ \mathbb{E}[(f(H_T; y_1, y_2) - q)^-] &= \varepsilon. \end{aligned}$$

Then the EL-optimal terminal wealth is

$$X_T^{EL} = f(H_T) = f(H_T; y_1, y_2).$$

**Proof.** In order to solve the optimization problem under EL-constraint, the common convex-duality approach is adapted by introducing the convex-conjugate of the utility function  $U$  with an additional term capturing the EL-constraint as it is shown in the following lemma which we prove in Appendix A.

**Lemma 7.2.** *Let  $z, y_1, y_2, q > 0$ . Then the solution of the optimization problem*

$$\max_{x>0} \{U(x) - y_1 z x - y_2 (x - q)^-\}$$

is  $x^* = f(z; y_1, y_2)$ .

Applying the above lemma pointwise for all  $z = H_T(\omega)$  it follows that  $\xi^* = f(H_T; y_1, y_2)$  is the solution of the maximization problem

$$\max_{\xi>0} \{U(\xi) - y_1 H_T \xi - y_2 (\xi - q)^-\}.$$

Obviously,  $\xi^*$  is  $\mathcal{F}_T$ -measurable and if  $y_1, y_2$  are chosen as solutions of the system of equations given in the proposition then it follows  $\xi^* = X_T^{EL}$ .

To complete the proof, let  $\eta$  be any admissible solution satisfying the static budget constraint and the EL-constraint (7.4). We have

$$\begin{aligned} \mathbb{E}[U(X_T^{EL})] - \mathbb{E}[U(\eta)] &= \mathbb{E}[U(X_T^{EL})] - \mathbb{E}[U(\eta)] - y_1 x + y_1 x - y_2 \varepsilon + y_2 \varepsilon \\ &\geq \mathbb{E}[U(X_T^{EL})] - \mathbb{E}[y_1 H_T X_T^{EL}] - y_2 \mathbb{E}[(X_T^{EL} - q)^-] \\ &\quad - \mathbb{E}[U(\eta)] + \mathbb{E}[y_1 H_T \eta] + y_2 \mathbb{E}[(\eta - q)^-] \\ &\geq 0, \end{aligned}$$

where the first inequality follows from the static budget constraint and the constraint for the risk holding with equality for  $X_T^{EL}$ , while holding with inequality for  $\eta$ . The last inequality is a consequence of the above lemma. Hence we obtain that  $X_T^{EL}$  is optimal.  $\blacksquare$

**Remark 7.3.** We have  $f(z; y_1, y_2) \rightarrow I(y_1 z)$  for  $y_2 \downarrow 0$ . This limit corresponds to  $\varepsilon \uparrow \varepsilon_{\max}$  and we derive the results for the unconstrained problem if we set  $y_2 = 0$  and  $f(z; y_1, 0) := I(y_1 z)$ .

We present in the following proposition the explicit expressions for the EL-optimal wealth and portfolio strategy before the horizon.

**Proposition 7.4.** Let the assumptions of Proposition 7.1 be fulfilled. Moreover, let  $U$  be the utility function given in (3.5).

(i) The EL-optimal wealth at time  $t < T$  before the horizon is given by

$$\begin{aligned} X_t^{EL} &= F(H_t, t) \\ F(z, t) &:= \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \left[ 1 - \Phi(-d_1(\underline{h}, z)) \right] \\ &\quad + qe^{-r(T-t)} \left[ \Phi(-d_2(\underline{h}, z)) - \Phi(-d_2(\bar{h}, z)) \right] \\ &\quad + G(z, \bar{h}), \end{aligned} \tag{7.5}$$

for  $z > 0$ . Thereby,  $\Phi(\cdot)$  is the standard-normal distribution function,  $y_1, y_2$  are as in Proposition 7.1,  $\Gamma(t), d_1, d_2$  are as in Proposition 6.2 and

$$\underline{h} = \frac{1}{y_1 q^\gamma} \text{ and } \bar{h} = \frac{q^{-\gamma} + y_2}{y_1},$$

$$G(z, \bar{h}) := \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{c_2(\bar{h}, z)} \frac{e^{-\frac{1}{2}(u-b)^2}}{(y_1 t e^{a+bu} - y_2)^{\frac{1}{\gamma}}} du,$$

$$c_2(\bar{h}, z) = \frac{1}{b}(\ln(\frac{\bar{h}}{z}) - a), \quad a := -(r + \frac{\|\kappa\|^2}{2})(T-t) \text{ and } b := -\|\kappa\|\sqrt{T-t}.$$

(ii) The EL-optimal fraction of wealth invested in stock at time  $t < T$  before the horizon is

$$\begin{aligned} \theta_t^{EL} &= \theta^N \Theta(H_t, t) \\ \Theta(z, t) &= \frac{1}{F(z, t)} \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \left[ 1 - \Phi(-d_1(\underline{h}, z)) + \frac{\gamma}{\|\kappa\|\sqrt{T-t}} \varphi(d_1(\underline{h}, z)) \right] \\ &\quad - \frac{q\gamma e^{-r(T-t)}}{F(z, t)\kappa\sqrt{T-t}} \varphi(d_2(\underline{h}, z)) \\ &\quad + \frac{y_1 z e^{(\|\kappa\|^2 - 2r)(T-t)}}{F(z, t)} \psi_0(c_2(\bar{h}, z), b, y_1 z e^a, y_2, 2b, 1, 1 + \frac{1}{\gamma}), \end{aligned}$$

for  $z > 0$ . Thereby,  $\theta^N = \frac{1}{\gamma}\sigma^{-1}\kappa = \frac{1}{\gamma}(\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1}_n)$  denotes the normal strategy,  $\varphi(\cdot)$  is the standard-normal probability density function, and  $\Theta(H_t, t)$  is the exposure to risky assets relative to the normal strategy and

$$\psi_0(\alpha, \beta, c_1, c_2, m, s, \delta) := \frac{1}{\sqrt{2\pi}s} \int_{-\infty}^{\alpha} \frac{\exp(-\frac{(u-m)^2}{2s^2})}{(c_1 e^{\beta u} - c_2)^\delta} du.$$

**Proof.**

(i) The computations and the arguments of this proof are the same as in Proposition 6.2 part (i), except that we can not compute the conditional expectation

$$J_2 = G(H_t, \bar{h}) := \mathbb{E} \left[ \frac{H_T}{H_t} I(y_1 H_t - y_2) 1_{\{\bar{h} \leq H_T\}} \middle| \mathcal{F}_t \right]$$

explicitly, but give it in terms of the integral  $G(z, \bar{h})$ .

- (ii) From Equation (7.5) it follows  $X_t^{EL} = F(H_t, t)$ . The process  $H = (H_t)_{t \in [0, T]}$  satisfies the SDE (4.7). Applying Itô's lemma to the function  $F(H_t, t)$  we find that the process  $X^{EL} = (X_t^{EL})_{t \in [0, T]}$  satisfies the SDE

$$dX_t^{EL} = \left[ F_t(H_t, t) - rF_z(H_t, t)H_t + \frac{\|\kappa\|^2}{2}F_{zz}(H_t, t)H_t^2 \right] dt - F_z(H_t, t)H_t \kappa^\top dW_t,$$

where  $F_z, F_{zz}$  and  $F_t$  denote the partial derivatives of  $F(z, t)$  w.r.t.  $z$  and  $t$ , respectively. Equating coefficients in front of  $dW_t$  in the above equation and Equation (4.8) leads to the following equality:

$$\theta_t^{EL} = -\sigma^{-1}\kappa \frac{F_z(H_t, t)H_t}{F(H_t, t)} = -\theta^N \gamma \frac{F_z(H_t, t)H_t}{F(H_t, t)}. \quad (7.6)$$

Computing the derivative  $F_z$  we get

$$\begin{aligned} F_z(z, t) &= -\frac{e^{\Gamma(t)}}{z\gamma(y_1 z)^{\frac{1}{\gamma}}} \left[ 1 - \Phi(-d_1(\underline{h}, z)) + \frac{\gamma}{\|\kappa\|\sqrt{T-t}} \varphi(d_1(\underline{h}, z)) \right] \\ &\quad + \frac{qe^{-r(T-t)}}{\|\kappa\|\sqrt{T-t}z} \left[ \varphi(d_2(\underline{h}, z)) - \varphi(d_2(\bar{h}, z)) \right] + \frac{\partial}{\partial z} G(z, \underline{h}). \end{aligned}$$

For the last term we have

$$\begin{aligned} \frac{\partial}{\partial z} G(z, \bar{h}) &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \frac{\partial}{\partial z} \left[ \int_{\infty}^{c_2(\bar{h}, z)} l(z, u) du \right] \\ &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \left[ \int_{\infty}^{c_2(\bar{h}, z)} \frac{\partial}{\partial z} l(z, u) du + \frac{\partial}{\partial z} c_2(\bar{h}, z) l(z, c_2(\bar{h}, z)) \right], \end{aligned}$$

where

$$l(z, u) = \frac{e^{-\frac{1}{2}(u-b)^2}}{(y_1 z e^{a+bu} - y_2)^{\frac{1}{\gamma}}}.$$

Finally, we get

$$\begin{aligned} \frac{\partial}{\partial z} G(z, \bar{h}) &= \frac{-y_1}{\gamma} e^{(\|\kappa\|^2 - 2r)(T-t)} \psi_0(c_2(\bar{h}, z), b, y_1 z e^a, y_2, 2b, 1, 1 + \frac{1}{\gamma}) \\ &\quad + \frac{qe^{r(T-t)}}{\|\kappa\|\sqrt{T-t}z} \varphi(-d_2(\bar{h}, z)). \end{aligned}$$

Substituting in (7.6), we get the final form of the optimal strategies before the horizon. ■

## 8 Optimization under Expected Utility Loss constraint

Since the objective of the portfolio manager is to maximize the expected utility from the terminal wealth, we find it interesting to deal with the portfolio optimization problem where the portfolio manager is confronted with a risk measured by a constraint of the type

$$\text{EUL}(G) := \mathbb{E}[G^-] = \mathbb{E}[(U(X_T) - U(q))^-] \leq \varepsilon, \quad (8.7)$$

where  $G = U(X_T) - U(q)$  and  $\varepsilon$  is a given bound for the Expected Utility Loss. The resulting constrained dynamic optimization problem reads as

$$\max_{\theta \in \mathcal{A}(x)} \mathbb{E}[U(X_T^\theta)] \quad (8.8)$$

$$\text{subject to } \mathbb{E}[(U(X_T^\theta) - U(q))^-] \leq \varepsilon$$

and its solutions are called EUL-optimal. The corresponding EUL-optimal wealth at time  $t \in [0, T]$  is denoted by  $X_t^{\text{EUL}}$  and the EUL-optimal strategy by  $\theta_t^{\text{EUL}}$ .

The corresponding static problem is formulated as

$$\max_{\xi \in \mathcal{B}(x)} \mathbb{E}[U(\xi)] \quad (8.9)$$

$$\text{subject to } \mathbb{E}[(U(\xi) - U(q))^-] \leq \varepsilon.$$

The following proposition characterizes the EUL-optimal terminal wealth which we denote as  $X_T^{\text{EUL}}$ .

**Proposition 8.1.** Let  $Q = q > 0$  be a fixed benchmark. Moreover, let for  $y_1, y_2 > 0$  be defined

$$\begin{aligned} \underline{h} = \underline{h}(y_1) &:= \frac{1}{y_1} U'(q), \\ \bar{h} = \bar{h}(y_1, y_2) &:= \frac{1 + y_2}{y_1} U'(q) = (1 + y_2) \underline{h} \quad \text{and} \\ f(z) = f(z; y_1, y_2) &:= \begin{cases} I(y_1 z) & \text{if } z < \underline{h}, \\ q & \text{if } \underline{h} \leq z < \bar{h}, \\ I(\frac{y_1}{1+y_2} z) & \text{if } \bar{h} \leq z, \end{cases} \end{aligned}$$

for  $z > 0$ . Finally, let the initial capital  $x > 0$  and the bound for the risk constraint  $\varepsilon$  be such that there are strictly positive and unique solutions  $y_1, y_2$  of the following system of equations

$$\begin{aligned} \mathbb{E}[H_T f(H_T; y_1, y_2)] &= x \\ \mathbb{E}[(U(f(H_T; y_1, y_2)) - U(q))^-] &= \varepsilon. \end{aligned}$$

Then the EUL-optimal terminal wealth is

$$X_T^{\text{EUL}} = f(H_T) = f(H_T; y_1, y_2).$$

**Proof.** The assumption on the existence of solutions  $y_1, y_2 > 0$  of the system of equations given in the proposition implies, that  $X_T^{\text{EUL}}$  fulfills the risk constraint with equality. In order to solve the optimization problem under the risk constraint, we adopt the common convex-duality approach by introducing the convex conjugate of the utility function  $U$  with an additional term capturing the risk constraint as it is shown in the following lemma. The proof can be found in Appendix B.

**Lemma 8.2.** *Let  $z, y_1, y_2, q > 0$ . Then the solution of the optimization problem*

$$\max_{x>0} \{U(x) - y_1zx - y_2(U(x) - U(q))^{-}\}$$

is  $x^* = f(z; y_1, y_2)$ .

Applying the above lemma pointwise for all  $z = H_T(\omega)$  it follows that  $\xi^* = f(H_T; y_1, y_2)$  is the solution of the maximization problem

$$\max_{\xi>0} \{U(\xi) - y_1H_T\xi - y_2(U(\xi) - U(q))^{-}\}.$$

Obviously,  $\xi^*$  is  $\mathcal{F}_T$ -measurable and if  $y_1, y_2$  are chosen as solutions of the system of equations given in the proposition then it follows  $\xi^* = X_T^{\text{EUL}}$ .

To complete the proof, let  $\eta$  be any admissible solution satisfying the static budget constraint and the EUL-constraint (8.7). We have

$$\begin{aligned} \mathbb{E}[U(X_T^{\text{EUL}})] - \mathbb{E}[U(\eta)] &= \mathbb{E}[U(X_T^{\text{EUL}})] - \mathbb{E}[U(\eta)] - y_1x + y_1x - y_2\varepsilon + y_2\varepsilon \\ &\geq \mathbb{E}[U(X_T^{\text{EUL}})] - y_1\mathbb{E}[H_TX_T^{\text{EUL}}] - y_2\mathbb{E}[(U(X_T^{\text{EUL}}) - U(q))^{-}] \\ &\quad - \mathbb{E}[U(\eta)] + y_1\mathbb{E}[H_T\eta] + y_2\mathbb{E}[(U(\eta) - U(q))^{-}] \\ &\geq 0, \end{aligned}$$

where the first inequality follows from the static budget constraint and the constraint for the risk holding with equality for  $X_T^{\text{EUL}}$ , while holding with inequality for  $\eta$ . The last inequality is a consequence of the above lemma. Hence we obtain that  $X_T^{\text{EUL}}$  is optimal.  $\blacksquare$

**Remark 8.3.** We have  $f(z; y_1, y_2) \rightarrow I(y_1z)$  for  $y_2 \downarrow 0$ . This limit corresponds to  $\varepsilon \uparrow \varepsilon_{\max}$  and we derive the results for the unconstrained problem if we set  $y_2 = 0$  and  $f(z; y_1, 0) := I(y_1z)$ .

In the following proposition we present the explicit expressions for the EUL-optimal wealth and portfolio strategies before the horizon.

**Proposition 8.4.** Let the assumptions of Proposition 8.1 be fulfilled. Moreover, let  $U$  be the utility function given in (3.5).

(i) The EUL-optimal wealth at time  $t < T$  before the horizon is given by

$$\begin{aligned} X_t^{\text{EUL}} &= F(H_t, t) \tag{8.10} \\ F(z, t) &:= \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}} - \left[ \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \Phi(-d_1(\underline{h}, z, t)) - qe^{-r(T-t)} \Phi(-d_2(\underline{h}, z, t)) \right] \\ &\quad + \left[ \frac{(1+y_2)^{\frac{1}{\gamma}} e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \Phi(-d_1(\bar{h}, z, t)) - qe^{-r(T-t)} \Phi(-d_2(\bar{h}, z, t)) \right], \end{aligned}$$

for  $z > 0$ . Thereby,  $\Phi(\cdot)$  is the standard-normal distribution function,  $y_1, y_2$  and  $\underline{h}, \bar{h}$  are as in Proposition 8.1, and

$$\begin{aligned} \Gamma(t) &:= \frac{1-\gamma}{\gamma} \left( r + \frac{\|\kappa\|^2}{2\gamma} \right) (T-t), \\ d_2(u, z, t) &:= \frac{\ln \frac{u}{z} + (r - \frac{\|\kappa\|^2}{2})(T-t)}{\|\kappa\| \sqrt{T-t}}, \\ d_1(u, z, t) &:= d_2(u, z, t) + \frac{1}{\gamma} \|\kappa\| \sqrt{T-t}. \end{aligned}$$

(ii) The EUL-optimal fraction of wealth invested in stock at time  $t < T$  before the horizon is

$$\begin{aligned} \theta_t^{\text{EUL}} &= \theta^N \Theta(H_t, t) \\ \text{where } \Theta(z, t) &:= \left( 1 - \frac{qe^{-r(T-t)}}{F(z, t)} \left[ \Phi(-d_2(\underline{h}, z, t)) - \Phi(-d_2(\bar{h}, z, t)) \right] \right), \end{aligned}$$

for  $z > 0$ . Thereby,  $\theta^N = \frac{1}{\gamma} \sigma^{-1} \kappa = \frac{1}{\gamma} (\sigma \sigma^\top)^{-1} (\mu - r \mathbf{1}_n)$  denotes the normal strategy and  $\Theta(H_t, t)$  is the exposure to risky assets relative to the normal strategy.

### Proof.

- (i) The process  $HX^{\text{EUL}}$  is an  $\mathcal{F}$ -martingale and the proof is as in part (i) of Proposition 6.2, if  $\underline{h}$  and  $\bar{h}$  are replaced appropriately. For detailed computations we refer to Appendix C.
- (ii) The same arguments as in the proof of Proposition 7.4 (ii), lead to following equality for the optimal trading strategy

$$\theta_t^{\text{EUL}} = -\sigma^{-1} \kappa \frac{F_z(H_t, t) H_t}{F(H_t, t)} = -\theta^N \gamma \frac{F_z(H_t, t) H_t}{F(H_t, t)}, \tag{8.11}$$

where  $F(z, t)$  is defined in Proposition 8.4. Further, formal evaluation of the

derivative  $F_z$  yields

$$\begin{aligned}
F_z(z, t) &= \frac{1}{\gamma z} \left[ -F(z, t) + qe^{-r(T-t)} \left( \Phi(-d_2(\underline{h}, z, t)) - \Phi(-d_2(\bar{h}, z, t)) \right) \right] \\
&\quad - \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}} \|\kappa\| \sqrt{T-tz}} \left[ \varphi(d_1(\underline{h}, z, t)) - (1+y_2)^{\frac{1}{\gamma}} \varphi(d_1(\bar{h}, z, t)) \right] \\
&\quad + \frac{qe^{-r(T-t)}}{\|\kappa\| \sqrt{T-tz}} \left[ \varphi(d_2(\underline{h}, z, t)) - \varphi(d_2(\bar{h}, z, t)) \right]. \tag{8.12}
\end{aligned}$$

Thereby,  $\varphi(\cdot)$  denotes the standard-normal probability density function. In Appendix E we show for stochastic benchmark  $Q = e^{\delta T} S_T$  that the terms in the second and third line add to zero. Following the same arguments in the case of deterministic benchmark  $q$ , we find that  $F_z$  reduces to the expression given in the first line (see [30]). Substituting into (8.11), we get the final form of the optimal strategy before the horizon. ■

The next proposition states two properties of the function  $\Theta(z, t)$  appearing in the definition of the above representation of the EUL-optimal strategy.

**Proposition 8.5.** Let the assumptions of Proposition 8.1 be fulfilled. Moreover, let  $U$  be the utility function given in (3.5). Then, for the function  $\Theta(z, t)$  defined in Proposition 8.4 (ii) there hold the following relations.

$$(i) \quad 0 < \Theta(z, t) < 1 \quad \text{for all } z > 0 \text{ and } t \in [0, T)$$

(ii)

$$\lim_{t \rightarrow T} \Theta(z, t) = \begin{cases} 1 & \text{if } z < \underline{h} \text{ or } z > \bar{h} \\ 0 & \text{if } \underline{h} < z < \bar{h}, \\ \frac{1}{2} & \text{if } z = \underline{h}, \bar{h} \end{cases}$$

**Proof.** Using Eq. (8.10) the function  $F(z, t)$  can be written as

$$\begin{aligned}
F(z, t) &= F_1(z, t) + F_2(z, t), \\
\text{where } F_1(z, t) &= \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \left[ 1 - \Phi(-d_1(\underline{h}, z, t)) + (1+y_2)^{\frac{1}{\gamma}} \Phi(-d_1(\bar{h}, z, t)) \right] \\
\text{and } F_2(z, t) &= qe^{-r(T-t)} \left[ \Phi(-d_2(\underline{h}, z, t)) - \Phi(-d_2(\bar{h}, z, t)) \right] \quad \text{for } z > 0.
\end{aligned}$$

From the other hand we have from Proposition 8.4

$$\Theta(z, t) = 1 - \frac{F_2(z, t)}{F(z, t)} = 1 - \frac{F_2(z, t)}{F_1(z, t) + F_2(z, t)}. \tag{8.13}$$

The terms  $F_1(z, t)$  and  $F_2(z, t)$  are strictly positive since  $y_2 > 0$  implies  $\underline{h} < \bar{h}$  and the functions  $d_{1/2}(u, \cdot, \cdot)$  are strictly increasing w.r.t.  $u$  and  $\Phi$  is strictly increasing, too. Hence we have  $0 < \Theta(z, t) < 1$  and it follows assertion (i).

For the proof of the second assertion we consider the limits of the following functions for  $t \rightarrow T$ .

	$z < \underline{h}$	$z = \underline{h}$	$\underline{h} < z < \bar{h}$	$z = \bar{h}$	$z > \bar{h}$
$d_{1/2}(\underline{h}, z, t)$	$+\infty$	0	$-\infty$	$-\infty$	$-\infty$
$d_{1/2}(\bar{h}, z, t)$	$+\infty$	$+\infty$	$+\infty$	0	$-\infty$
$\Phi(-d_{1/2}(\underline{h}, z, t))$	0	$\frac{1}{2}$	1	1	1
$\Phi(-d_{1/2}(\bar{h}, z, t))$	0	0	0	$\frac{1}{2}$	1
$F_1(z, t)$	$\frac{1}{(y_1 z)^{\frac{1}{\gamma}}}$	$\frac{q}{2}$	0	$\frac{q}{2}$	$\left(\frac{1+y_2}{y_1 z}\right)^{\frac{1}{\gamma}}$
$F_2(z, t)$	0	$\frac{q}{2}$	$q$	$\frac{q}{2}$	0

Thereby, the relations  $\left(\frac{1}{y_1 \underline{h}}\right)^{\frac{1}{\gamma}} = q$  and  $\left(\frac{1}{y_1 \bar{h}}\right)^{\frac{1}{\gamma}} = \frac{q}{(1+y_2)^{\frac{1}{\gamma}}}$  have been used. Substituting these limits into (8.13) yields the assertion. ■

**Remark 8.6.** The second assertion of Proposition 8.5 shows that the lower and upper bounds for  $\Theta(z, t)$  given in the first assertion can not be improved. The given bounds are reached (depending on the value of  $z$ ) asymptotically if time  $t$  approaches the horizon  $T$ .

From the proposition it follows that the EUL-optimal fraction of wealth  $\theta_T^{\text{EUL}}$  invested in the stock at the horizon is equal to the normal strategy  $\theta^N$  in the bad and good states and equal to 0 in the intermediate states of the market, which are described by  $H_T$ . Before horizon  $T$ ,  $\theta_t^{\text{EUL}}$  is always strictly positive and never exceeds the normal strategy  $\theta^N$ .

## 9 Numerical results

This section illustrates the findings of the preceding sections with an example. Table 13.1 shows the parameters for the portfolio optimization problem and the underlying Black-Scholes model of the financial market. In this example the aim is to maximize the expected logarithmic utility ( $\gamma = 1$ ) of the terminal wealth  $X_T$  of the portfolio with the horizon  $T = 20$  years. The shortfall level  $q$  is set to be 80% of the terminal wealth of a pure bond portfolio, i.e.,  $q = 0.8xe^{rT}$ . We bound the shortfall probability  $P(X_T < q)$  by  $\varepsilon = 10\%$ , i.e., we consider the optimization with the VaR constraint described in Section 6.

First the solution of the static problem is considered, it leads to the optimal terminal wealth  $\xi^{\text{VaR}}$ . Figure 9.1 shows the probability density function and its cumulated counterpart - the distribution function - of this random variable. For the sake of comparison we also give the corresponding functions for the terminal wealth resulting from portfolios managed by the

- pure bond strategy  $\theta_t \equiv \theta^0 = 0$ ,



stock	$\mu = 7\%, \quad \sigma = 20\%$
bond	$r = 4\%$
horizon	$T = 20$
initial wealth	$x = 1$
utility function	$U(x) = \ln x, \quad \text{i.e., } \gamma = 1$
shortfall level	$q = 0.8 x e^{rT} = 1.78 \dots$
constraint	$\text{VaR}_\varepsilon(X_T - q) \leq 0 \quad \Leftrightarrow \quad P(X_T < q) \leq \varepsilon$
shortfall probability	$\varepsilon = 10\%$

Table 9.1: Parameters of the optimization problem

- pure stock strategy  $\theta_t \equiv \theta^1 = 1$ ,
- optimal strategy of the unconstrained problem  $\theta_t \equiv \theta^B = \frac{\mu - r}{\gamma \sigma^2} = 0.75$   
(see Example 4.6).

Additionally, on the horizontal axes the expected terminal wealths  $E[X_T]$  for the considered portfolios are marked.

While in case of the pure bond portfolio the distribution of the terminal wealth  $X_T^{\theta^0}$  is concentrated in the single point  $x e^{rT}$ , the terminal wealth  $X_T^{\theta^1}$  and  $X_T^{\theta^B}$  are absolutely continuous. It holds

$$e^{rT} = X_T^{\theta^0} < E[X_T^{\theta^B}] < E[X_T^{\theta^1}] = e^{\mu T}.$$

We mention that  $\xi^B = X_T^{\theta^B}$  maximizes the expected utility  $E[U(X_T^{\theta^B})]$  and not the expected terminal wealth  $E[X_T^{\theta^B}]$  itself, therefore the latter inequality is not a contradiction. For parameter sets fulfilling  $\theta^B = \frac{\mu - r}{\gamma \sigma^2} > 1$  the reverse inequality can be observed.

The distribution of the optimal terminal wealth  $X_T^{\text{VaR}}$  for the constrained problem contains a discrete as well as an absolutely continuous part. This follows from the representation of  $X_T^{\text{VaR}}$  in Proposition 6.1, which indicates that the probability  $P(\underline{h} \leq H_T < \bar{h}) = 0.1711 \dots$  is concentrated in the single point  $q$ . In the density plot this probability mass built up at the shortfall level  $q$  is marked by a vertical line at  $q$ . It is noted that there is a gap in the support of the absolutely continuous distribution, since an interval  $(q_0, q) = (1.1343, 1.7804)$  of values below the shortfall level  $q$  (small losses) carries no probability while the interval  $(0, q_0]$  (large losses) carries the maximum allowed probability of  $\varepsilon = 10\%$ . This effect demonstrates a serious drawback of the VaR constraint which bounds only the probability of the losses but does not care about the magnitude of losses.

The comparison of the expected terminal wealth yields that the VaR-optimal portfolio reaches an expected terminal wealth  $E X_T^{\text{VaR}}$  which is very close below of  $E X_T^{\theta^B}$  from the optimal portfolio of the unconstrained problem.

The solution of the representation problem, i.e., the optimal strategy  $\theta_t^* = \theta_t^{\text{VaR}}$ , is shown in Figure 9.2. Again we give for the sake of comparison the strategies of

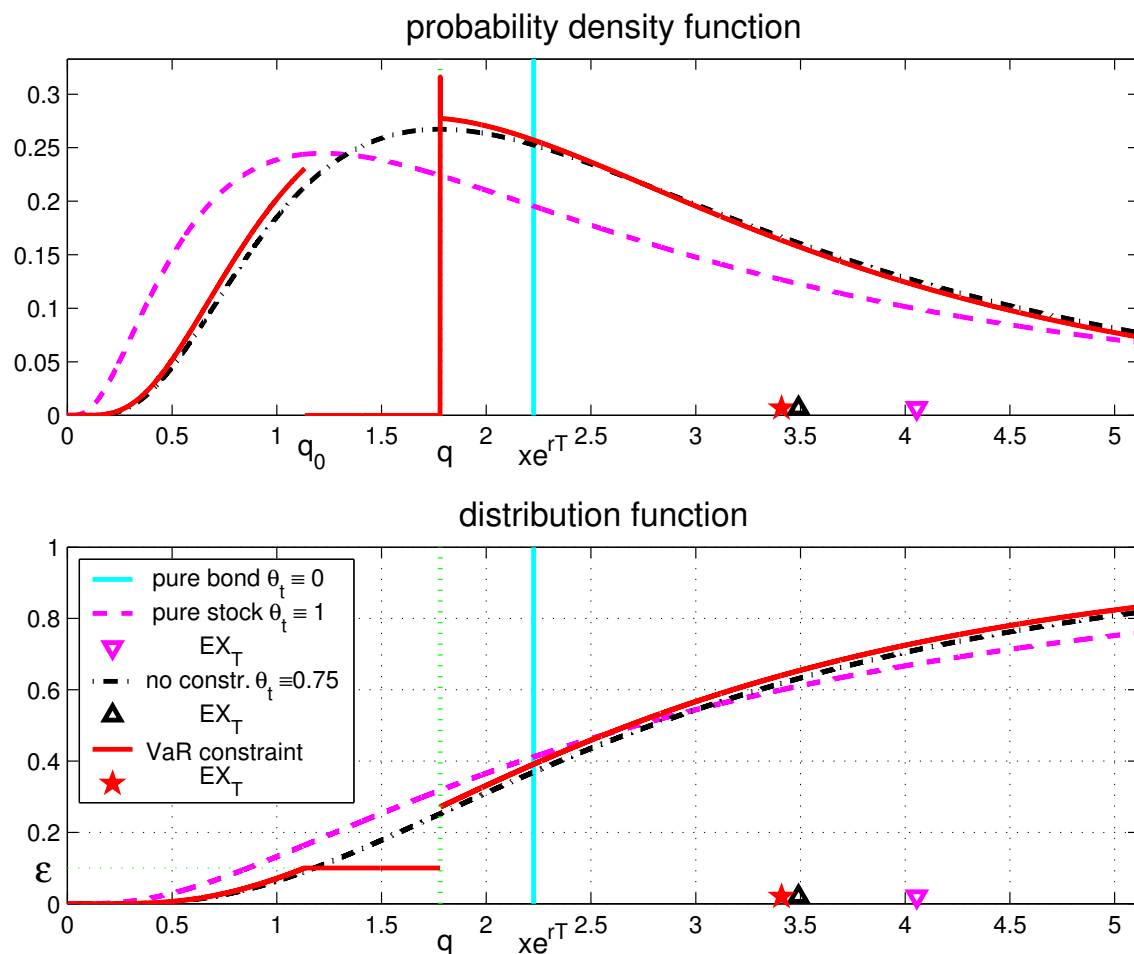


Figure 9.1: Distribution of the VaR-optimal terminal wealth

the other portfolios considered in Figure 9.1, i.e., the strategies  $\theta^0 \equiv 0$ ,  $\theta^1 \equiv 1$  and  $\theta^B \equiv \frac{\mu-r}{\gamma\sigma^2} = 0.75$ , which are constants. Contrary to this, the optimal strategy  $\theta_t^{\text{VaR}}$  is a feedback strategy, i.e., it is a function of time  $t$  and the state  $X_t$  which is the wealth at time  $t$ . Proposition 6.2 (ii) gives an equivalent representation of  $\theta_t^{\text{VaR}}$  in terms of  $t$  and the state price density  $H_t$ . Since  $H_t$  can be expressed in terms of  $t$  and  $S_t$  the optimal strategy can be written also as a function of  $t$  and the stock price  $S_t$  at time  $t$ . Figure 9.2 shows the dependence of  $\theta_t^{\text{VaR}}$  on the stock price  $S_t$  for three instants  $t = 0.25T = 5$ ,  $t = 0.75T = 15$  and the time just before the horizon  $T = 20$ . Moreover the dependence of the VaR-optimal strategy on time  $t$  and stock price  $S$  is visualized.

It can be seen that at the horizon  $T$  the optimal strategies  $\theta_T^{\text{VaR}}$  and  $\theta^B$  of the constrained and unconstrained problem, respectively, coincide for small and large stock prices, i.e., for  $S_T \in (0, 0.8639) \cup (1.5759, \infty)$ . In case of medium stock prices ( $S_T \in (0.8639, 1.5759)$ ) it holds  $\theta_t^{\text{VaR}} \rightarrow 0$  for  $t \rightarrow T$ , which indicates that in this case the complete capital is invested in the riskless bond, in order to ensure that the terminal wealth exceeds  $q$  with the required probability  $1 - \varepsilon$ . For prior instants  $t$  in case of very small stock prices the relation  $\theta_t^{\text{VaR}} > \theta_t^B$  can be observed. This seems

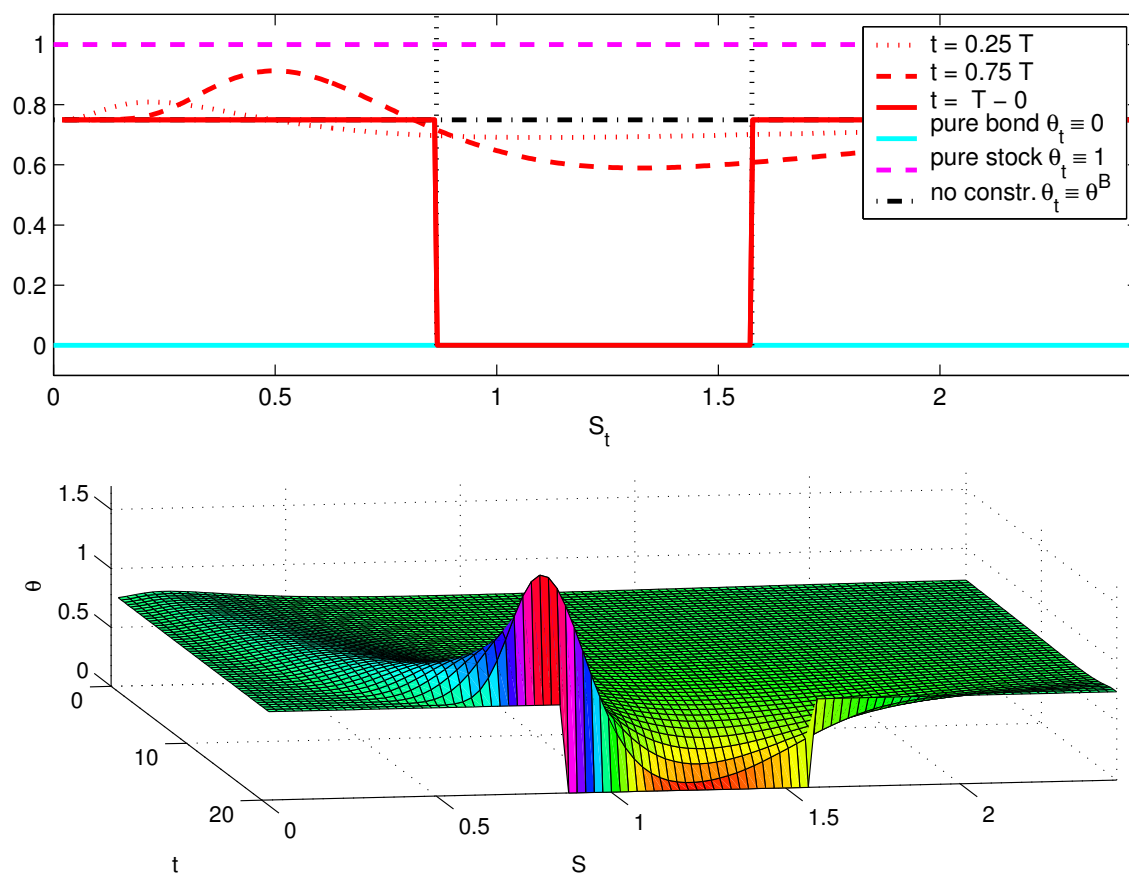


Figure 9.2: VaR-optimal strategy  $\theta^{\text{VaR}}$  as a function of time  $t$  and the stock price  $S$

to be very risky and not rational but corresponds to the above described form of the distribution of the terminal wealth which concentrates the maximum of the allowed probability  $\varepsilon$  in the region of very small values of  $X_T$ , i.e., in a region of large losses.

Measuring the shortfall risk using the shortfall probability leads in case of the VaR-optimal portfolio to

$$P(X_T^{\text{VaR}} < q) = \varepsilon = 0.1 \quad \text{or} \quad \text{VaR}_{0.1}(X_T - q) = 0.$$

Using the Expected Shortfall as a risk measure one obtains

$$\text{EL}(X_T^{\text{VaR}} - q) = \text{E}[(X_T^{\text{VaR}} - q)^-] = 0.0926 \dots =: \bar{\varepsilon}.$$

For the sake of comparison we present results for the EL-optimal portfolio which maximizes the expected utility of the terminal wealth  $\text{EU}(X_T^{\text{EL}})$  but satisfies the constraint  $\text{EL}(X_T^{\text{EL}} - q) \leq \bar{\varepsilon}$  instead of  $P(X_T^{\text{VaR}} < q) = \varepsilon = 0.1$ .

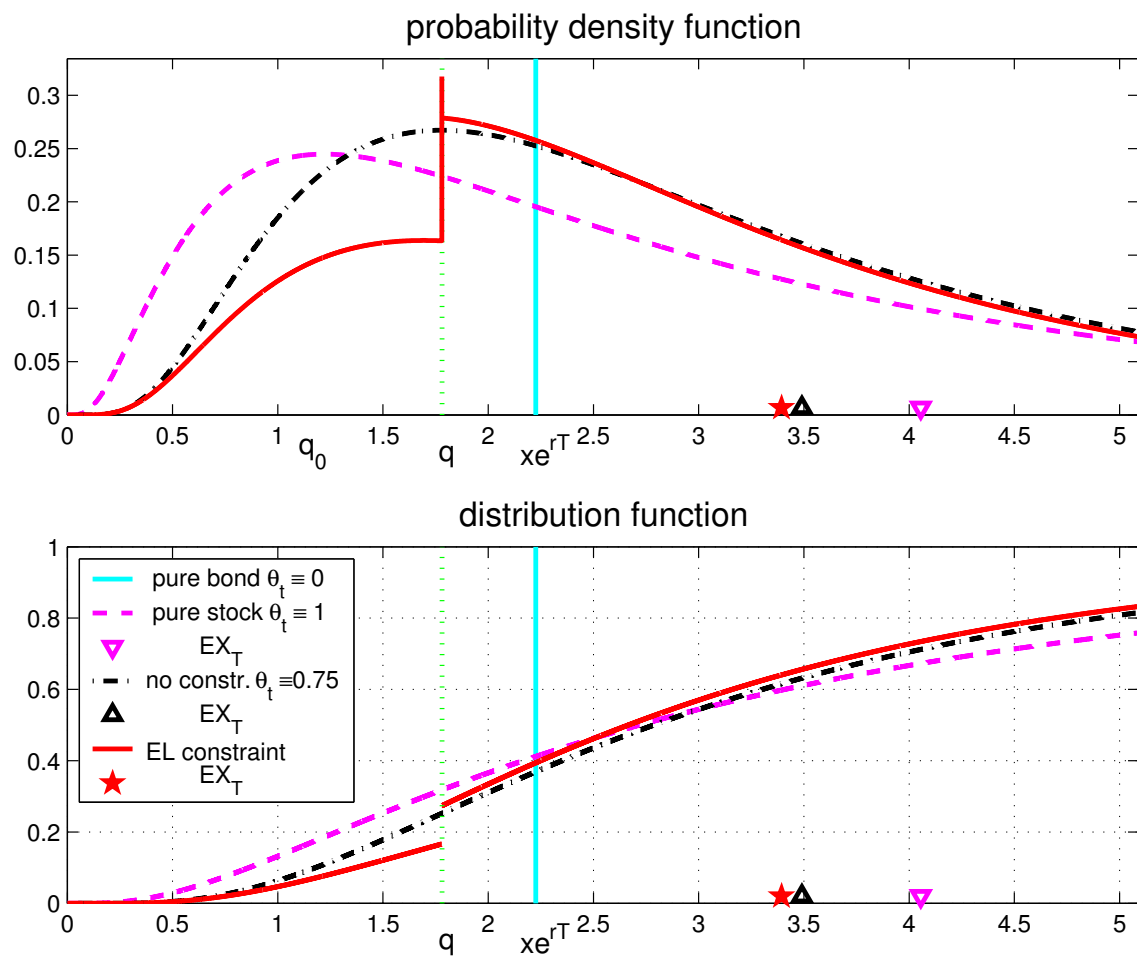


Figure 9.3: Distribution of the EL-optimal terminal wealth

Figure 9.3 shows the probability density and distribution function of the EL-optimal terminal wealth  $\xi^{\text{EL}} = X_T^{\text{EL}}$ . As in the previous example there is a discrete as well as an absolutely continuous part of the distribution. The single point  $q$  carries the probability  $P(\underline{h} \leq H_T < \bar{h}) = 0.1073\dots$ . Contrary to the VaR-optimal terminal wealth now there is no gap in the support of the distribution.

While both (VaR- and EL-) optimal portfolios possess the same Expected Loss  $\bar{\varepsilon}$  the shortfall probability of the EL-optimal terminal wealth is significantly higher, it holds

$$P(X_T^{\text{EL}} < q) = 0.1664\dots > 0.1 = P(X_T^{\text{VaR}} < q).$$

On the other hand there is nearly no difference in the expected terminal wealths since

$$E[X_T^{\text{VaR}}] = 3.4097\dots \approx 3.3938\dots = E[X_T^{\text{EL}}].$$

Both values are close to the expected optimal terminal  $E[X_T^B] = 3.4903$  wealth of the unconstrained problem.

Figure 9.4 is the analogue to Figure 9.2 and shows the EL-optimal strategy  $\theta^{\text{EL}}$  as a function of time  $t$  and stock price  $S$ . There is a similar behavior for medium and large values of  $S$ . Differences can be observed for small values of  $S$  and if time  $t$  approaches the horizon  $T$ . For  $t \rightarrow T$  the strategy  $\theta_t^{\text{EL}}$  does not tend to the value  $\theta^B$  of the optimal strategy of the unconstrained problem but remains larger. Moreover, the region of medium stock prices  $(1.1432, 1.5870)$  characterized by  $\theta_t^{\text{EL}} \rightarrow 0$  for  $t \rightarrow T$  is smaller than in case of the VaR-optimal strategy  $\theta_t^{\text{VaR}}$ , where this region is the interval  $(0.8639, 1.5759)$ .

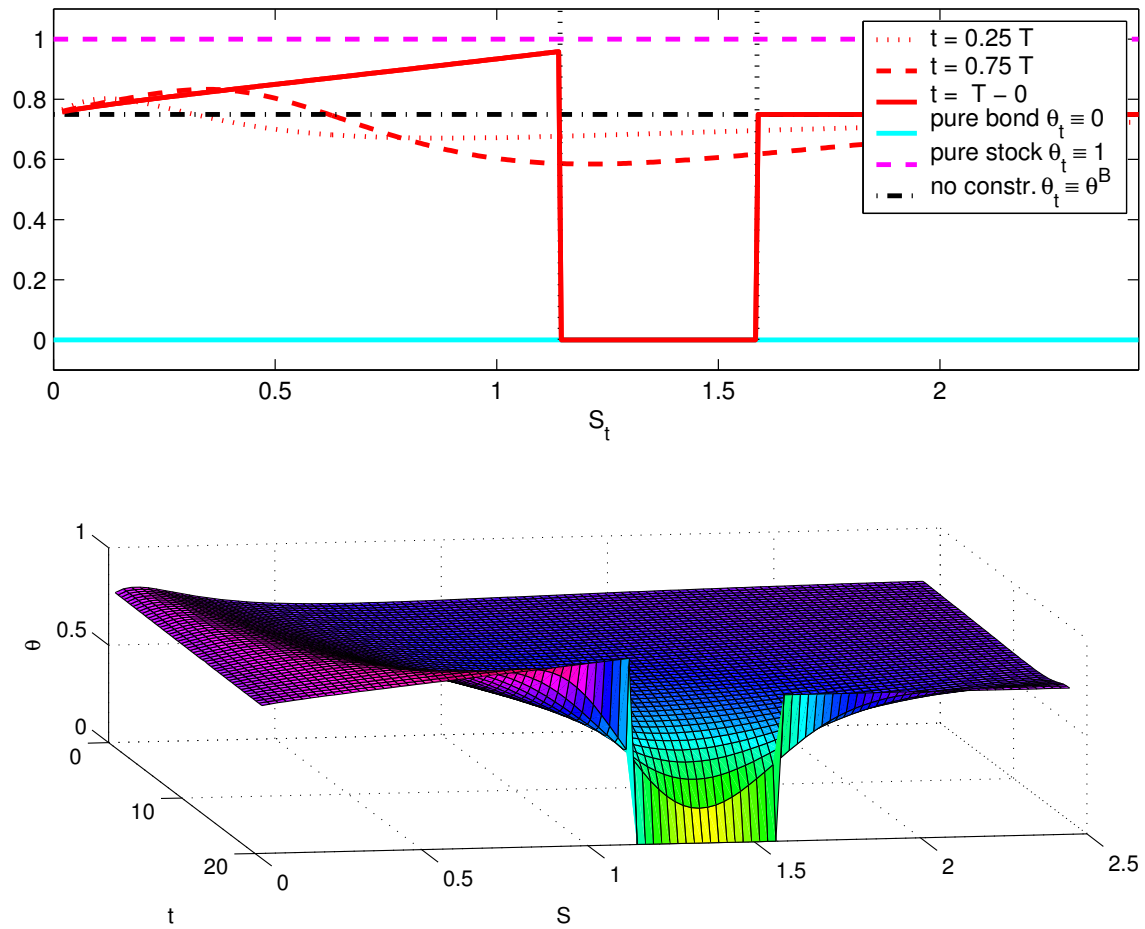


Figure 9.4: EL-optimal strategy  $\theta^{\text{EL}}$  as a function of time  $t$  and the stock price  $S$

## Chapter III

# Portfolio optimization with stochastic benchmark

### 10 The aim of this chapter

In this chapter we investigate the impact of adding a Utility Expected Loss constraint to the problem of portfolio manager who aims to beat the return of a given portfolio. More precisely, we deal with a portfolio manager who manages the relative performance, or tracking error of his portfolio along with other objectives. For a given benchmark  $Q$  representing the performance of a portfolio or an index or any economic indicator. Contrary to Section 8, now the benchmark  $Q$  is not a constant but is a random variable and chosen to be proportional to the result of an investment in a pure stock portfolio, i.e.,

$$Q = e^{\delta T} X_T^{\theta=1} = e^{\delta T} x S_T, \quad \delta \in \mathbb{R}.$$

The real number  $\delta$  measures the over- ( $\delta > 0$ ) or under-performance ( $\delta < 0$ ) of the stock market in terms of the annual logarithmic return. We define the tracking error of the portfolio manager's horizon wealth  $X$  relative to the benchmark  $Q$  as

$$G = G(X_T, Q) = U(X_T) - U(Q),$$

where the random variable  $G$  can be interpreted as the utility gain of the terminal wealth relative to the benchmark. Moreover, the shortfall risk is quantified by assigning to the random variable  $G$  a real valued risk measure  $\rho(G)$  given by

$$\rho(G) = \text{EUL}(G) := \mathbb{E}[G^-] = \mathbb{E}[(U(X_T) - U(Q))^-].$$

The portfolio optimization problem is formulated with a risk constraint bounding the Expected Utility Loss with a given  $\varepsilon$ . Contrary to the case of a benchmarked money market studied in Chapter II, we find here that the decision of the portfolio manager depends on the sensitivity of the benchmark to economic conditions.

## 11 Benchmarking the stock market

In this section we consider in addition to one risk-free bond with a constant interest rate  $r$ , only one risky stock. The shortfall is related now to the event, that the terminal wealth  $X_T$  of the portfolio is smaller than the benchmark  $Q = e^{\delta T} x S_T$  and we measure the shortfall risk using the Expected Utility Loss.

For the representation of the subsequent results we have to consider three cases of the underlying Black-Scholes model of the economy. These three cases result if we compare the quantity  $\nu := \frac{\sigma}{\kappa} = \frac{\sigma^2}{\mu-r}$  with  $\frac{1}{\gamma}$ . Thereby,  $\nu$  is referred to as the sensitivity of the benchmark to economic conditions (see Basak, Shapiro, Tepla [5]) and  $\gamma$  is the parameter of the CRRA utility function characterizing the portfolio managers risk aversion. Using the relation

$$\theta^N = \frac{\mu - r}{\gamma \sigma^2} = \frac{\kappa}{\gamma \sigma} = \frac{1}{\gamma \nu},$$

where  $\theta^N$  is the normal strategy which is the optimal strategy in the unconstrained optimization problem (see Example 4.6), the three cases are

- a)  $\nu < \frac{1}{\gamma} \iff \theta^N > 1$
- b)  $\nu > \frac{1}{\gamma} \iff \theta^N < 1$
- c)  $\nu = \frac{1}{\gamma} \iff \theta^N = 1$ .

Since the benchmark  $Q$  is related to the stock price  $S_T$  it can be expressed in terms of the state price density  $H_T$  at time  $T$ . The SDEs (4.6) and (4.7) imply

$$S_T = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T\right) \quad \text{and} \quad H_T = \exp\left(-\left(r + \frac{\kappa^2}{2}\right)T - \kappa W_T\right)$$

which gives

$$Q = e^{\delta T} x S_T = A H_T^{-\nu} \quad \text{where} \quad A := x \exp\left(\left[\delta + \left(\mu - \frac{\sigma^2}{2}\right) - \left(r + \frac{\kappa^2}{2}\right)\nu\right]T\right).$$

### 11.1 Optimization under Expected Utility Loss

In this section we solve the optimization problem of a risk manager, who is required to limit the magnitude of shortfall by maintaining his tracking error relative to the stock market return to be above some prespecified level  $\varepsilon$  over an investment horizon  $[0, T]$ . The dynamic optimization problem of the EUL-risk manager can be restated in the case of stochastic benchmark as the following static variational problem:

$$\begin{aligned} & \max_{\xi \in \mathcal{B}} \mathbb{E}[U(\xi)] \\ & \text{subject to} \quad \mathbb{E}[(U(\xi) - U(AH_T^{-\nu}))^-] \leq \varepsilon. \end{aligned}$$

The tracking error complicates the optimization by linking the nature of the non-concavity to the sensitivity of the benchmark to economic conditions. The next proposition characterizes the form of the EUL -optimal terminal wealth.



**Proposition 11.1.** Let  $Q = xe^{\delta T} S_T$  with  $\delta \in \mathbb{R}$  be a random benchmark and  $U$  be the utility function given in (3.5). Moreover, let for  $y_1, y_2 > 0$  be defined

$$\begin{aligned}\underline{h} = \underline{h}(y_1) &:= \left( \frac{1}{y_1 A^\gamma} \right)^{\frac{1}{1-\gamma\nu}}, \\ \bar{h} = \bar{h}(y_1, y_2) &:= \left( \frac{1+y_2}{y_1 A^\gamma} \right)^{\frac{1}{1-\gamma\nu}} = (1+y_2)^{\frac{1}{1-\gamma\nu}} \underline{h} \quad \text{and}\end{aligned}$$

(a) for economies with  $\nu < 1/\gamma$ :

$$f(z) = f(z; y_1, y_2) := \begin{cases} I(y_1 z) & \text{if } z < \underline{h}, \\ Az^{-\nu} & \text{if } \underline{h} \leq z < \bar{h}, \\ I\left(\frac{y_1}{1+y_2} z\right) & \text{if } \bar{h} \leq z, \end{cases}$$

(b) for economies with  $\nu > 1/\gamma$ :

$$f(z) = f(z; y_1, y_2) := \begin{cases} I\left(\frac{y_1}{1+y_2} z\right) & \text{if } z < \bar{h}, \\ Az^{-\nu} & \text{if } \bar{h} \leq z < \underline{h}, \\ I(y_1 z) & \text{if } \underline{h} \leq z, \end{cases}$$

for  $z > 0$ . Finally, let the initial capital  $x > 0$  and the bound for the risk constraint  $\varepsilon$  be such that there are strictly positive and unique solutions  $y_1, y_2$  of the following system of equations

$$\begin{aligned}\mathbb{E}[H_T f(H_T; y_1, y_2)] &= x \\ \mathbb{E}[(U(f(H_T; y_1, y_2)) - U(AH_T^{-\nu}))^-] &= \varepsilon.\end{aligned}$$

Then the EUL-optimal terminal wealth is

$$X_T^{\text{EUL}} = f(H_T) = f(H_T; y_1, y_2).$$

The proof is analogous to the proof of Proposition 8.1 if Lemma 8.2 is replaced by the following lemma which is proven in Appendix D.

**Lemma 11.2.** *Let  $z, y_1, y_2 > 0$ . Then the solution of the optimization problem*

$$\max_{x>0} \{U(x) - y_1 z x - y_2 (U(x) - U(Az^{-\nu}))^-\}$$

is  $x^* = f(z; y_1, y_2)$ .

In the following proposition we present the explicit expressions for the EUL-optimal wealth and portfolio strategies before the horizon.

**Proposition 11.3.** Let the assumptions of Proposition 11.1 be fulfilled.

(i) The EUL-optimal wealth at time  $t < T$  before the horizon is given by

$$X_t^{\text{EUL}} = F(H_t, t),$$

where  $F(z, t)$  for  $z > 0$  is defined by

(a) for economies with  $\nu < 1/\gamma$ :

$$F(z, t) = \frac{e^{\Gamma_1(t)}}{(y_1 z)^{\frac{1}{\gamma}}} - \left[ \frac{e^{\Gamma_1(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \Phi(-d_1(\underline{h}, z, t)) - A \frac{e^{\Gamma_2(t)}}{z^\nu} \Phi(-d_2(\underline{h}, z, t)) \right] \quad (11.1)$$

$$+ \left[ (1 + y_2)^{\frac{1}{\gamma}} \frac{e^{\Gamma_1(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \Phi(-d_1(\bar{h}, z, t)) - A \frac{e^{\Gamma_2(t)}}{z^\nu} \Phi(-d_2(\bar{h}, z, t)) \right].$$

(b) for economies with  $\nu > \frac{1}{\gamma}$ :

$$F(z, t) = (1 + y_2)^{\frac{1}{\gamma}} \frac{e^{\Gamma_1(t)}}{(y_1 z)^{\frac{1}{\gamma}}} + \left[ \frac{e^{\Gamma_1(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \Phi(-d_1(\underline{h}, z, t)) - A \frac{e^{\Gamma_2(t)}}{z^\nu} \Phi(-d_2(\underline{h}, z, t)) \right]$$

$$- \left[ (1 + y_2)^{\frac{1}{\gamma}} \frac{e^{\Gamma_1(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \Phi(-d_1(\bar{h}, z, t)) - A \frac{e^{\Gamma_2(t)}}{z^\nu} \Phi(-d_2(\bar{h}, z, t)) \right].$$

Hereby,  $\Phi(\cdot)$  is the standard-normal distribution function,  $y_1, y_2$  and  $\underline{h}, \bar{h}, A$  are as in Proposition 11.1, and

$$\Gamma_1(t) := \frac{1 - \gamma}{\gamma} \left( r + \frac{\kappa^2}{2\gamma} \right) (T - t)$$

$$\Gamma_2(t) := \frac{\sigma - \kappa}{\kappa} \left( r + \frac{\sigma\kappa}{2} \right) (T - t)$$

$$d_1(u, z, t) := \frac{\ln \frac{u}{z} + (r - \frac{\kappa^2}{2})(T - t)}{\kappa\sqrt{T - t}} + \frac{\kappa}{\gamma} \sqrt{T - t}$$

$$d_2(u, z, t) := \frac{\ln \frac{u}{z} + (r - \frac{\kappa^2}{2})(T - t)}{\kappa\sqrt{T - t}} + \sigma\sqrt{T - t}.$$

(ii) The EUL-optimal fraction of wealth invested in stock at time  $t < T$  before the horizon is

$$\theta_t^{\text{EUL}} = \theta^N \Theta(H_t, t).$$

Hereby,  $\theta^N = \frac{\kappa}{\gamma\sigma} = \frac{\mu - r}{\gamma\sigma^2}$  denotes the normal strategy and  $\Theta(z, t)$  is the exposure to risky assets relative to the normal strategy, which is defined for  $z > 0$  as follows

(a) for economies with  $\nu < \frac{1}{\gamma}$ :

$$\Theta(z, t) := 1 - (1 - \gamma\nu) \frac{Ae^{\Gamma_2(t)}}{z^\nu F(z, t)} \left[ \Phi(-d_2(\underline{h}, z, t)) - \Phi(-d_2(\bar{h}, z, t)) \right],$$

(b) for economies with  $\nu > \frac{1}{\gamma}$ :

$$\Theta(z, t) := 1 + (1 - \gamma\nu) \frac{Ae^{\Gamma_2(t)}}{z^\nu F(z, t)} \left[ \Phi(-d_2(\underline{h}, z, t)) - \Phi(-d_2(\bar{h}, z, t)) \right].$$

**Remark 11.4.** The case of an economy with  $\nu = \frac{1}{\gamma}$  is considered in Proposition 11.5.

**Proof.** We give the proof for case (a), i.e., for economies with  $\nu < \frac{1}{\gamma}$ , the proof for case (b) is similar.

- (i) The proof is similar to Proposition 8.4 (i), if the deterministic benchmark  $q$  is replaced by the stochastic benchmark  $Q = xe^{\delta T} S_T = AH_T^{-\nu}$ . For more details, we refer to [30].
- (ii) Following the same arguments as in Proposition 8.4 (ii), we obtain the following equality for the optimal trading strategy

$$\theta_t^{\text{EUL}} = -\frac{\kappa F_z(H_t, t)H_t}{\sigma F(H_t, t)} = -\theta^N \gamma \frac{F_z(H_t, t)H_t}{F(H_t, t)}, \quad (11.2)$$

where  $F(z, t)$  is given in Proposition 11.3. Formal evaluation of the derivative  $F_z$  yields

$$\begin{aligned} F_z(z, t) = & -\frac{1}{\gamma z} \left[ F(z, t) - (1 - \gamma\nu) \frac{Ae^{\Gamma_2(t)}}{z^\nu} \left( \Phi(-d_2(\underline{h}, z, t)) - \Phi(-d_2(\bar{h}, z, t)) \right) \right] \\ & - \frac{e^{\Gamma_1(t)}}{(y_1 z)^{\frac{1}{\gamma}} \kappa \sqrt{T - tz}} \left[ \varphi(d_1(\underline{h}, z, t)) - (1 + y_2)^{\frac{1}{\gamma}} \varphi(d_1(\bar{h}, z, t)) \right] \\ & + \frac{Ae^{\Gamma_2(t)}}{z^\nu \kappa \sqrt{T - tz}} \left[ \varphi(d_2(\underline{h}, z, t)) - \varphi(d_2(\bar{h}, z, t)) \right]. \end{aligned} \quad (11.3)$$

In Appendix E we show that the terms in the second and third line add to zero, hence  $F_z$  reduces to the expression given in the first line. Substituting into Equation (11.2), we get the final form of the optimal strategies before the horizon.  $\blacksquare$

Finally, the case of an economy with  $\nu = \frac{1}{\gamma}$  is considered. In this case the optimal strategy of the unconstrained optimization problem is  $\theta^N = 1$ , i.e., the optimal portfolio is a pure stock portfolio and  $\theta^N$  is a "buy-and-hold" strategy. The following proposition is proven in Appendix F.

**Proposition 11.5.** Let  $U$  be the utility function given in (3.5) and  $\nu = \frac{1}{\gamma}$ . Moreover, let

$$y := y^N = \frac{1}{x^\gamma} e^{(1-\gamma)(r + \frac{\kappa^2}{2\gamma})T} = \frac{1}{x^\gamma} e^{(1-\gamma)\frac{\mu+r}{2}T}.$$

Then the EUL-optimal terminal wealth is  $\xi^{\text{EUL}} = \xi^N = I(y^N H_T)$  and the EUL-optimal strategy coincides with the normal strategy, i.e.,  $\theta_t^{\text{EUL}} \equiv \theta^N = 1$ , provided that the risk constraint

$$\mathbb{E}[(U(\xi^N) - U(Q))^-] \leq \varepsilon$$

is fulfilled. Otherwise, there is no admissible solution.

## 12 Properties of the optimal portfolio

The next proposition collects two properties of the exposure to risky assets  $\Theta(z, t)$  relative to the normal strategy  $\theta^N$  given in Proposition 11.3.

**Proposition 12.1.** Let the assumptions of Proposition 11.1 be fulfilled. Then, for the function  $\Theta(z, t)$  defined in Proposition 11.3 (ii) there hold the following relations.

- (i) (a) for economies with  $\nu < \frac{1}{\gamma}$ :  $\frac{1}{\theta^N} < \Theta(z, t) < 1$   
 (b) for economies with  $\nu > \frac{1}{\gamma}$ :  $1 < \Theta(z, t) < \frac{1}{\theta^N}$   
 for all  $z > 0$  and  $t \in [0, T)$ .

(ii)

$$\lim_{t \rightarrow T} \Theta(z, t) = \begin{cases} 1 & \text{if } z < \min(\underline{h}, \bar{h}) \text{ or } z > \max(\underline{h}, \bar{h}) \\ \frac{1}{\theta^N} & \text{if } \min(\underline{h}, \bar{h}) < z < \max(\underline{h}, \bar{h}), \\ \frac{1}{2}(1 + \frac{1}{\theta^N}) & \text{if } z = \underline{h}, \bar{h} \end{cases}$$

**Proof.** We prove the claim for economies with  $\nu < \frac{1}{\gamma}$ , economies with  $\nu > \frac{1}{\gamma}$  are treated similarly. Using Eq. (11.1) the function  $F(z, t)$  can be written as

$$\begin{aligned} F(z, t) &= F_1(z, t) + F_2(z, t) \\ \text{where } F_1(z, t) &= \frac{e^{\Gamma_1(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \left[ 1 - \Phi(-d_1(\underline{h}, z, t)) + (1 + y_2)^{\frac{1}{\gamma}} \Phi(-d_1(\bar{h}, z, t)) \right] \\ \text{and } F_2(z, t) &= \frac{A e^{\Gamma_2(t)}}{z^\nu} \left[ \Phi(-d_2(\underline{h}, z, t)) - \Phi(-d_2(\bar{h}, z, t)) \right] \quad \text{for } z > 0. \end{aligned}$$

From the other hand we have from Proposition 11.3

$$\Theta(z, t) = 1 - (1 - \gamma\nu) \frac{F_2(z, t)}{F(z, t)} = 1 - \left(1 - \frac{1}{\theta^N}\right) \frac{F_2(z, t)}{F_1(z, t) + F_2(z, t)}.$$

Thereby, the relation  $\gamma\nu = \frac{1}{\theta^N}$  has been used. The terms  $F_1(z, t)$  and  $F_2(z, t)$  are strictly positive since  $y_2 > 0$  implies  $\underline{h} < \bar{h}$  and the functions  $d_{1/2}(u, \dots)$  are strictly increasing w.r.t.  $u$ . Moreover,  $\Phi$  is strictly increasing. Hence we have  $\frac{1}{\theta^N} < \Theta(z, t) < 1$  and it follows assertion (i).

The proof of the second assertion is analogous to the proof of Proposition 8.5. ■

**Remark 12.2.** As in Proposition 8.5 dealing with the case of a deterministic benchmark the second assertion of Proposition 12.1 shows that the lower and upper bounds for  $\Theta(z, t)$  given in the first assertion can not be improved. The given bounds are reached (depending on the value of  $z$ ) asymptotically if time  $t$  approaches the horizon  $T$ .

From the proposition it follows that the EUL-optimal fraction of wealth  $\theta_T^{\text{EUL}}$  invested in the stock at the horizon is equal to the normal strategy  $\theta^N$  in the bad and good states and equal to 1 in the intermediate states of the market, which are described by  $H_T$ . Before horizon  $T$ ,  $\theta_t^{\text{EUL}}$  is always bounded by 1 and the normal strategy  $\theta^N$ .

## 13 Numerical results

This section illustrates the findings of the preceding sections with an example. Table 13.1 shows the parameters for the portfolio optimization problem and the underlying Black-Scholes model of the financial market. In this example the aim is to maximize the expected logarithmic utility ( $\gamma = 1$ ) of the terminal wealth  $X_T$  of the portfolio with the horizon  $T = 20$  years. The benchmark  $Q$  is set to be equal to 100% of the terminal wealth of a pure stock portfolio, i.e., we set  $\delta = 0$  and  $Q = xS_T$ . We bound the Expected Utility Loss by  $\varepsilon = 0.02$ .

stock	$\mu = 9\%, \quad \sigma = 20\%$
bond	$r = 4\%$
horizon	$T = 20$
initial capital	$x = 1$
utility function	$U(z) = \ln z, \quad \text{i.e., } \gamma = 1$
benchmark	$Q = xS_T, \delta = 0$
constraint	$\varepsilon = 0.02$

Table 13.1: Parameters of the optimization problem

For the present case of a logarithmic utility function  $U(z) = \ln(z)$  the risk constraint can be reformulated in terms of the annual logarithmic return  $\varrho(z) = \frac{1}{T} \ln \frac{z}{x}$  where  $x$  is the initial capital (see Remark 5.2). It holds

$$\mathbb{E}[(U(X_T^{\text{EUL}}) - U(Q))^-] \leq \varepsilon = 0.02 \iff \mathbb{E}[(\varrho(X_T^{\text{EUL}}) - \varrho(Q))^-] \leq \frac{\varepsilon}{T},$$

i.e., we bound the expected Loss of the annual logarithmic return by 0.1%.

The chosen parameters correspond to an economy with

$$\nu = \frac{\sigma}{\kappa} = \frac{\sigma^2}{\mu - r} = 0.8 < \frac{1}{\gamma} = 1 \quad \text{or equivalently} \quad \theta^N = \frac{\kappa}{\gamma\sigma} = \frac{\mu - r}{\gamma\sigma^2} = 1.25 > 1.$$

This is case (a) in the propositions of Subsection 11. The form of the EUL-optimal terminal wealth  $X_T^{\text{EUL}}$  as a function  $f(H_T)$  of the state price density  $H_T$  at the horizon  $T$  is given in Proposition 11.1 (a). For the interpretation of the result it seems to be more convenient to express  $X_T^{\text{EUL}}$  as a function  $\tilde{f}(S_T)$  of the terminal stock price  $S_T$ . This dependence can be easily derived from the following relation which follows from the SDEs (4.6) and (4.7)

$$H_t = G(S_t, t) := \exp\left(\left[\frac{1}{\nu}\left(\mu - \frac{\sigma^2}{2}\right) - \left(r + \frac{\kappa^2}{2}\right)\right]t\right) S_t^{-\frac{1}{\nu}} \quad \text{for } t \in [0, T] \quad (13.4)$$

which implies

$$S_T = g(H_T) := \exp\left(\left[\left(\mu - \frac{\sigma^2}{2}\right) - \left(r + \frac{\kappa^2}{2}\right)\nu\right]T\right) H_T^{-\nu}.$$

Figure 13.1 shows the EUL-terminal wealth  $X_T^{\text{EUL}}$  as a function of the terminal stock price  $S_T$ . Moreover, the terminal wealth of the

- pure bond portfolio  $X_T^{\theta_t \equiv 0}$ ,
- pure stock portfolio  $X_T^{\theta_t \equiv 1} = S_T = Q$ , which is equal to the chosen benchmark  $Q$ ,
- the optimal portfolio of the unconstrained problem  $X_T^N = X_T^{\theta^N}$ , (see Example 4.6).

are drawn as functions of  $S_T$ .

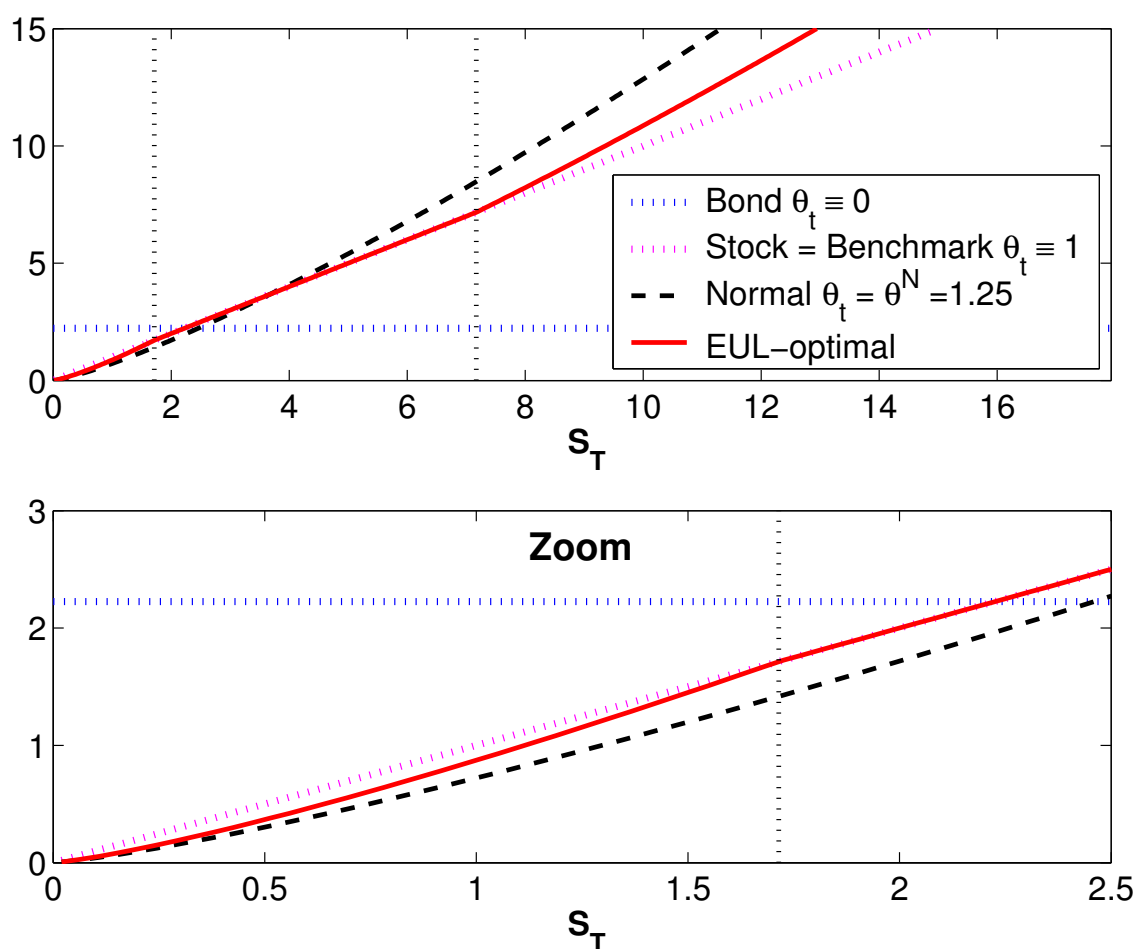


Figure 13.1: EUL-optimal terminal wealth

It can be seen, that for states with large  $S_T$ , i.e.,  $S_T > g(\bar{h})$ , the EUL-optimal portfolio overperforms the stock market, it holds  $X_T^{\text{EUL}} > Q = S_T$ . For states with intermediate stock prices, i.e.,  $g(\underline{h}) \leq S_T \leq g(\bar{h})$ , the EUL-optimal wealth coincides with the benchmark  $Q$ . An underperformance or shortfall of the EUL-optimal portfolio occurs in the states with small stock prices, i.e.,  $S_T < g(\underline{h})$ . On the other hand, it can

	$X_T^{\theta_t \equiv 0}$ pure bond	$X_T^{\theta_t \equiv 1} = Q$ pure stock	$X_T^N$ normal	$X_T^{\text{EUL}}$ EUL-optimal
$E[U(X_T)]$	0.800	1.400	1.425	1.415
$E[\varrho(X_T)]$	0.040	0.070	0.071	0.071
$E[X_T]$	2.226	6.050	7.758	6.893
$E[(U(X_T) - U(Q))^-]$	0.734	0.000	0.077	0.020

Table 13.2: Expected utility  $E[U(X_T)]$ , expected annual logarithmic return  $E[\varrho(X_T)]$ , expected terminal wealth  $E[X_T]$  and Expected Utility Loss  $E[(U(X_T) - U(Q))^-]$

be seen that the terminal wealth  $X_T^N$ , resulting from the optimization without risk constraint, underperforms the stock market in considerable more states of the market.

Table 13.2 gives the expected utilities of the terminal wealth  $E[U(X_T)]$  (which have to be maximized in the optimization), the corresponding expected annual return  $E[\varrho(X_T)]$ , the expected terminal wealth  $E[X_T]$  and the risk measure Expected Utility Loss for the above considered portfolios. The comparison of these values demonstrates, that imposing an additional risk constraint leads only to small losses of the expected utility but to considerable gains of the risk measure.

Figure 13.2 shows the probability density functions of the random terminal wealth  $X_T^{\theta_t \equiv 0}$ ,  $X_T^{\theta_t \equiv 1} = S_T = Q$ ,  $X_T^N$  and  $X_T^{\text{EUL}}$  considered above. While the terminal wealth of the pure bond portfolio  $X_T^{\theta_t \equiv 0}$  is concentrated in the single point  $e^{rT}$ , the random variables  $Q$  and  $X_T^N$  possess logarithmic normal distributions. The distribution of  $X_T^{\text{EUL}}$  is also absolutely continuous but its probability density function shows jumps at  $g(\underline{h})$  and  $g(\bar{h})$ . This is a consequence of the form of  $X_T^{\text{EUL}}$  as a function of  $H_T$  given in Proposition 11.1 (a).

Next, the EUL-optimal strategy  $\theta_t^{\text{EUL}}$ , which leads to the terminal wealth  $X_T^{\text{EUL}}$  discussed above, is considered. This strategy is a feedback strategy, i.e., it depends on time  $t$  as well as of the state of the financial market, which can be expressed in terms of the state price density  $H_t$ , the stock price  $S_t$  or of the wealth  $X_t$  of the portfolio at time  $t$ , respectively. In Proposition 11.3, (ii) the EUL-optimal strategy has been given as a function of time  $t$  and state price density  $H_t$ . This is convenient from the mathematical point of view. For practical purposes it seems to be more convenient to express the strategy in terms of  $t$  and the stock price  $S_t$  at time  $t$ , i.e.,  $\theta_t^{\text{EUL}} = \theta^{\text{EUL}}(t, S_t)$ . Such a representation can be easily derived from the representation given in Proposition 11.3, (ii) using the relation  $H_t = G(t, S_t)$  given in (13.4). The value of  $\theta_t^{\text{EUL}}$  gives the EUL-optimal fraction of wealth, which at time  $t$  has to be invested in the stock, if the stock price  $S_t$  is observed.

The upper plot in Figure 13.3 shows  $\theta^{\text{EUL}}(t, S_t)$  as a function of the stock price  $S_t$  for the fixed times  $t = \frac{1}{4}T = 5$  years,  $t = \frac{3}{4}T = 15$  years and its limit for  $t \rightarrow T - 0 = 20$  years, (i.e., the time just before the horizon). Moreover, the upper plot displays the strategies of the pure bond and stock portfolio, which are no feedback strategies but constants, namely  $\theta_t \equiv 0$  and  $\theta_t \equiv 1$ , respectively, and the normal strategy  $\theta_t = \theta^N = \frac{\mu - r}{\gamma \sigma^2} = 1.25$ , which is also constant.

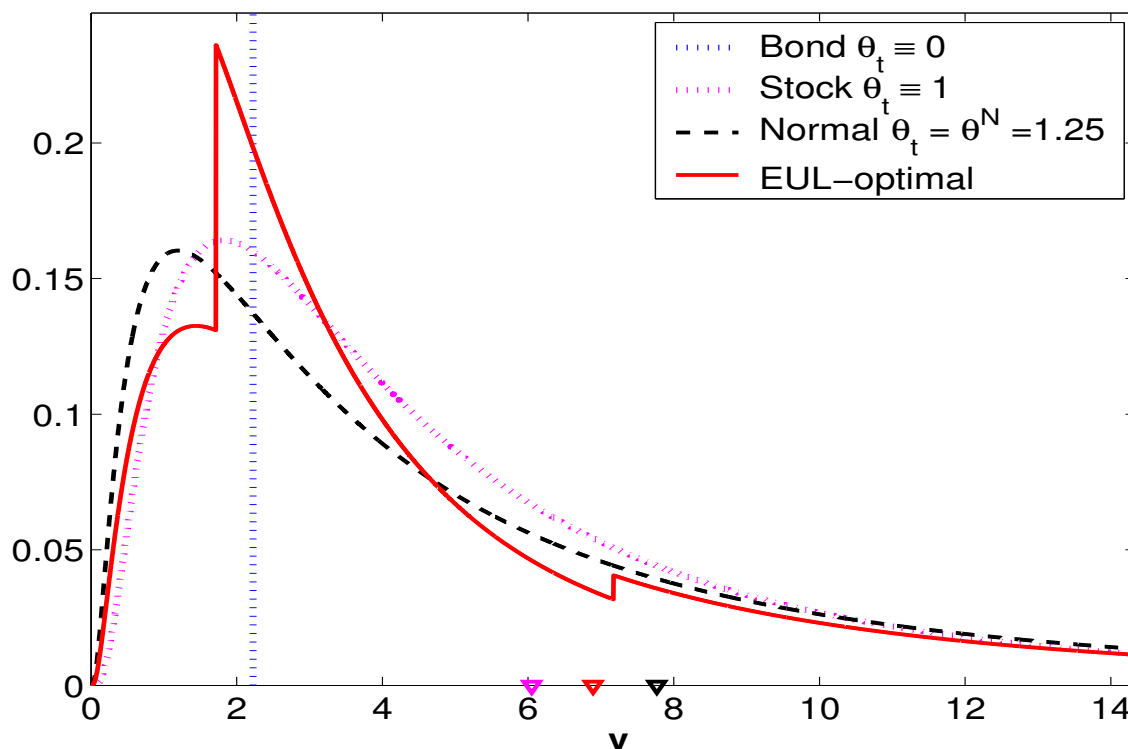


Figure 13.2: Probability density functions of terminal wealth

The lower plot shows a 3D-plot of  $\theta^{\text{EUL}}(t, S_t)$  as a function of both time  $t$  and stock price  $S_t$ . For  $t = \frac{1}{4}T, \frac{3}{4}T$  and  $T - 0$  one gets the plots of the upper picture as intersection of the 3D-plot with planes parallel to the  $(S, \theta)$ -plane. Moreover, the lower picture contains the path of a simulated stock price  $S_t = S_t(\omega)$ ,  $0 \leq t < T$ , in the  $(S, t)$ -plane and the corresponding path of the EUL-optimal strategy  $\theta_t^{\text{EUL}}(\omega) = \theta^{\text{EUL}}(t, S_t(\omega))$ .

The figure shows that the EUL-optimal fraction of wealth  $\theta_t^{\text{EUL}}$  which has to be invested in the stock is bounded from below by 1, i.e., the strategy of the pure stock portfolio. It is bounded from above by  $\theta^N = 1.25$ , i.e., the normal strategy. Let us note, that  $\theta_t > 1$  corresponds to a short position in the bond, since the fraction of wealth invested in the bond is  $1 - \theta_t^{\text{EUL}} < 0$ . For the present parameters of the financial market, especially the large positive difference of the mean return  $\mu = 9\%$  of the stock and the risk-free interest rate  $r = 4\%$  relative to the volatility of  $\sigma = 20\%$ , it is optimal to borrow money and invest this money into the stock earning the "high" mean return  $\mu$  while paying the "low" interest rate  $r$ .

If time  $t$  approaches the horizon then the strategy tends to the normal strategy  $\theta^N$  in the states with small and large stock prices. In the states with intermediate stock prices the EUL-optimal strategy tends to 1, in order to reach the corresponding EUL-optimal terminal wealth, which is in this case the stock price, i.e.,  $X_T^{\text{EUL}} = Q = S_T$ .



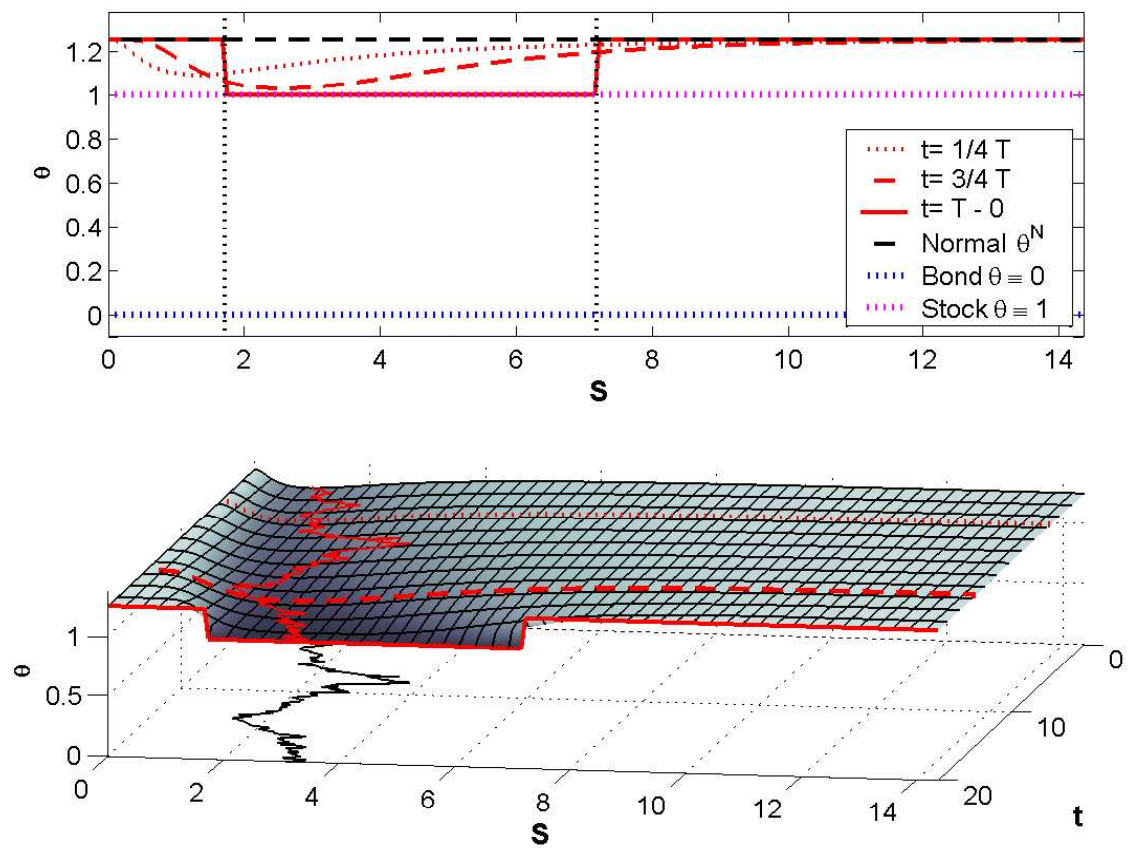


Figure 13.3: EUL-optimal strategy  $\theta_t^{\text{EUL}} = \theta^{\text{EUL}}(t, S_t)$  as a function of time  $t$  and stock price  $S_t$

Next, we change the value of the mean return of the stock  $\mu$  from 9% to the lower value 7%. The other parameters remain unchanged. Now we have an economy with

$$\nu = \frac{\sigma}{\kappa} = \frac{\sigma^2}{\mu - r} = 1.33 > \frac{1}{\gamma} = 1 \quad \text{or equivalently} \quad \theta^N = \frac{\kappa}{\gamma\sigma} = \frac{\mu - r}{\gamma\sigma^2} = 0.75 < 1.$$

This is case (b) in the propositions of Subsection 11.

Figure 13.4 shows the terminal wealth of the EUL-optimal portfolio and of the other considered portfolios as a function of the terminal stock price  $S_T$ . Note that in case (b) contrary to case (a) it holds  $\underline{h} > \bar{h}$  and consequently  $g(\underline{h}) < g(\bar{h})$ .

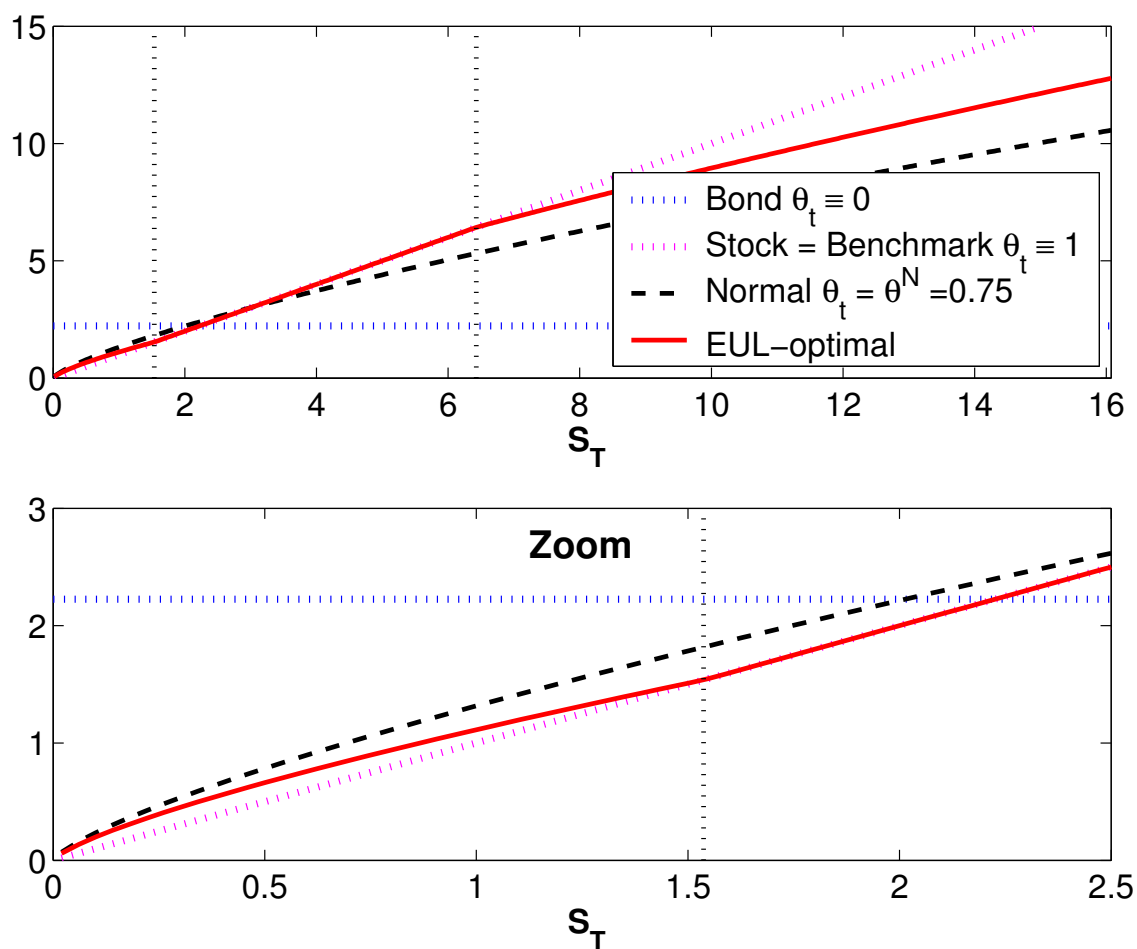


Figure 13.4: EUL-optimal terminal wealth

It can be seen, that contrary to the first example where  $\mu = 9\%$ , for states with large  $S_T$ , i.e.,  $S_T > g(\underline{h})$ , the EUL-optimal portfolio underperforms the stock market, it holds  $X_T^{\text{EUL}} < Q = S_T$ . These are the states where the shortfall happens. For states with intermediate stock prices, i.e.,  $g(\bar{h}) \leq S_T \leq g(\underline{h})$ , the EUL-optimal wealth coincides with the benchmark  $Q$ . An overperformance of the EUL-optimal portfolio occurs in the states with small stock prices, i.e.,  $S_T < g(\bar{h})$ . As in the first example it can be seen, that the terminal wealth  $X_T^N$ , resulting from the optimization without

	$X_T^{\theta_t \equiv 0}$ pure bond	$X_T^{\theta_t \equiv 1} = Q$ pure stock	$X_T^N$ normal	$X_T^{\text{EUL}}$ EUL-optimal
$E[U(X_T)]$	0.800	1.000	1.025	1.015
$E[\varrho(X_T)]$	0.040	0.050	0.051	0.051
$E[X_T]$	2.226	4.055	3.489	3.791
$E[(U(X_T) - U(Q))^-]$	0.466	0.000	0.077	0.020

Table 13.3: Expected utility  $E[U(X_T)]$ , expected annual logarithmic return  $E[\varrho(X_T)]$ , expected terminal wealth  $E[X_T]$  and Expected Utility Loss  $E[(U(X_T) - U(Q))^-]$

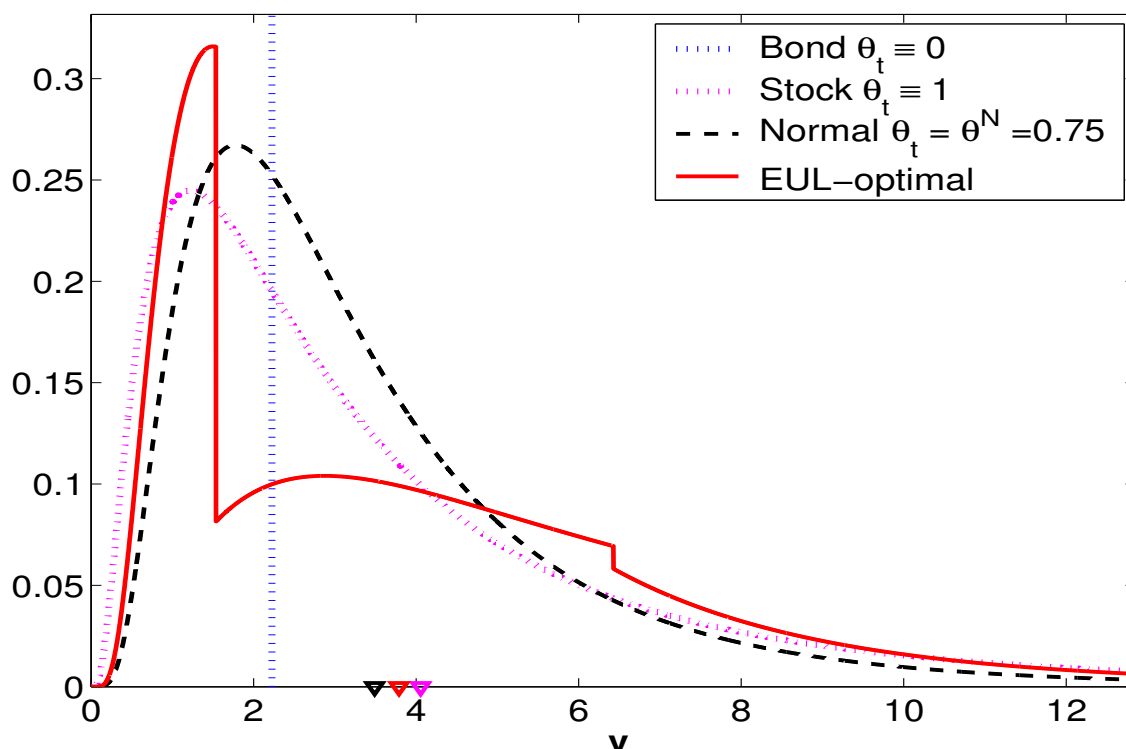


Figure 13.5: Probability density functions of terminal wealth

risk constraint, underperforms the stock market in considerable more states of the market.

Table 13.3 gives the expected utilities of the terminal wealth  $E[U(X_T)]$ , the corresponding expected annual logarithmic return  $E[\varrho(X_T)]$ , the expected terminal wealth  $E[X_T]$  and the risk measure Expected Utility Loss for the above considered portfolios. Once again the comparison of these values demonstrates, that imposing an additional risk constraint leads only to small losses of the expected utility but to considerable gains of the risk measure.

Figure 13.5 shows the probability density functions of the random terminal wealth  $X_T^{\theta_t \equiv 0}$ ,  $X_T^{\theta_t \equiv 1} = S_T = Q$ ,  $X_T^N$  and  $X_T^{\text{EUL}}$  considered above.

Figure 13.6 is analogous to Figure 13.3 and shows  $\theta^{\text{EUL}}(t, S_t)$  as a function of time  $t$  and the stock price  $S_t$ . In the upper picture three times  $t = \frac{1}{4}T = 5$  years,  $t = \frac{3}{4}T = 15$

years and the time just before the horizon  $t = T - 0 = 20$  years are fixed. The lower picture shows a 3D-plot of  $\theta^{\text{EUL}}(t, S_t)$  as a function of both time  $t$  and stock price  $S_t$ . The figure shows that the EUL-optimal fraction of wealth  $\theta_t^{\text{EUL}}$  is bounded from below by  $\theta^N = 0.75$ , i.e., the normal strategy. It is bounded from above by 1, i.e., the strategy of the pure stock portfolio. Contrary to the first example with the larger mean return  $\mu = 9\%$  of the stock, now it is not necessary to go short into the bond, since we have  $0 \leq 1 - \theta^{\text{EUL}} \leq 1 - \theta^N = 0.25$ . We note, that in the present example according to Proposition 12.1 (ii), the fraction of wealth invested in the risky stock by the EUL-optimal portfolio manager is in all states of the financial markets at least as much as the corresponding fraction of the manager following the normal strategy  $\theta^N$ . Due to this riskier strategy the EUL-optimal portfolio manager reaches a slightly smaller expected utility of the terminal wealth but he satisfies the risk constraint, which is violated by the manager following the normal strategy.

As in the first example with  $\mu = 9\%$  we observe, that if time  $t$  approaches the horizon then the strategy tends to the normal strategy  $\theta^N$  in the states with small and large stock prices. In the states with intermediate stock prices the EUL-optimal strategy tends to 1, in order to reach the corresponding EUL-optimal terminal wealth, which is in this case the stock price, i.e.,  $X_T^{\text{EUL}} = Q = S_T$ .

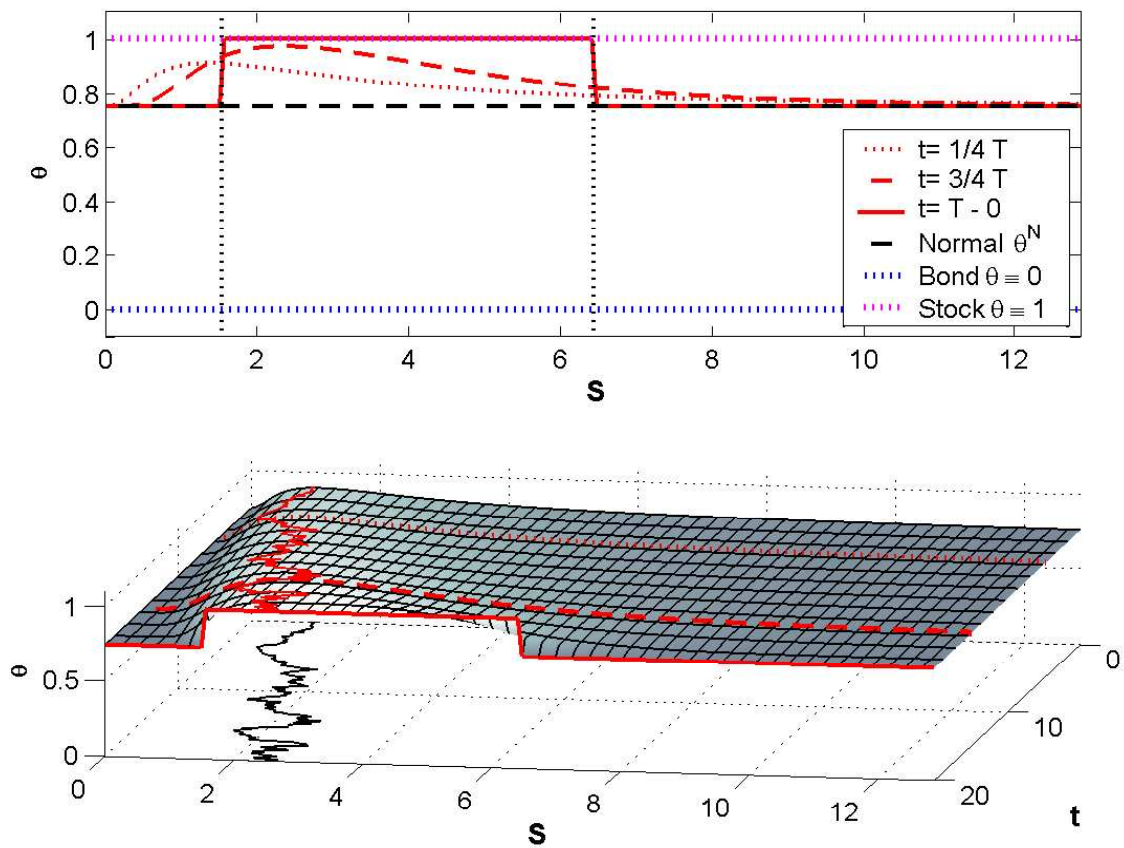


Figure 13.6: EUL-optimal strategy  $\theta_t^{\text{EUL}} = \theta^{\text{EUL}}(t, S_t)$  as a function of time  $t$  and stock price  $S_t$

## Chapter IV

# The case of partial information

In this chapter we extend the model (4.6) to the case of a non constant and random drift which is not directly observable. We deal with a portfolio manager who is starting with an initial capital and wishes to maximize his expected utility of terminal wealth by choosing a portfolio strategy based only on information about the asset-prices in a given financial market. More precisely, it is assumed that the portfolio manager cannot directly observe the stock appreciation rates or the driving Brownian motion, he can only observe past and present stock prices. This fact called partial information is modeled by requiring that all investment decisions in this market have to be adapted to the augmented filtration generated by the observed stock price process. Basically, the utility maximization problem in a setting of partial information can be solved only if the dynamics of the drift process is specified. We suggest a drift as a continuous time Markov chain independent of the driving Brownian motion, and we add to the portfolio optimization problem a risk constraint bounding the Expected Utility Loss with a given  $\varepsilon$ , i.e., a constraint of the type

$$\text{EUL}(G) := \mathbb{E}[G^-] = \mathbb{E}[(U(X_T^\pi) - U(q))^-] \leq \varepsilon,$$

where the shortfall level  $q$  is assumed to be constant.

## 14 An HMM for the stock return

We consider in this chapter a financial market which consists of *one risk-free asset*, whose price process is assumed for simplicity to be equal to 1 at each date, and  $n$  stocks whose *prices*  $S = (S_t)_{t \in [0, T]}$ ,  $S_t = (S_t^1, \dots, S_t^n)^\top$  evolve according to

$$dS_t^i = S_t^i [\mu_t^i dt + \sum_{j=1}^n \sigma_{ij} dW_t^j], \quad S_0^i = s_0^i \in \mathbb{R}, \quad i = 1, \dots, n, \quad (14.1)$$

where  $s_0 = (s_0^1, \dots, s_0^n)^\top$  is the constant initial price vector. The volatility matrix  $\sigma = (\sigma_{ij})_{i,j=1, \dots, n}$  is assumed to be constant satisfying the non-degeneracy condition

$$x^\top \sigma \sigma^\top x \geq \delta x^\top x \quad \text{for all } x \in \mathbb{R}^n,$$

where  $\delta > 0$  is a given constant. Let  $\mathcal{F}^S = (\mathcal{F}_t^S)_{t \in [0, T]}$  be the  $P$ -augmented filtration generated by the price process  $S$ .

The *return process*  $R = (R_t)_{t \in [0, T]}$  associated with the stocks is defined by

$$dR_t = (\text{Diag}(S_t))^{-1} dS_t, \quad \text{or equivalently} \quad R_t = \int_0^t \mu_s ds + \int_0^t \sigma dW_s. \quad (14.2)$$

The Hidden Markov Model has a finite set of states, each of which is associated with a (generally multidimensional) probability distribution. Transitions among the states are governed by a set of probabilities called transition probabilities. In a particular state an outcome or observation can be generated, according to the associated probability distribution. It is only the outcome, not the state visible to an external observer and therefore states are “hidden” to the outside; hence the name Hidden Markov Model.

**Definition 14.1.**

We assume that  $\mu = (\mu_t)_{t \in [0, T]}$ , the *drift process* of the return is given by

$$\mu_t = BY_t, \quad t \in [0, T], \quad (14.3)$$

where  $Y = (Y_t)_{t \in [0, T]}$  is a stationary, irreducible, *continuous time Markov chain* independent of  $W$  with state space  $\{e_1, \dots, e_d\}$ , the standard unit vectors in  $\mathbb{R}^d$ , and  $B = (B_{ik})_{i=1, \dots, n; k=1, \dots, d}$  is given by  $B_{ik} = b_i^k$ . The columns of the *state matrix*  $B \in \mathbb{R}^{n \times d}$  contain the  $d$  possible states of  $\mu_t$ . To take the unit vectors as state space is convenient since it allows us to give filtering equations for  $Y$  and not only its transition probabilities. Further  $Y$  is characterized by its *rate matrix*  $Q \in \mathbb{R}^{d \times d}$ , where  $Q_{kl} = \lim_{t \searrow 0} \frac{1}{t} P(Y_t = e_l | Y_0 = e_k)$ ,  $k \neq l$ , is the jump rate or transition rate from  $e_k$  to  $e_l$ , and  $\lambda_k = -Q_{kk} = \sum_{l=1, l \neq k}^d Q_{kl}$  is the rate of leaving  $e_k$ . Hence in state  $e_k$  the waiting time for the next jump is exponentially distributed with parameter  $\lambda_k$ , and  $Q_{kl}/\lambda_k$  is the probability that the chain jumps to  $e_l$  when leaving  $e_k$ .

The special feature of this chapter is that we consider the case of *partial information*, and this is the situation where the drift process and the Brownian motion associated with the stock prices are not observable to a portfolio manager acting in this market. He can only observe the stock prices and hence we have a hidden Markov model (HMM) for the stock returns. The fact that portfolio managers have only a partial information will be modeled by requiring that investment decisions and all processes appearing in this market have to be adapted to the filtration generated by the stock price, which is smaller than the original filtration.

Similarly to the case of full information, we now introduce the new measure  $\tilde{P}$  which is the *risk-neutral* probability measure we need for the optimization. This measure coincides with the reference measure used in filtering due to the structure of the drift given in Equation (14.3). Let us define the martingale density process  $Z = (Z_t)_{t \in [0, T]}$

$$Z_t = \exp\left(-\int_0^t (\sigma^{-1} B Y_s)^\top dW_s - \frac{1}{2} \int_0^t \|\sigma^{-1} B Y_s\|^2 ds\right), \quad (14.4)$$

and the risk-neutral probability measure  $\tilde{P}$  by  $d\tilde{P} = Z_T dP$ . The process  $(\sigma^{-1}BY_t)_{t \in [0, T]}$  is bounded and hence Girsanov's Theorem implies that the  $n$ -dimensional process

$$\tilde{W}_t = W_t + \int_0^t \sigma^{-1}BY_s ds, \quad t \in [0, T], \quad (14.5)$$

defines a  $\tilde{P}$ -Brownian motion. We denote by  $\tilde{\mathbb{E}}$  the expectation operator corresponding to the measure  $\tilde{P}$ .

**Remark 14.2.** For a constant drift, i.e., only  $d = 1$  state, the processes  $Z$  and  $H$  coincide (provided  $r = 0$ ). Further, the analysis presented in Chapter II to solve the portfolio optimization problem was used in a way that uses only the local martingale property of  $Z$  and avoids the use of  $\tilde{P}$  altogether.

The boundedness of the process  $(\sigma^{-1}BY_t)_{t \in [0, T]}$  implies in addition the following lemma.

**Lemma 14.3.**  $Z$  and  $Z^{-1} = (Z_t^{-1})_{t \in [0, T]}$  are  $P$ - and  $\tilde{P}$ -martingales, respectively. Moreover, for all  $p \in [1, \infty)$ ,  $t \in [0, T]$

$$\tilde{\mathbb{E}}[|Z_t|^p] < \infty, \quad \tilde{\mathbb{E}}[|Z_t^{-1}|^p] < \infty \quad \text{and} \quad \tilde{\mathbb{E}}\left[\left|\int_0^t Z_s^{-1} ds\right|^p\right] < \infty.$$

On the other hand we have the following decompositions for the return process:

$$dR_t = \mu_t dt + \sigma dW_t, \quad \text{and} \quad dR_t = \sigma d\tilde{W}_t. \quad (14.6)$$

Relations (14.2) and (14.6) imply that  $S$ ,  $R$ , and  $\tilde{W}$  generate the same filtration  $\mathcal{F}^S$ . Moreover, this filtration is continuous (see Karatzas and Shreve [42], Corollary 2.7.8 or Lakner [52], Proposition 4.1). As a consequence we have the following key lemma which allows to state a martingale representation theorem for  $(\tilde{P}, \mathcal{F}^S)$ -local martingales with respect to the Brownian motion  $\tilde{W}$ . We refer to Proposition 4.2 in [52] for a detailed proof.

**Lemma 14.4.** Let  $M = (M_t)_{t \in [0, T]}$  be any  $(\tilde{P}, \mathcal{F}^S)$ -local martingale. Then, there exists an  $\mathbb{R}^n$ -valued process  $\gamma = (\gamma_t)_{t \in [0, T]}$  which is  $\mathcal{F}^S$ -adapted and a.s. square integrable such that

$$M_t = M_0 + \int_0^t \gamma_s^\top d\tilde{W}_s, \quad 0 \leq t \leq T.$$

Moreover, if  $M$  is square integrable, then  $\gamma$  is unique in  $L^2$ .

## 15 The optimization problem under partial information

We shall now define an investment decision for a portfolio manager acting in a market with partial information. We find it convenient to use for the investment decision at



time  $t$  the amount of money  $\pi_t^i$  defined for each stock  $i$  by  $\pi_t^i = \psi_t^i S_t^i$ , where  $\psi_t^i$  is the number of shares held by the portfolio manager in the asset  $i$ . In the case of full information studied in the preceding chapters, we used the portfolio strategy  $\theta_t^i$  defined in equivalent way by  $\theta_t^i = \frac{\pi_t^i}{X_t}$  for each stock  $i$ .

**Definition 15.1.**

A trading strategy  $\pi = (\pi_t)_{t \in [0, T]}$  is an  $n$ -dimensional  $\mathcal{F}^S$ -adapted, measurable process which satisfies  $\int_0^T \|\pi_t\|^2 dt < \infty$ ,  $\int_0^T |\mu_t^\top \pi_t| dt < \infty$ . For initial capital  $x > 0$  the corresponding wealth process  $X^\pi = (X_t^\pi)_{t \in [0, T]}$  is defined by

$$dX_t^\pi = \pi_t^\top (\mu_t dt + \sigma dW_t), \quad X_0 = x. \quad (15.7)$$

A trading strategy  $\pi$  is called admissible, if  $P(X_t^\pi > 0, \text{ for all } t \in [0, T]) = 1$ . We shall denote by  $\mathcal{A}_p$  the set of all admissible strategies.

Itô's rule implies that for any trading strategy  $\pi$ , the wealth  $X_t^\pi$  has the form

$$dX_t^\pi = \pi_t^\top \sigma d\widetilde{W}_t, \quad t \in [0, T]. \quad (15.8)$$

**The optimization problem**

Let  $U$  be a utility function given in Definition 4.2 Chapter I. Our objective in this chapter is to analyse the behavior of a portfolio manager when he has only partial information. We consider a portfolio manager who wants to maximize the expected utility from the terminal wealth, and who is confronted with a risk measured by a constraint of the type

$$\text{EUL}(G) := \mathbb{E}[G^-] = \mathbb{E}[(U(X_T^\pi) - U(q))^-] \leq \varepsilon, \quad (15.9)$$

where  $\varepsilon$  is a given bound for the Expected Utility Loss and  $q$  is a given real number representing the shortfall level. This case is related to a fixed non-random benchmark  $Q = q$  and a typical choice is

$$q = e^{\delta T} X_T^{\pi \equiv 0} = x e^{\delta T}, \quad \delta \in \mathbb{R}.$$

We refer to Section 5 for a detailed information about the benchmark. The special feature in this chapter is that the portfolio manager's decisions have to be adapted to the augmented filtration  $\mathcal{F}^S$  generated by the stock price process. In the case of partial information, we call the solutions to the resulting constrained dynamic optimization problem

$$\max_{\pi \in \mathcal{A}_p} \mathbb{E}[U(X_T^\pi)] \quad (15.10)$$

$$\text{subject to } \mathbb{E}[(U(X_T^\pi) - U(q))^-] \leq \varepsilon$$

EUL-optimal. The corresponding EUL-optimal wealth at time  $t \in [0, T]$  is denoted by  $X_t^{EUL}$  and the EUL-optimal strategy by  $\pi_t^{EUL}$ .

In economies with partial information the process  $Z$  can not be a driving process

since its not adapted to the filtration  $\mathcal{F}^S$ . A natural idea to find such a process in economies where the portfolio manager has only partial information, is to project the density process  $Z$  to the filtration  $\mathcal{F}^S$  and to consider instead the conditional density process  $\zeta = (\zeta_t)_{t \in [0, T]}$ ,

$$\zeta_t = \mathbb{E}[Z_t | \mathcal{F}_t^S]. \quad (15.11)$$

In a similar way to the case of full information studied in Chapter II, the portfolio optimization problem under partial information can be solved using the martingale duality approach introduced by Karatzas, Lehoczky, Shreve [38], or Cox, Huang [11]. The basic difference is that instead of  $Z$  we have to use the  $\mathcal{F}^S$ -adapted process  $\zeta$ . Define

$$\mathcal{B}_p(x) := \{\xi > 0 : \xi \text{ is } \mathcal{F}_T^S \text{-measurable and } \mathbb{E}[\zeta_T \xi] \leq x\}.$$

The corresponding static problem reads as

$$\begin{aligned} & \max_{\xi \in \mathcal{B}_p(x)} \mathbb{E}[U(\xi)] \\ & \text{subject to } \mathbb{E}[(U(\xi) - U(q))^-] \leq \varepsilon. \end{aligned} \quad (15.12)$$

The following proposition characterizes the EUL-optimal terminal  $X_T^{EUL}$ .

**Proposition 15.2.** Let  $q > 0$  be a fixed benchmark. Moreover, let for  $y_1, y_2 > 0$  be defined

$$\begin{aligned} \underline{h} = \underline{h}(y_1) & := \frac{1}{y_1} U'(q), \\ \bar{h} = \bar{h}(y_1, y_2) & := \frac{1 + y_2}{y_1} U'(q) = (1 + y_2) \underline{h} \quad \text{and} \\ f(z) = f(z; y_1, y_2) & := \begin{cases} I(y_1 z) & \text{if } z < \underline{h}, \\ q & \text{if } \underline{h} \leq z < \bar{h}, \\ I(\frac{y_1}{1+y_2} z) & \text{if } \bar{h} \leq z, \end{cases} \end{aligned}$$

for  $z > 0$ . Finally, let the initial capital  $x > 0$  and the bound for the risk constraint  $\varepsilon > 0$  be such that there are strictly positive and unique solutions  $y_1, y_2$  of the following system of equations

$$\begin{aligned} \mathbb{E}[\zeta_T f(\zeta_T; y_1, y_2)] & = x \\ \mathbb{E}[(U(f(\zeta_T; y_1, y_2)) - U(q))^-] & = \varepsilon. \end{aligned}$$

Then the EUL-optimal terminal wealth is

$$X_T^{EUL} = f(\zeta_T) = f(\zeta_T; y_1, y_2). \quad (15.13)$$

**Proof.** The proof is quite similar to the proof of Proposition 8.1 of Chapter II. We just need to consider the  $\mathcal{F}^S$ -adapted conditional density process  $\zeta$  instead of  $Z$ . ■

**Remark 15.3.** Under full information the optimal terminal wealth  $X_T^{EUL}$  given in Proposition 8.1 of Chapter II was expressed in terms of  $H_T$  which is not  $\mathcal{F}_T^S$ -measurable. Hence the optimal terminal wealth given in Proposition 15.2 under partial information, is a sub-optimal value under full information

$$\max_{\pi \in \mathcal{A}_p} \mathbb{E}[U(X_T^\pi)] \leq \max_{\pi \in \mathcal{A}} \mathbb{E}[U(X_T^\pi)].$$

## 16 HMM filtering results

In this section we present results of HMM filtering which we will use to determine the optimal trading strategy. In particular, we have to find a good estimator for the drift process given the observation  $R$ . Equations (14.2) and (14.3) show that we are in the classical situation of HMM filtering with signal  $Y$  and observation  $R$ , see [21]. We determine the filter  $\mathbb{E}[Y_t | \mathcal{F}_t^S]$  for  $Y_t$ , which is an  $L^2$ -optimal estimator.

### Definition 16.1.

The filter  $\eta = (\eta_t)_{t \in [0, T]}$  for  $Y$  and the unnormalized filter  $\mathcal{E} = (\mathcal{E}_t)_{t \in [0, T]}$  for  $Y$  are defined by

$$\eta_t = \mathbb{E}[Y_t | \mathcal{F}_t^S] \quad \text{and} \quad \mathcal{E}_t = \tilde{\mathbb{E}}[Z_T^{-1} Y_t | \mathcal{F}_t^S].$$

Bayes's law implies

$$\mathcal{E}_t = \frac{\mathbb{E}[Z_T Z_T^{-1} Y_t | \mathcal{F}_t^S]}{\mathbb{E}[Z_T | \mathcal{F}_t^S]} = \frac{\mathbb{E}[Y_t | \mathcal{F}_t^S]}{\zeta_t} = \zeta_t^{-1} \eta_t. \quad (16.14)$$

Moreover, the following assertions are fulfilled:  $\mathbf{1}_d^\top \eta_t = 1$ ,  $\mathbf{1}_d^\top \mathcal{E}_t = \zeta_t^{-1}$ ,  $\eta_t^k \in [0, 1]$ , and  $\eta_t^k \geq 0$ , with  $k = 1, \dots, d$ .

The optimal strategy will be expressed in terms of the unnormalized filter  $\mathcal{E}$  and its Malliavin derivatives. So we need a characterization of the unnormalized filter  $\mathcal{E}$  which is given by the following theorem in terms of a linear stochastic differential equation.

### Theorem 16.2.

The unnormalized filter  $\mathcal{E}$  is given by

$$\mathcal{E}_t = \mathbb{E}[Y_0] + \int_0^t Q^\top \mathcal{E}_s ds + \int_0^t \text{Diag}(\mathcal{E}_s) B^\top (\sigma \sigma^\top)^{-1} dR_s, \quad t \in [0, T].$$

**Remark 16.3.** This theorem proved by Elliott (see Theorem 4 in [21]), is extended by Sass and Haussman [66] to non trivial volatility  $\sigma$  and stochastic  $B$  where

$B = (B_t)_{t \in [0, T]}$  is an  $\mathbb{R}^{n \times d}$ -valued,  $\mathcal{F}^S$ -progressively measurable process satisfying  $\int_0^T \|B\| dt < \infty$ . Since  $R$  and  $S$  generate the same continuous filtration, Theorem 16.2 allows to use the filter with respect to  $\mathcal{F}^S$ . Moreover, this theorem can be established as in [21] via the filter equation for  $\eta$ . For a direct proof of Theorem 16.2, we refer to Theorem 4 in [21].

Formula(15.13) for the optimal terminal wealth involves the random variable  $\zeta_T$ . A complete representation of the conditional state density process  $\zeta$  is given in the following corollary.

**Corollary 16.4.**

$\zeta$  and  $\zeta^{-1} = (\zeta_t^{-1})_{t \in [0, T]}$  are continuous  $\mathcal{F}^S$ -martingales with respect to  $P$  and  $\tilde{P}$ , respectively. Furthermore,  $\zeta_t^{-1} = \tilde{\mathbb{E}}[Z_t^{-1} | \mathcal{F}_t^S]$

$$\zeta_t^{-1} = 1 + \int_0^t (B\mathcal{E}_s)^\top (\sigma\sigma^\top)^{-1} dR_s, \quad t \in [0, T].$$

**Proof.** Lemma 14.3 implies the martingale property. The representation of  $\zeta^{-1}$  is a consequence of Theorem 16.2 using the fact that  $\zeta_t^{-1} = \mathbf{1}_d^\top \mathcal{E}_t$  and  $Q\mathbf{1}_d = 0$ . ■

## 17 Malliavin derivative

In the particular contexts of utility maximization from investment, the generalized Clark's formula leads directly to representations of optimal portfolios for this task. However, the expressions obtained are fairly general and hard to manipulate further, as they involve functional derivatives of the Malliavin type, stochastic integrals, and conditional expectations under an auxiliary probability measure. When specialized to the case of logarithmic utility, or to a financial market with quite simple coefficients, the Clark's formula leads to very explicit expressions for the optimal portfolios in feedback form on the current level of wealth. In this section we introduce the gradient operator  $D$  acting on the subset of the class of functionals of  $\{\widetilde{W}_t, 0 \leq t \leq T\}$  called  $\mathbb{D}_{1,1}$ . For the exact definitions of the space  $\mathbb{D}_{1,1}$  and the operator  $D$  see Appendix G, and for detailed results we refer to Ocone and Karatzas [61], Nualart and Pardoux [59]. Because it is a crucial tool and a key result in obtaining the expressions of our optimal portfolios, we start this section by stating the generalized version of Clark's formula which allows for elements of  $\mathbb{D}_{1,1}$  to be represented in form of stochastic integrals. For the proof, see (Karatzas et al., [40]).

**Theorem 17.1.**

For every  $F \in \mathbb{D}_{1,1}$  we have

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[(D_t F)^\top | \mathcal{F}_t] dW_t. \quad (17.15)$$

**Remark 17.2.**

i) From (17.15) it follows also that

$$F = \mathbb{E}[F] + \int_0^t \mathbb{E}[(D_s F)^\top | \mathcal{F}_s] dW_s, \quad 0 \leq t \leq T. \quad (17.16)$$

ii) We need the extension of Clark's formula in (17.15) from  $\mathbb{D}_{2,1}$  to  $\mathbb{D}_{1,1}$  in order to represent the  $\tilde{P}$ -martingales  $\tilde{\mathbb{E}}[F | \mathcal{F}_t]$  as stochastic integrals with respect to the process  $\tilde{W}$  of 14.5 using Bayes's formula  $\tilde{\mathbb{E}}[F | \mathcal{F}_t] = (Z_t^{-1})\mathbb{E}[F Z_T | \mathcal{F}_t]$  and then applying the Clark's formula to  $F Z_T$ , here the process  $Z$  is given by (14.4) and  $d\tilde{P} = Z_T dP$ . To deal with the Sobolev's space  $\mathbb{D}_{1,1}$  in Theorem 17.1 instead of  $\mathbb{D}_{2,1}$  is therefore useful for avoiding unnecessarily restrictive moment bounds on the random variables  $F$  and  $DF$ . For example, if  $F \in L^2(\tilde{P})$ , it does not follow that  $F Z_T \in L^2(P)$ . However,

$$\mathbb{E}[|F Z_T|^p] = \tilde{\mathbb{E}}[|F|^p Z_T^{p-1}] \leq \left( \tilde{\mathbb{E}}[F^2] \right)^{\frac{p}{2}} \left( \tilde{\mathbb{E}}[Z_T^{2(p-1)/(2-p)}] \right)^{1-(p/2)} < \infty$$

if  $1 \leq p < 2$ .

Let us now provide some preliminary results which are proved by Sass and Hausman in [65] and in which we use the results of Appendix G for the Brownian motion  $\tilde{W}$  and its corresponding measure  $\tilde{P}$ .

**Proposition 17.3.** It holds  $\mathcal{E}_t^k \in \mathbb{D}, k = 1, \dots, d$ . Further, for every  $u \in [0, T]$  we have  $D_t \mathcal{E}_u = 0$  for  $t \in (u, T]$ , and

$$D_t \mathcal{E}_u = \sigma^{-1} B \text{Diag}(\mathcal{E}_t) + \int_t^u (D_t \mathcal{E}_s) Q ds + \int_t^u (D_t \mathcal{E}_s) \text{Diag}(B^\top (\sigma \sigma^\top)^{-1}) dR_s,$$

for all  $t \in [0, u]$ .

**Proof.** The result follows by applying Proposition G.2 of Appendix G to the  $\mathbb{R}^d$ -valued process  $(\mathcal{E}_t^1, \dots, \mathcal{E}_t^d)$ . ■

**Remark 17.4.** For  $n = 1$ , we set  $B = b$ . Then we have

$$D_t \mathcal{E}_u = \frac{b}{\sigma} \mathcal{E}_t + \int_t^u Q(D_t \mathcal{E}_s) ds + \frac{b}{\sigma^2} \int_t^u D_t \mathcal{E}_s dR_s.$$

Therefore, the process  $(D_t \mathcal{E}_u)_{u \in [t, T]}$  satisfies the same stochastic differential equation as  $(\mathcal{E}_u)_{u \in [t, T]}$  with initial value  $\frac{b}{\sigma} \mathcal{E}_t$ .

As a consequence of Lemma 14.3, and using the fact that  $\zeta_T^{-1} = \tilde{\mathbb{E}}[Z_T^{-1} | \mathcal{F}_t^S]$  we have the following lemma.

**Lemma 17.5.** *For all  $p \in [1, \infty)$  we have*

$$\tilde{\mathbb{E}}[|\zeta_T|^p] < \infty \quad \text{and} \quad \tilde{\mathbb{E}}[|\zeta_T^{-1}|^p] < \infty.$$

The chain rule given in G.3 of Appendix G, is instrumental in obtaining the following lemma.

**Lemma 17.6.**  *$\zeta_T$  and  $\zeta_T^{-1}$  lie in  $\mathbb{D}$  and for  $t \in [0, T]$*

$$D_t \zeta_T^{-1} = \sigma^{-1} B \mathcal{E}_t + \int_t^T (D_t \mathcal{E}_s) B^\top dR_s, \quad (17.17)$$

$$D_t \zeta_T = -\zeta_T^2 D_t \zeta_T^{-1}. \quad (17.18)$$

**Proof.** We have  $\zeta_T^{-1} = \mathbf{1}_d \mathcal{E}_T \in \mathbb{D}$ , this implies that  $D_t \zeta_T^{-1} = (D_t \mathcal{E}_t) \mathbf{1}_d$ . Using the fact that  $Q \mathbf{1}_d = 0$ , the representation (17.17) follows from Proposition 17.3.

For  $g : (0, \infty) \mapsto \mathbb{R}$ ,  $g(x) = \frac{1}{x}$ , we have  $\zeta_T = g(\zeta_T^{-1})$ . Lemma 17.5 implies that  $g(\zeta_T^{-1}) = \zeta_T \in \cap L^p(\tilde{\mathcal{P}})$ , and  $g'(\zeta_T^{-1}) = -\zeta_T^2 \in \cap L^p(\tilde{\mathcal{P}})$  and the representation (17.18) follows from the chain rule in Proposition G.3 in Appendix G. ■

In this section we prove Theorem 17.7, where we use approximation arguments to show that the optimal terminal wealth (15.13) has a Malliavin derivative. We recall that the optimal terminal wealth is given as  $X_T^{EUL} = f(\zeta_T) = f(\zeta_T; y_1, y_2)$  where

$$f(z) = f(z; y_1, y_2) := \begin{cases} I(y_1 z) & \text{if } z < \underline{h}, \\ q & \text{if } \underline{h} \leq z < \bar{h}, \\ I(\frac{y_1}{1+y_2} z) & \text{if } \bar{h} \leq z. \end{cases}$$

**Theorem 17.7.**

*If strictly positive and unique solutions  $y_1, y_2$  of*

$$\mathbb{E}[\zeta_T f(\zeta_T; y_1, y_2)] = x_0, \quad \mathbb{E}[(U(f(\zeta_T; y_1, y_2)) - U(q x_0))^-] = \varepsilon \quad (17.19)$$

*exist. Further, if  $I'(y_1 \zeta_T) \in L^q(\tilde{\mathcal{P}})$  for some  $q > 1$ , then the optimal terminal wealth  $X_T^{EUL} = f(\zeta_T)$  given in (15.13) lies in  $\mathbb{D}_{1,1}$  and for  $t \in [0, T]$*

$$D_t f(\zeta_T) = f'(\zeta_T) D_t \zeta_T, \quad (17.20)$$

where

$$f'(z) = f'(z; y_1, y_2) := \begin{cases} y_1 I'(y_1 z) & \text{if } z < \underline{h}, \\ 0 & \text{if } \underline{h} \leq z < \bar{h}, \\ \frac{y_1}{1+y_2} I'(\frac{y_1}{1+y_2} z) & \text{if } \bar{h} \leq z, \end{cases}$$

$y_1, y_2, \underline{h}$  and  $\bar{h}$  are as in Proposition 15.2.

**Proof.** Let us denote by  $\|\cdot\|_p$  and  $\|\cdot\|_{L^2}$  the norms in  $L^p = L^p(\Omega, \tilde{P})$  and  $L^2((0, T))$ , respectively. Obviously we can find  $f_n \in C^1((0, \infty))$ ,  $n \in \mathbb{N}$  such that

$$0 < f_n \leq f, \quad f_n \rightarrow f \quad (n \rightarrow \infty) \quad (17.21)$$

and

$$|f'_n(z)|^q < y_1^q |I'(y_1 z)|^q + \left(\frac{y_1}{1+y_2}\right)^q \left|I'\left(\frac{y_1 h}{1+y_2}\right)\right|^q, \quad f'_n \rightarrow f' \quad (n \rightarrow \infty), \quad (17.22)$$

the convergence meant pointwise in both cases.

The condition  $E[\zeta_T f(\zeta_T; y_1, y_2)] = x_0$  implies  $f(\zeta_T) \in L^1$ , hence by (17.21)  $f_n(\zeta_T) \in L^1$ ,  $n \in \mathbb{N}$ . So we get by the Dominated Convergence Theorem

$$\|f_n(\zeta_T) - f(\zeta_T)\|_1 \rightarrow 0 \quad (n \rightarrow \infty). \quad (17.23)$$

From the condition in the theorem we have  $I'(y_1 \zeta_T) \in L^q$ , hence  $f'(\zeta_T) \in L^q$  and by (17.22) also  $f'_n(\zeta_T) \in L^q$ ,  $n \in \mathbb{N}$ . So

$$\|f'_n(\zeta_T) - f'(\zeta_T)\|_q \rightarrow 0 \quad (n \rightarrow \infty). \quad (17.24)$$

By Lemma 17.6 we know that  $\zeta_T \in \mathbb{D} = \bigcap_{p>1} \mathbb{D}_{p,1}$  with the derivative as given in Equations (17.17) and (17.18). So the conditions of the chain rule G.4 given in Appendix G are fulfilled. Therefore

$$Df_n(\zeta_T) = f'_n(\zeta_T) D\zeta_T.$$

Further  $\zeta_T \in \mathbb{D}$  implies  $\|D\zeta_T\|_{L^2} \|p\|_p < \infty$  for all  $p > 1$ , in particular for  $p = \frac{q}{q-1}$ . Using Hölder's Inequality and (17.24) we obtain

$$\begin{aligned} \|Df_n(\zeta_T) - f'(\zeta_T) D\zeta_T\|_{L^2} \|1\|_1 &= \|f'_n(\zeta_T) - f'(\zeta_T)\| \|D\zeta_T\|_{L^2} \|1\|_1 \\ &\leq \|f'_n(\zeta_T) - f'(\zeta_T)\|_q \|D\zeta_T\|_{L^2} \|\frac{q}{q-1}\|_{\frac{q}{q-1}} \\ &\rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Since  $D$  is a closed operator on  $\mathbb{D}_{1,1}$  it follows in combination with (17.23) that  $f(\zeta_T) \in \mathbb{D}_{1,1}$  with  $Df(\zeta_T) = f'(\zeta_T) D\zeta_T$ . Actually we have shown that  $f_n(\zeta_T) \rightarrow f(\zeta_T)$  in the norm  $\|F\|_{1,1} = \|F\|_1 + \|DF\|_{L^2} \|1\|_1$  under which the space  $\mathbb{D}_{1,1}$  is closed. ■

## 18 Optimal Trading Strategies

In Proposition 15.2 we derive the form of the optimal terminal wealth as  $X_T^{EUL} = f(\zeta_T) = f(\zeta_T; y_1, y_2)$  by proceeding similarly to the case of full information studied in Chapter II. Then we can proceed as in [65, Section 4] to find the optimal trading strategy.

**Theorem 18.1.**

Let the assumptions of Theorem 17.7 be fulfilled. Then the optimal strategy of the problem (15.10) is given by

$$\pi_t^{EUL} = (\sigma\sigma^\top)^{-1}\tilde{\mathbb{E}}[f'(\zeta_T)\sigma D_t\zeta_T | \mathcal{F}_t^S], \quad 0 \leq t \leq T, \quad (18.1)$$

where

$$\begin{aligned} D_t\zeta_T &= -\zeta_T^2 \sigma^{-1} \left( B \mathcal{E}_t + \int_t^T (\sigma D_t\mathcal{E}_s) B^\top (\sigma\sigma^\top)^{-1} dR_s \right), \\ \sigma D_t\mathcal{E}_s &= B \text{Diag}(\mathcal{E}_t) + \int_t^s (\sigma D_t\mathcal{E}_u) Q du + \int_t^s (\sigma D_t\mathcal{E}_u) \text{Diag} (B^\top (\sigma\sigma^\top)^{-1}) dR_u. \end{aligned}$$

**Proof.** The form of the optimal terminal wealth  $X_T^{EUL} = f(\zeta_T; y_1, y_2)$  follows from Proposition 15.2. By Theorem 17.7 this optimal terminal wealth has a Mallivain derivative. On the other hand, Lemma 17.6 provides  $\zeta_T \in \mathbb{D}$ . The conditions of Proposition G.4 are fulfilled for  $\zeta$  and  $f$ . Therefore,  $X_T^{EUL} = f(\zeta_T; y_1, y_2) \in \mathbb{D}_{1,1}$  with

$$D_t X_T^{EUL} = f'(\zeta_T) D_t \zeta_T. \quad (18.2)$$

Using Clark's Formula in Theorem 17.1, we get for the terminal wealth

$$X_T^{EUL} = x + \int_0^T \tilde{\mathbb{E}}[(D_t X_T^{EUL})^\top | \mathcal{F}_t^S] d\tilde{W}_t. \quad (18.3)$$

Equating (15.8) with (18.3) we obtain

$$\pi_t^{EUL} = (\sigma\sigma^\top)^{-1}\tilde{\mathbb{E}}[f'(\zeta_T)\sigma D_t\zeta_T | \mathcal{F}_t^S], \quad 0 \leq t \leq T,$$

where

$$\begin{aligned} D_t\zeta_T &= -\zeta_T^2 \sigma^{-1} \left( B \mathcal{E}_t + \int_t^T (\sigma D_t\mathcal{E}_s) B^\top (\sigma\sigma^\top)^{-1} dR_s \right), \\ \sigma D_t\mathcal{E}_s &= B \text{Diag}(\mathcal{E}_t) + \int_t^s (\sigma D_t\mathcal{E}_u) Q du + \int_t^s (\sigma D_t\mathcal{E}_u) \text{Diag} (B^\top (\sigma\sigma^\top)^{-1}) dR_u, \end{aligned}$$

are given by Lemma 17.6 and Proposition 17.3, respectively. ■

**Remark 18.2.** We point out that the optimal strategy given in Theorem 18.1 is given in terms of the processes  $\zeta$ ,  $\mathcal{E}$  and  $D\mathcal{E}$  which are all  $\mathcal{F}^S$ -adapted and due to Theorem 16.2, Corollary 16.4 and Proposition 17.3, their dynamics can be expressed in terms of the return process  $R$  and the parameters  $B$ ,  $Q$ ,  $\sigma\sigma^\top$ .

In what follows we shall derive a convenient representation of the optimal trading strategy for the important utility functions given in Chapter I Equation (3.5): The *logarithmic utility*  $U_1(x) = \ln(x)$ , and the *power utility*  $U_\gamma(x) = x^{1-\gamma}/1-\gamma$  for  $\gamma > 0$ ,  $\gamma \neq 1$ . These cover the whole range of risk behavior. Here  $\gamma$  is the *Arrow-Pratt*



*index* of risk aversion. So suppose  $\gamma$  is fixed and assume that the parameters  $y_1, y_2$  in (17.19) exist. Denoting the inverse of the derivative  $U'$  by  $I$  we have

$$U'(x) = x^{-\gamma}, \quad I(y) = y^{-\frac{1}{\gamma}}, \quad I'(y) = -\frac{1}{\gamma}y^{-\frac{1}{\gamma}-1},$$

in particular  $I(xy) = I(x)I(y)$ . Then we can write for all  $t \in [0, T]$

$$X_T^{EUL} = I(y_1)I(\zeta_t)G_{t,T}, \quad (18.4)$$

where  $G_{t,T} = g(\zeta_t, \zeta_{t,T})$ ,  $\zeta_{t,T} = \zeta_T/\zeta_t$ , and

$$g(z_1, z_2) = \begin{cases} I(z_2), & z_2 \in (0, \frac{h}{z_1}], \\ I(\frac{h}{z_1}), & z_2 \in (\frac{h}{z_1}, \frac{h}{z_1}), \\ I(\frac{1}{1+y_2} z_2), & z_2 \in [\frac{h}{z_1}, \infty). \end{cases}$$

In particular  $X_T^{EUL} = I(y_1)G_{0,T}$ . So from condition (17.19) we get

$$I(y_1) = \frac{x}{\tilde{\mathbb{E}}[G_{0,T}]}. \quad (18.5)$$

In combination with (18.4) this implies

$$X_t^{EUL} = \tilde{\mathbb{E}}[X_T^{EUL} | \mathcal{F}_t^S] = I(y_1\zeta_t)\tilde{\mathbb{E}}[G_{t,T} | \mathcal{F}_t^S] = \frac{xI(\zeta_t)\tilde{\mathbb{E}}[G_{t,T} | \mathcal{E}_t]}{\tilde{\mathbb{E}}[G_{0,T}]},$$

where we used that  $\mathcal{E}_t$  is a sufficient statistic to compute  $G_{t,T}$  as can be seen from its definition observing that  $\zeta_t = 1/\mathbf{1}_d^\top \mathcal{E}_t$ . Thus by (18.4), (18.5)

$$X_T^{EUL} = \frac{xI(\zeta_t)G_{t,T}}{\tilde{\mathbb{E}}[G_{0,T}]} = \frac{X_t^{EUL}G_{t,T}}{\tilde{\mathbb{E}}[G_{t,T} | \mathcal{E}_t]}. \quad (18.6)$$

By this representation and the representation of  $\pi_t^{EUL}$  in Theorem 18.1 we get the following proposition.

**Proposition 18.3.** For all  $\gamma > 0$  and for all  $0 \leq t \leq T$

$$\begin{aligned} \theta_t^{EUL} &= \frac{\pi_t^{EUL}}{X_t^{EUL}} \\ &= \frac{(\sigma\sigma^\top)^{-1}}{\gamma} \left( \tilde{\mathbb{E}}[C_{t,T} | X_t^{EUL}, \mathcal{E}_t] B \eta_t + \tilde{\mathbb{E}} \left[ C_{t,T} \int_t^T (\sigma D_t \mathcal{E}_{t,s}) B^\top (\sigma\sigma^\top)^{-1} dR_s \mid X_t^{EUL}, \mathcal{E}_t \right] \right) \end{aligned} \quad (18.7)$$

where  $\mathcal{E}_{t,s} = \mathcal{E}_s \zeta_t$ ,  $C_{t,T} = \frac{\zeta_{t,T} G_{t,T}}{\tilde{\mathbb{E}}[G_{t,T} | \mathcal{E}_t]} - \zeta_{t,T} q_t \mathbf{1}_{\{h\zeta_t^{-1} < \zeta_{t,T} < \bar{h}\zeta_t^{-1}\}}$ , and  $q_t = \frac{q}{X_t^{EUL}}$ .

**Proof.** Using the definition of  $q_t$  and (18.6)

$$\begin{aligned}
f'(\zeta_T) &= -\frac{1}{\gamma} \zeta_T^{-1} (f(\zeta_T) - q \mathbf{1}_{\{\zeta_T \in (\underline{h}, \bar{h})\}}) \\
&= -\frac{X_t^{EUL}}{\gamma} \zeta_T^{-1} \left( \frac{X_T^{EUL}}{X_t^{EUL}} - q_t \mathbf{1}_{\{\zeta_T \in (\underline{h}, \bar{h})\}} \right) \\
&= -\frac{X_t^{EUL}}{\gamma} \zeta_T^{-1} \left( \frac{G_{t,T}}{\tilde{\mathbb{E}}[G_{t,T} | \mathcal{F}_t^S]} - q_t \mathbf{1}_{\{\zeta_T \in (\underline{h}, \bar{h})\}} \right) \\
&= -\frac{X_t^{EUL}}{\gamma} \zeta_t \zeta_T^{-2} C_{t,T}.
\end{aligned}$$

Using the representation of  $D_t \zeta_T$  in Theorem 18.1 we get

$$\begin{aligned}
f'(\zeta_T) \sigma D_t \zeta_T &= \frac{X_t^{EUL}}{\gamma} \zeta_t C_{t,T} \left( B \mathcal{E}_t + \int_t^T (\sigma D_t \mathcal{E}_s) B^\top (\sigma \sigma^\top)^{-1} dR_s \right) \\
&= \frac{X_t^{EUL}}{\gamma} C_{t,T} \left( B \eta_t + \int_t^T (\sigma D_t \mathcal{E}_{t,s}) B^\top (\sigma \sigma^\top)^{-1} dR_s \right),
\end{aligned}$$

where we used  $\mathcal{E}_t \zeta_t = \eta_t$ . ■

**Remark 18.4.** (i) We obtain in a market model with partial information the optimal strategy (Proposition 18.3) which is expressed in terms of the unnormalized filter  $\mathcal{E}$ , its Malliavin derivative  $D\mathcal{E}$ , the state price density  $\zeta$ , and the parameters of the model. These quantities are all  $\mathcal{F}^S$ -adapted. Moreover, the filters and derivatives can be approximated very well using Euler scheme because of the linear structure of the equations in Theorem 16.2, Corollary 16.4 and in Proposition 17.3. So  $(\mathcal{E}_t, X_t^{EUL})$  is a sufficient statistic for the calculation of  $\pi_t^{EUL}$ , and the Markovian nature of this statistic is very helpful for Monte Carlo simulations we need to compute the second term of  $\pi_t^{EUL}$  see [30].

(ii) The correction factor  $C_{t,T}$  replaces  $(\tilde{\mathbb{E}}[\zeta_{t,T}^{-\frac{1}{\gamma}} | \mathcal{F}_t^S])^{-1} \zeta_{t,T}^{\frac{\gamma-1}{\gamma}}$  given by Sass and Haussmann [65, Proposition 4.10] in the case of unconstrained problem. In particular this shows that the solutions coincide for  $q \rightarrow 0$ . For constant  $\mu$  we get the solution in [30, Section 4.1], see Corollary 18.5 below.

(iii) Given  $X_t^{EUL}$  and  $q_t = \frac{q}{X_t^{EUL}}$  it is optimal to continue trading in  $t$  with the same strategy as when starting at 0 with  $x$ . If we would start at  $t$  with a different wealth level the strategy (in terms of the risky fraction) would no longer be optimal. This applies also to the full information case, but it is different from the unconstrained case in e.g. [65] where the optimal risky fraction is independent of the current wealth.

Let us examine the particular case of a constant drift which leads to the situation of full information studied in Chapter II Section 8. We restrict for the sake of simplicity to the 1-dimensional case. Obviously in this case we have  $B = \mu$  with  $Y = 1$ , and the  $P$ -augmented filtration  $\mathcal{F}^S$  generated by the stock price, coincides with the  $P$ -augmented natural filtration  $\mathcal{F}$  of the Brownian motion  $W$ . It turns out that the conditional state price density  $\zeta$  coincides with the state price density  $Z$ , further it results  $\mathcal{E}_s = Z_s^{-1} = e^{\frac{1}{2}\kappa^2 s + \kappa W_s}$  for the unnormalized filter, where  $\kappa = \frac{\mu}{\sigma}$ .

**Corollary 18.5.**

For a constant drift the optimal strategy obtained in the case of partial information coincides with the optimal strategy obtained in the case of full information. More precisely, it results

$$\begin{aligned} \frac{\pi_t^{EUL}}{X_t^{EUL}} &= \theta^N \left( 1 - \frac{qx}{X_t^{EUL}} \left[ \Phi(-d_2(\underline{h}, Z_t, t)) - \Phi(-d_2(\bar{h}, Z_t, t)) \right] \right) \\ &= \theta_t^{EUL}, \end{aligned}$$

where

$$\begin{aligned} d_2(u, z, t) &:= \frac{\ln \frac{u}{z} + (-\frac{\kappa^2}{2})(T-t)}{\kappa\sqrt{T-t}}, \\ d_1(u, z, t) &:= d_2(u, z, t) + \frac{1}{\gamma}\kappa\sqrt{T-t}, \\ \theta^N &= \frac{\kappa}{\gamma\sigma} = \frac{\mu}{\gamma\sigma^2}. \end{aligned}$$

Thereby,  $\theta^N$  denotes the normal strategy of the non-risk manager, and  $\kappa = \frac{\mu}{\sigma}$  is the market price of risk.

**Proof.** Computing the Malliavin derivative of  $\mathcal{E}_s$  we get using the usual chain rule

$$\begin{aligned} D_t \mathcal{E}_s &= D_t(e^{\frac{1}{2}\kappa^2 s + \kappa W_s}) = e^{\frac{1}{2}\kappa^2 s} D_t(e^{\kappa W_s}) \\ &= e^{\frac{1}{2}\kappa^2 s} \kappa e^{\kappa W_s} D_t W_s = \kappa Z_s^{-1} \mathbf{1}_{[0,t]}(s). \end{aligned}$$

On the other hand, the dynamics of  $Z^{-1}$  is given by

$$dZ_t^{-1} = \kappa Z_t^{-1} d\widetilde{W}_t. \quad (18.8)$$

Computing the stochastic integral with respect to the return process we get

$$\begin{aligned} \int_t^T (\sigma D_t \mathcal{E}_{t,s}) B^\top (\sigma \sigma^\top)^{-1} dR_s &= \int_t^T \sigma (\zeta_t \sigma^{-1} \mu Z_s^{-1}) \mu \sigma^{-2} dR_s \\ &= \int_t^T \zeta_t \sigma^{-2} \mu^2 Z_s^{-1} \sigma d\widetilde{W}_s = \kappa^2 \sigma \zeta_t \int_t^T Z_s^{-1} d\widetilde{W}_s \\ &= \kappa \sigma \zeta_t \int_t^T \kappa Z_s^{-1} d\widetilde{W}_s = \kappa \sigma \zeta_t (Z_T^{-1} - Z_t^{-1}), \end{aligned}$$

where the last equality follows from Equation (18.8). It results

$$\begin{aligned} \widetilde{\mathbb{E}} \left[ C_{t,T} \int_t^T (\sigma D_t \mathcal{E}_{t,s}) B^\top (\sigma \sigma^\top)^{-1} dR_s \mid \mathcal{E}_t \right] &= \widetilde{\mathbb{E}} \left[ \zeta_t C_{t,T} \kappa \sigma (Z_T^{-1} - Z_t^{-1}) \mid \mathcal{F}_t \right] \\ &= \mu \widetilde{\mathbb{E}} \left[ (C_{t,T} Z_T^{-1} Z_t - C_{t,T}) \mid \mathcal{F}_t \right]. \end{aligned}$$

Substituting in Formula (18.7) and using the representation of  $C_{t,T}$  in Proposition 18.3, we get

$$\begin{aligned}
\frac{\pi_t^{EUL}}{X_t^{EUL}} &= \frac{\sigma^{-2}\mu}{\gamma} \tilde{\mathbb{E}} \left[ C_{t,T} Z_T^{-1} Z_t \mid \mathcal{F}_t \right] \\
&= \frac{\sigma^{-2}\mu}{\gamma} \tilde{\mathbb{E}} \left[ \frac{G_{t,T}}{\tilde{\mathbb{E}}[G_{t,T} \mid \mathcal{E}_t]} - q_t \mathbf{1}_{\{\underline{h}\zeta_t^{-1} < \zeta_{t,T} < \bar{h}\zeta_t^{-1}\}} \mid \mathcal{F}_t \right] \\
&= \frac{\sigma^{-2}\mu}{\gamma} \left\{ 1 - \tilde{\mathbb{E}} \left[ \frac{qx}{X_t^{EUL}} \mathbf{1}_{\{\underline{h} < Z_T < \bar{h}\}} \mid \mathcal{F}_t \right] \right\} \\
&= \frac{\sigma^{-2}\mu}{\gamma} \left\{ 1 - \frac{qx}{X_t^{EUL}} Z_t^{-1} \mathbb{E} \left[ Z_T \mathbf{1}_{\{\underline{h} < Z_T < \bar{h}\}} \mid \mathcal{F}_t \right] \right\} \\
&= \theta^N \left( 1 - \frac{qx}{X_t^{EUL}} \left[ \Phi(-d_2(\underline{h}, Z_t, t)) - \Phi(-d_2(\bar{h}, Z_t, t)) \right] \right) \\
&= \theta_t^{EUL},
\end{aligned}$$

where  $\theta_t^{EUL}$  is the fraction of wealth we have obtained in the case of full information studied in Chapter II, see Proposition 8.4 (ii). Here,  $\theta^N = \frac{\kappa}{\gamma\sigma} = \frac{\mu}{\gamma\sigma^2}$  denotes the normal strategy. ■

# Chapter V

## Appendix

### A Proof of Lemma 7.2

Let  $z > 0$  and consider the function  $h(x) := U(x) - y_1zx - y_2(x - q)^-$ . Defining the two functions

$$\begin{aligned} h_1(x) &:= U(x) - y_1zx \\ h_2(x) &:= U(x) - y_1zx + y_2(x - q) = U(x) - (y_1z - y_2)x - y_2q, \end{aligned}$$

the function  $h$  can be written as

$$h(x) = \begin{cases} h_1(x) & \text{for } x \geq q, \\ h_2(x) & \text{for } x < q. \end{cases} \quad (\text{A.1})$$

Since  $h_1$  and  $h_2$  are strictly concave and continuously differentiable, the function  $h$  is a continuous and strictly concave function which is differentiable in  $(0, q)$  and  $(q, \infty)$  and possesses different one-sided derivatives in the point  $x = q$  which are  $h'(q - 0) = h'_2(q)$  and  $h'(q + 0) = h'_1(q)$ .

The functions  $h_1$  and  $h_2$  attain their maximum values at  $x_1 = I(y_1z)$  and  $x_2 := I(y_1z - y_2)$ , respectively. Since the function  $I(\cdot)$  is strictly decreasing and  $y_2 > 0$  it follows  $x_1 < x_2$ . To find the maximum of  $h$  one has to consider the following three cases.

(i)  $q < x_1$ :

Since  $U'$  is strictly decreasing we have  $U'(q) > U'(x_1) = U'(I(y_1z)) = y_1z$ , hence  $z < \frac{U'(q)}{y_1} = \underline{h}$ . Considering the one-sided derivatives at  $x = q$  one obtains

$$\begin{aligned} h'(q - 0) = h'_2(q) = U'(q) - (y_1z - y_2) &> U'(q) - y_1 \frac{U'(q)}{y_1} + y_2 > 0 \\ \text{and } h'(q + 0) = h'_1(q) = U'(q) - y_1z &> U'(q) - y_1 \frac{U'(q)}{y_1} = 0, \end{aligned}$$

i.e., the function  $h$  is increasing at  $x = q$ . It follows that the function  $h$  attains its maximum on  $(q, \infty)$  where  $h(x) = h_1(x)$ , i.e., the maximum is at  $x = x_1 = I(y_1z)$ .

(ii)  $\mathbf{x}_1 \leq \mathbf{q} < \mathbf{x}_2$ :

Now the relation  $q \geq x_1$  implies  $z \geq \underline{h}$  while  $q > x_2$  leads to

$$U'(q) < U'(x_2) = U'(I(y_1z - y_2)) = y_1z - y_2,$$

i.e.,  $z < \frac{U'(q)+y_2}{y_1} = \bar{h}$ , which gives  $\underline{h} \leq z < \bar{h}$ . It follows that

$$h'(q-0) = h'_2(q) = U'(q) - (y_1z - y_2) > U'(q) - y_1 \frac{U'(q) + y_2}{y_1} + y_2 = 0$$

$$\text{and } h'(q+0) = h'_1(q) = U'(q) - y_1z \leq U'(q) - y_1 \frac{U'(q)}{y_1} = 0.$$

From the strict concavity of  $h$  we deduce that

$$\begin{aligned} h'(x) = h'_2(x) &> h'_2(q) > 0 & \text{for } x < q \\ h'(x) = h'_1(x) &< h'_1(q) \leq 0 & \text{for } x > q. \end{aligned}$$

Thus the function  $h$  is strictly increasing for  $x < q$  and strictly decreasing for  $x > q$ , hence  $h$  attains its maximum at  $x^* = q$ .

The relations

$$\frac{y_1}{1+y_2}z < u'(q) \leq y_1z$$

imply

$$\underline{h} \leq z < \bar{h} = \frac{1+y_2}{y_1}u'(q). \quad (\text{A.2})$$

(iii)  $\mathbf{q} \geq \mathbf{x}_2$ :

This case is equivalent to  $z \geq \bar{h} = \frac{U'(q)+y_2}{y_1}$ . For the one-sided derivatives at  $x = q$  one obtains

$$h'(q-0) = h'_2(q) = U'(q) - (y_1z - y_2) \leq U'(q) - y_1 \frac{U'(q) + y_2}{y_1} + y_2 = 0$$

$$\text{and } h'(q+0) = h'_1(q) = U'(q) - y_1z \leq U'(q) - y_1 \frac{U'(q) + y_2}{y_1} = -y_2 < 0.$$

It follows that the function  $h$  is decreasing at  $x = q$  attains its maximum on  $(0, q)$  where  $h(x) = h_2(x)$  and hence the maximum is at  $x = x_2 = I(y_1z - y_2)$ .

## B Proof of Lemma 8.2

Consider the function

$$h(x) := U(x) - y_1zx - y_2(U(x) - U(q))^-.$$

Defining the two functions

$$h_1(x) := U(x) - y_1zx$$

$$h_2(x) := U(x) - y_1zx + y_2(U(x) - U(q)) = (1+y_2)U(x) - y_1zx - y_2U(q),$$

the function  $h$  can be written as

$$h(x) = \begin{cases} h_1(x) & \text{for } x \geq q, \\ h_2(x) & \text{for } x < q. \end{cases}$$

Since  $h_1$  and  $h_2$  are strictly concave and continuously differentiable, the function  $h$  is a continuous and strictly concave function which is differentiable in  $[0, q)$  and  $(q, \infty)$  and possesses different one-sided derivatives in the point  $x = q$  which are  $h'(q-0) = h'_2(q)$  and  $h'(q+0) = h'_1(q)$ .

The functions  $h_1$  and  $h_2$  attain its maximum values at  $x_1 := I(y_1 z)$  and  $x_2 := I(\frac{y_1}{1+y_2} z)$ , respectively. Since the function  $I(\cdot)$  is strictly decreasing and  $y_2 > 0$  it follows  $x_1 < x_2$ . To find the maximum of  $h$  one has to consider the following three cases.

(i)  $q < x_1$ :

Since  $U'$  is strictly decreasing we have  $U'(q) > U'(x_1) = U'(I(y_1 z)) = y_1 z$ . Considering the one-sided derivatives at  $x = q$  one obtains

$$\begin{aligned} h'(q-0) = h'_2(q) &= (1+y_2)U'(q) - y_1 z > (1+y_2)y_1 z - y_1 z = y_1 y_2 z > 0 \\ \text{and } h'(q+0) = h'_1(q) &= U'(q) - y_1 z > y_1 z - y_1 z = 0, \end{aligned}$$

i.e., the function  $h$  is increasing at  $x = q$ . It follows that the function  $h$  attains its maximum on  $(q, \infty)$  where  $h(x) = h_1(x)$ , i.e., the maximum is at  $x^* = x_1 = I(y_1 z)$ .

Solving the inequality  $U'(q) > y_1 z$  for  $z$  it yields

$$z < \frac{u'(q)}{y_1} = \underline{h}. \tag{B.3}$$

(ii)  $x_1 \leq q < x_2$ :

Now the relation  $q \geq x_1$  implies  $U'(q) \leq y_1 z$  while  $q < x_2$  leads to

$$U'(q) > U'(x_2) = U' \left( I \left( \frac{y_1}{1+y_2} z \right) \right) = \frac{y_1}{1+y_2} z.$$

For the one-sided derivatives at  $x = q$  we find

$$\begin{aligned} h'(q-0) = h'_2(q) &= (1+y_2)U'(q) - y_1 z > (1+y_2)\frac{y_1}{1+y_2} z - y_1 z = 0 \\ \text{and } h'(q+0) = h'_1(q) &= U'(q) - y_1 z \leq y_1 z - y_1 z = 0. \end{aligned}$$

From the strict concavity of  $h$  we deduce that  $h'(x) = h'_1(x) < h'_1(q) < 0$  for  $x > q$ . Thus the function  $h$  is strictly increasing for  $x < q$  and strictly decreasing for  $x > q$ , hence  $h$  attains its maximum at  $x^* = q$ .

The relations

$$\frac{y_1}{1+y_2} z < U'(q) \leq y_1 z$$

imply

$$\underline{h} \leq z < \bar{h} = \frac{1 + y_2}{y_1} U'(q) \quad (\text{B.4})$$

(iii)  $\mathbf{q} \geq \mathbf{x}_2$ :

In this case we have  $U'(q) \leq U'(x_2) = \frac{y_1}{1+y_2}z$ , For the one-sided derivatives at  $x = q$  one obtains

$$\begin{aligned} h'(q-0) &= h'_2(q) = (1+y_2)U'(q) - y_1z \leq y_1z - y_1z = 0 \\ \text{and } h'(q+0) &= h'_1(q) = U'(q) - y_1z \leq \frac{y_1z}{1+y_2} - y_1z < 0. \end{aligned}$$

It follows that the function  $h$  is decreasing at  $x = q$  attains its maximum on  $(0, q)$  where  $h(x) = h_2(x)$  and hence the maximum is at  $x^* = x_2 = I(\frac{1+y_2}{y_1}z)$ .

Solving the inequality  $U'(q) \leq U'(x_2) = \frac{y_1}{1+y_2}z$  for  $z$  it follows

$$z \geq \frac{1 + y_2}{y_1} U'(q) = \bar{h}. \quad (\text{B.5})$$

## C Computation of the current terminal wealth in Proposition 8.4

The state price density  $H_t$  is the solution of the (SDE) (4.7) and its terminal value  $H_T$  at the horizon  $T$  can be expressed in terms of the value  $H_t$  at time  $t \leq T$  by

$$\begin{aligned} H_T &= H_t \exp\left(-\left(r + \frac{\|\kappa\|^2}{2}\right)(T-t) - \kappa^\top(W_T - W_t)\right) \\ &= H_t \exp(a + b\eta). \end{aligned}$$

Thereby  $a = -(r + \frac{\|\kappa\|^2}{2})(T-t)$ ,  $b = -\|\kappa\|\sqrt{T-t}$  and  $\eta$  is a standard Gaussian random variable, which is independent of  $\mathcal{F}_t$ .

Applying Itô's lemma together with Equations (4.7) and (4.8) implies that the process  $HX^{\text{EUL}}$  is an  $\mathcal{F}$ -martingale. As a consequence we get

$$\begin{aligned} X_t^{\text{EUL}} &= \mathbb{E}\left[\frac{H_T}{H_t} X_T^{\text{EUL}} \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\frac{H_T}{H_t} I(y_1 H_T) \mathbf{1}_{\{H_T < \underline{h}\}} \middle| \mathcal{F}_t\right] + \mathbb{E}\left[\frac{H_T}{H_t} q \mathbf{1}_{\{\underline{h} \leq H_T < \bar{h}\}} \middle| \mathcal{F}_t\right] + \\ &\quad \mathbb{E}\left[\frac{H_T}{H_t} I\left(\frac{y_1}{1+y_2} H_T\right) \mathbf{1}_{\{\bar{h} \leq H_T\}} \middle| \mathcal{F}_t\right] \end{aligned} \quad (\text{C.6})$$

Using this representation and the facts that  $H_t$  is  $\mathcal{F}_t$ -measurable and  $\eta$  is independent of  $\mathcal{F}_t$ , the conditional expectations in Eq. (C.6) can be written in the form

$$\frac{c}{H_t} \mathbb{E}[g(H_t, \eta) | \mathcal{F}_t] = \frac{c}{H_t} \psi(H_t) \quad \text{with} \quad \psi(z) = \mathbb{E}[g(z, \eta)], \text{ for } z \in (0, \infty),$$



where  $g$  is a measurable function and  $c$  is a real constant. Applying this relation, the three conditional expectations can be evaluated as follows.

i) For the first term of Equation (C.6), it holds

$$\mathbb{E}\left[\frac{H_T}{H_t}I(y_1 H_T)1_{\{H_T < \underline{h}\}}\middle|\mathcal{F}_t\right] = \frac{y_1^{-\frac{1}{\gamma}}}{H_t}\mathbb{E}[g(H_t, \eta)|\mathcal{F}_t] = \frac{y_1^{-\frac{1}{\gamma}}}{H_t}\psi(H_t)$$

with  $g(z, x) = z^\lambda e^{\lambda(a+bx)}1_{\{ze^{a+bx} < \underline{h}\}}$ , where  $\lambda = 1 - \frac{1}{\gamma}$ . Computing  $\psi$  we get

$$\begin{aligned}\psi(z) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} z^\lambda e^{a\theta + b\lambda x} e^{-\frac{1}{2}x^2} 1_{\{ze^{a+bx} < \underline{h}\}} dx \\ &= \frac{z^\lambda}{\sqrt{2\pi}} e^{a\lambda + \frac{1}{2}b^2\lambda^2} \int_{\frac{\ln(\frac{\underline{h}}{z}) - a}{b}}^{\infty} e^{-\frac{1}{2}(x - b\lambda)^2} dx \\ &= \frac{z^\lambda}{\sqrt{2\pi}} e^{a\lambda + \frac{1}{2}b^2\lambda^2} \int_{\frac{\ln(\frac{\underline{h}}{z}) - a}{b} - b\lambda}^{\infty} e^{-\frac{1}{2}x^2} dx \\ &= z^\lambda e^{\Gamma(t)} \left[ 1 - \Phi\left(\frac{\ln(\frac{\underline{h}}{z}) - a}{b} - b\lambda\right) \right] = z^\lambda e^{\Gamma(t)} [1 - \Phi(-d_1(\underline{h}, z, t))].\end{aligned}$$

Finally we get

$$\mathbb{E}\left[\frac{H_T}{H_t}I(y_1 H_T)1_{\{H_T < \underline{h}\}}\middle|\mathcal{F}_t\right] = \frac{y_1^{-\frac{1}{\gamma}}}{H_t}\psi(H_t) = (y_1 H_t)^{-\frac{1}{\gamma}} e^{\Gamma(t)} [1 - \Phi(-d_1(\underline{h}, H_t, t))].$$

ii) For the second term of Equation (C.6) we obtain

$$\mathbb{E}\left[\frac{H_T}{H_t}q1_{\{\underline{h} \leq H_T < \bar{h}\}}\middle|\mathcal{F}_t\right] = \frac{q}{H_t}\mathbb{E}[g(H_t, \eta)|\mathcal{F}_t] = \frac{q}{H_t}\psi(H_t),$$

where  $g(z, x) = ze^{a+bx}1_{\{\underline{h} \leq ze^{a+bx} < \bar{h}\}}$ . Computing  $\psi$  we get

$$\begin{aligned}\psi(z) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} ze^{a+bx} e^{-\frac{1}{2}x^2} 1_{\{\underline{h} \leq ze^{a+bx} < \bar{h}\}} dx \\ &= \frac{z}{\sqrt{2\pi}} e^{a + \frac{1}{2}b^2} \int_{\frac{\ln(\frac{\bar{h}}{z}) - a}{b}}^{\frac{\ln(\frac{\underline{h}}{z}) - a}{b}} e^{-\frac{1}{2}(x - b)^2} dx \\ &= \frac{z}{\sqrt{2\pi}} e^{a + \frac{1}{2}b^2} \int_{\frac{\ln(\frac{\bar{h}}{z}) - a}{b} - b}^{\frac{\ln(\frac{\underline{h}}{z}) - a}{b} - b} e^{-\frac{1}{2}x^2} dx \\ &= ze^{-r(T-t)} \left[ \Phi\left(\frac{\ln(\frac{\underline{h}}{z}) - a}{b} - b\right) - \Phi\left(\frac{\ln(\frac{\bar{h}}{z}) - a}{b} - b\right) \right] \\ &= ze^{-r(T-t)} [\Phi(-d_2(\underline{h}, z, t)) - \Phi(-d_2(\bar{h}, z, t))],\end{aligned}$$

and we have

$$\mathbb{E}\left[\frac{H_T}{H_t}q1_{\{\underline{h} \leq H_T < \bar{h}\}}\middle|\mathcal{F}_t\right] = \frac{q}{H_t}\psi(H_t) = qe^{-r(T-t)} [\Phi(-d_2(\underline{h}, H_t, t)) - \Phi(-d_2(\bar{h}, H_t, t))].$$

iii) For the third term we obtain

$$\mathbb{E} \left[ \frac{H_T}{H_t} I \left( \frac{y_1}{1+y_2} H_T \right) 1_{\{H_T \geq \bar{h}\}} \middle| \mathcal{F}_t \right] = \frac{\left( \frac{y_1}{1+y_2} \right)^{-\frac{1}{\gamma}}}{H_t} \mathbb{E} [g(H_t, \eta) | \mathcal{F}_t] = \frac{\left( \frac{y_1}{1+y_2} \right)^{-\frac{1}{\gamma}}}{H_t} \psi(H_t)$$

with  $g(z, x) = z^\lambda e^{\lambda(a+bx)} 1_{\{ze^{a+bx} \geq \bar{h}\}}$  and  $\lambda = 1 - \frac{1}{\gamma}$ . Computing  $\psi$  we get

$$\begin{aligned} \psi(z) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} z^\lambda e^{a\lambda+bx} e^{-\frac{1}{2}x^2} 1_{\{ze^{a+bx} \geq \bar{h}\}} dx \\ &= \frac{z^\lambda}{\sqrt{2\pi}} e^{a\lambda+\frac{1}{2}b^2\lambda^2} \int_{-\infty}^{\frac{\ln(\frac{\bar{h}}{z})-a}{b}} e^{-\frac{1}{2}(x-b\lambda)^2} dx \\ &= \frac{z^\lambda}{\sqrt{2\pi}} e^{a\lambda+\frac{1}{2}b^2\lambda^2} \int_{-\infty}^{\frac{\ln(\frac{\bar{h}}{z})-a}{b}-b\lambda} e^{-\frac{1}{2}x^2} dx \\ &= z^\lambda e^{\Gamma(t)} \left[ \Phi \left( \frac{\ln(\frac{\bar{h}}{z})-a}{b} - b\lambda \right) \right] = z^\lambda e^{\Gamma(t)} [\Phi(-d_1(\bar{h}, z, t))]. \end{aligned}$$

Finally we obtain

$$\mathbb{E} \left[ \frac{H_T}{H_t} I \left( \frac{y_1}{1+y_2} H_T \right) 1_{\{H_T \geq \bar{h}\}} \middle| \mathcal{F}_t \right] = \left( \frac{y_1}{1+y_2} H_t \right)^{-\frac{1}{\gamma}} e^{\Gamma(t)} [\Phi(-d_1(\bar{h}, H_t, t))].$$

## D Proof of Lemma 11.2

The proof is analogous to the above proof of Lemma 8.2 if the quantity  $q$  is substituted by  $q = q(z) = Az^{-\nu}$ . The dependence of  $q$  on  $z$  affects only the solutions of the inequalities given in (B.3), (B.4), and (B.5). Taking into account the dependence of  $q$  on  $z$  we get the following solutions.

(i)  $\mathbf{q} < \mathbf{x}_1$ :

The inequality  $U'(q) = (Az^{-\nu})^{-\gamma} = A^{-\gamma} z^{\gamma\nu} > y_1 z$  is fulfilled

$$(a) \text{ for } \nu < \frac{1}{\gamma} \quad \text{if } z < \left( \frac{1}{y_1 A^\gamma} \right)^{\frac{1}{1-\gamma\nu}} = \underline{h},$$

$$(b) \text{ for } \nu > \frac{1}{\gamma} \quad \text{if } z > \underline{h}$$

(ii)  $\mathbf{x}_1 \leq \mathbf{q} < \mathbf{x}_2$ :

The inequalities  $\frac{y_1}{1+y_2} z < U'(q) = (Az^{-\nu})^{-\gamma} = A^{-\gamma} z^{\gamma\nu} \leq y_1 z$  are fulfilled

$$(a) \text{ for } \nu < \frac{1}{\gamma} \quad \text{if } \underline{h} \leq z < \bar{h} = \left( \frac{1+y_2}{y_1 A^\gamma} \right)^{\frac{1}{1-\gamma\nu}},$$

$$(b) \text{ for } \nu > \frac{1}{\gamma} \quad \text{if } \bar{h} \leq z < \underline{h}$$

(iii)  $\mathbf{q} \geq \mathbf{x}_2$ :

The inequality  $\frac{y_1}{1+y_2}z \geq U'(q) = (Az^{-\nu})^{-\gamma} = A^{-\gamma}z^{\gamma\nu}$  is fulfilled

(a) for  $\nu < \frac{1}{\gamma}$  if  $z \geq \bar{h}$ ,

(b) for  $\nu > \frac{1}{\gamma}$  if  $z \leq \bar{h}$ .

## E Evaluation of the derivative $\mathbf{F}_z(\mathbf{z}, t)$ in Eq. (11.3)

We rewrite Equation (11.3) as follows

$$F_z(z, t) = -\frac{1}{\gamma z} \left[ F(z, t) - (1 - \gamma\nu) \frac{Ae^{\Gamma_2(t)}}{z^\nu} \left( \Phi(-d_2(\underline{h}, z, t)) - \Phi(-d_2(\bar{h}, z, t)) \right) \right] \\ + \frac{1}{z\kappa\sqrt{T-t}}(V_1 + V_2),$$

where

$$V_1 = -\frac{e^{\Gamma_1(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \left[ \varphi(d_1(\underline{h}, z, t)) - (1 + y_2)^{\frac{1}{\gamma}} \varphi(d_1(\bar{h}, z, t)) \right],$$

and

$$V_2 = \frac{Ae^{\Gamma_2(t)}}{z^\nu} \left[ \varphi(d_2(\underline{h}, z, t)) - \varphi(d_2(\bar{h}, z, t)) \right].$$

For simplicity we make the following notation:

$$a = \kappa\sqrt{T-t}, \quad d_1(\underline{h}, z, t) = \underline{d}_1, \quad d_2(\underline{h}, z, t) = \underline{d}_2, \quad d_1(\bar{h}, z, t) = \bar{d}_1, \quad d_2(\bar{h}, z, t) = \bar{d}_2.$$

Since  $\underline{d}_1 = \underline{d}_2 + \frac{a}{\gamma}$ , this implies that  $\varphi(\underline{d}_1) = \varphi(\underline{d}_2) e^{-\frac{1}{2}\frac{a^2}{\gamma^2}} e^{-\frac{a}{\gamma}\underline{d}_2}$ . Moreover, it holds

$$-\frac{\ln(1+y_2)\bar{d}_1}{a} = -\frac{\ln(1+y_2)\underline{d}_2}{a} + \ln(1+y_2)^{-\frac{1}{\gamma}}.$$

As consequence we obtain

$$V_1 = -\frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \varphi(\underline{d}_1) \left[ 1 - (1 + y_2)^{\frac{1}{\gamma}} e^{-\frac{1}{2}\left(\frac{\ln(1+y_2)}{a}\right)^2} e^{-\frac{\ln(1+y_2)}{a}\underline{d}_1} \right] \\ = -\frac{e^{\Gamma(t) - \frac{1}{2}\frac{a^2}{\gamma^2} - \frac{a}{\gamma}\underline{d}_2}}{(y_1 z)^{\frac{1}{\gamma}}} \varphi(\underline{d}_2) \left[ 1 - e^{-\frac{1}{2}\left(\frac{\ln(1+y_2)}{a}\right)^2} e^{-\frac{\ln(1+y_2)}{a}\underline{d}_2} \right].$$

From the other hand we have

$$\Gamma(t) - \frac{1}{2}\frac{a^2}{\gamma^2} - \frac{a}{\gamma}\underline{d}_2 = \Gamma(t) - \frac{1}{2}\frac{a^2}{\gamma^2} - \frac{1}{\gamma}\left(r - \frac{\kappa^2}{2}\right)(T-t) + \ln\left(\frac{h}{z}\right)^{-\frac{1}{\gamma}} \\ = -r(T-t) + \ln\left(\frac{h}{z}\right)^{-\frac{1}{\gamma}},$$

which implies that

$$\begin{aligned} V_1 &= -\frac{e^{-r(T-t)}\left(\frac{h}{z}\right)^{-\frac{1}{\gamma}}}{(y_1 z)^{\frac{1}{\gamma}}}\varphi(\underline{d}_2)\left[1 - e^{-\frac{1}{2}\left(\frac{\ln(1+y_2)}{a}\right)^2}e^{-\frac{\ln(1+y_2)}{a}\underline{d}_2}\right] \\ &= -qe^{-r(T-t)}\varphi(\underline{d}_2)\left[1 - e^{-\frac{1}{2}\left(\frac{\ln(1+y_2)}{a}\right)^2}e^{-\frac{\ln(1+y_2)}{a}\underline{d}_2}\right] \\ &= -V_2. \end{aligned}$$

Hence, the claim is proved.

## F Proof of Proposition 11.5

Following the argumentation of Lemma 11.2 we get the following solutions  $z$  of the stated inequalities.

(i)  $\mathbf{q} < \mathbf{x}_1$ :

The inequality  $U'(q) = (Az^{-\nu})^{-\gamma} = A^{-\gamma}z^{\gamma\nu} > y_1z$  is fulfilled for all  $z > 0$  if  $y_1A^\gamma \in (0, 1)$ . Otherwise there is no solution.

(ii)  $\mathbf{x}_1 \leq \mathbf{q} < \mathbf{x}_2$ :

The inequalities  $\frac{y_1}{1+y_2}z < U'(q) = (Az^{-\nu})^{-\gamma} = A^{-\gamma}z^{\gamma\nu} \leq y_1z$  are fulfilled for all  $z > 0$  if  $y_1A^\gamma \in [1, 1 + y_2)$ . Otherwise there is no solution.

(iii)  $\mathbf{q} \geq \mathbf{x}_2$ :

The inequality  $\frac{y_1}{1+y_2}z \geq U'(q) = (Az^{-\nu})^{-\gamma} = A^{-\gamma}z^{\gamma\nu}$  is fulfilled for all  $z > 0$  if  $y_1A^\gamma \in [1 + y_2, \infty)$ . Otherwise there is no solution.

Provided an optimal solution exists Lemma 11.2 gives the following form of the optimal terminal wealth which depends on the value of  $y_1A^\gamma$

$$\xi^* = \begin{cases} I(y_1H_T) & \text{if } y_1A^\gamma \in (0, 1) \\ AH_T^{-\nu} = Q & \text{if } y_1A^\gamma \in [1, 1 + y_2) \\ I\left(\frac{y_1}{1+y_2}H_T\right) & \text{if } y_1A^\gamma \in [1 + y_2, \infty). \end{cases}$$

In order to check the existence of an optimal solution one has to check whether there exist real numbers  $y_1 > 0$  and  $y_2 \geq 0$  such that it holds

$$\begin{aligned} \mathbb{E}[H_T\xi^*] &= x \\ \mathbb{E}[(U(\xi^*) - U(Q))^-] &\leq \varepsilon. \end{aligned}$$

In the first case the parameter  $y_1$  can to be chosen such that the budget constraint is fulfilled with equality, i.e.,  $\mathbb{E}[H_T\xi^*] = \mathbb{E}[H_T I(y_1H_T)] = x$  which gives  $y_1 = y^N$ . It can be observed that  $\xi^*$  coincides with the optimal terminal wealth of the problem

without risk constraint. If the risk constraint is fulfilled (not necessarily with equality) then it is the optimal solution since there is no other terminal wealth exceeding the expected utility of  $\xi^*$ .

In the second case  $\xi^*$  coincides with the benchmark  $Q$ . Using  $\nu = \frac{1}{\gamma}$ ,  $I(z) = z^{-\frac{1}{\gamma}} = z^{-\nu}$  and setting  $A = \tilde{y}^{-\nu}$  we get

$$\xi^* = AH_T^{-\nu} = (\tilde{y}H_T)^{-\nu} = I(\tilde{y}H_T).$$

For the budget constraint this implies

$$x = \mathbb{E}[H_T \xi^*] = \mathbb{E}[H_T I(\tilde{y}H_T)]$$

For the Expected Utility Loss we get  $\mathbb{E}[(U(\xi^*) - U(Q))^-] = \mathbb{E}[(U(Q) - U(Q))^-] = 0 < \varepsilon$ .

If the parameters of our model which are involved in  $\tilde{y}$  are such that the budget constraint is fulfilled with equality then the optimal terminal wealth is  $\xi^* = I(\tilde{y}H_T)$  since the risk constraint is always fulfilled. Moreover it coincides with the optimal terminal wealth of the problem without risk constraint.

For other parameters there is no optimal solution.

In the third case the optimal terminal wealth is  $\xi^* = I\left(\frac{y_1}{1+y_2}H_T\right)$  if it satisfies

$$\begin{aligned} \mathbb{E}[H_T \xi^*] &= x \\ \mathbb{E}[(U(\xi^*) - U(Q))^-] &\leq \varepsilon. \end{aligned}$$

For the first equation we get

$$\mathbb{E}[H_T \xi^*] = \mathbb{E}\left[H_T I\left(\frac{y_1}{1+y_2}H_T\right)\right].$$

From the considerations of the problem without risk constraint it is known, that the equation is fulfilled for  $\frac{y_1}{1+y_2} = y^N$ . For the risk constraint we use that in the considered case it holds  $\xi^* \leq Q = AH_T^{-\nu}$  (see case (iii) of Lemma 11.2 which holds for  $y_1 A^\gamma \in [1 + y_2, \infty)$ ). From this property it follows for the Expected Utility Loss

$$\begin{aligned} \mathbb{E}[(U(\xi^*) - U(Q))^-] &= \mathbb{E}[U(Q) - U(\xi^*)] \\ &= \mathbb{E}[U(AH_T^{-\nu}) - U(I(y^N H_T))]. \end{aligned}$$

If the numbers  $y_1$  and  $y_2$  are such that  $\frac{y_1}{1+y_2} = y^N$  and  $y_1 A^\gamma \in [1 + y_2, \infty)$  then the Expected Utility Loss depends on the parameters of the financial market (as  $\mu, r, \sigma, x$ ) via  $A, \nu, H_T, y^N$  and the parameter  $\gamma$  of the utility function via  $I$  but not directly on  $y_1$  and  $y_2$ . So, if to a given  $\varepsilon$  and a  $y^N$  following from the solution of the first equation the risk constraint is fulfilled then the optimal terminal wealth is

$$\xi^* = I\left(\frac{y_1}{1+y_2}H_T\right) = I(y^N H_T).$$

Hence, it coincides with the optimal terminal wealth of the problem without risk constraint.

On the other hand, if the risk constraint is not fulfilled, then there is no admissible solution and consequently no optimal solution.

The given form of  $y^N$  follows directly from the representation of  $y^N$  given in Example 4.6 for  $\nu = \frac{1}{\gamma}$ . Moreover in this case we have  $\theta^N = \frac{\gamma\kappa}{\sigma} = \frac{1}{\gamma\nu} = 1$ .

## G Malliavin derivative

Consider a complete probability space  $(\Omega, \mathcal{H}, P)$  and let  $W = (W_t)_{t \in [0, T]}$  be an  $\mathbb{R}^n$ -valued standard Brownian motion defined on it,  $W_t = (W_t^1, \dots, W_t^n)^\top$ . We shall denote by  $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]}$  the  $P$ -augmentation of the natural filtration  $\mathcal{F}^W = (\mathcal{F}_t^W)_{t \in [0, T]}$ ,  $\mathcal{F}_t^W = \sigma(W_s; 0 \leq s \leq t)$  which satisfies the usual conditions of right-continuity and completion by  $P$ -negligible sets.

We shall need to recall the definition of the Malliavin derivative (Ocone and Karatzas [61]), see also Nualart and Pardoux [59]. Denote by  $C_b^\infty(\mathbb{R}^m)$  the set of  $C^\infty$  functions  $f : \mathbb{R}^m \mapsto \mathbb{R}$  which are bounded and have bounded derivatives of all orders. Let  $\mathcal{S}$  be the class of smooth functionals, i.e., random variables of the form

$$F(\omega) = f(W_{t_1}(\omega), \dots, W_{t_d}(\omega)),$$

where  $(t_1, \dots, t_d) \in [0, T]^d$  and the function  $f(x_1^1, \dots, x_1^n, \dots, x_d^1, \dots, x_d^n)$  belongs to  $C_b^\infty(\mathbb{R}^{dn})$ . The gradient  $DF(\omega)$  of the smooth functional  $F$  is defined as the  $(L^2([0, T]))^n$ -valued random variable  $DF = (D^1F, \dots, D^nF)$  with components

$$D^i F(\omega)(t) = \sum_{j=1}^d \frac{\partial}{\partial x_j^i} f(W_{t_1}(\omega), \dots, W_{t_n}(\omega)) \mathbf{1}_{[0, t_j]}(t) \quad i = 1, \dots, n. \quad (\text{G.7})$$

Finally, let  $\|\cdot\|$  denote the  $L^2([0, T])$  norm;  $|\cdot|$  will be reserved for the Euclidean norm on  $\mathbb{R}^d$ ,  $d \geq 1$ . For each  $p \geq 1$ , we introduce the norm

$$\|F\|_{p,1} = \left( \mathbb{E} \left[ |F|^p + \left( \sum_{j=1}^n \|D^j F\|^2 \right)^{p/2} \right] \right)^{1/p} \quad (\text{G.8})$$

on  $\mathcal{S}$ , and we denote by  $\mathbb{D}_{p,1}$  the Banach space which is the closure of  $\mathcal{S}$  under  $\|\cdot\|_{p,1}$ .

**Remark G.1.** It is proved that  $DF$  is well-defined on  $\mathbb{D}_{p,1}$  by closure for any  $p \geq 1$ , see [68], Lemma 2.1. Given  $F \in \mathbb{D}_{p,1}$ , on can find a measurable process  $(t, \omega) \mapsto D_t F(\omega)$  such that for a.e.  $\omega \in \Omega$ ,  $D_t F(\omega) = DF(\omega)(t)$  hold for almost all  $t \in [0, T]$ .  $D_t F(\omega)$  is defined uniquely up to sets of measure zero on  $[0, T] \times \Omega$ . In general, if  $X : \Omega \mapsto L^2([0, T])$  is measurable, there exists a  $\mathcal{B}([0, T]) \otimes \mathcal{F}$  measurable random variable,  $\tilde{X} = (\tilde{X}(t, \omega))_{(t, \omega) \in [0, T] \times \Omega}$ , such that  $\tilde{X}(\cdot, \omega) = X(\omega)$  holds almost surely.

For  $F = (F_1, \dots, F_d)^\top \in (\mathbb{D}_{p,1})^d$  the Malliavin derivative  $DF$  is defined by setting  $(DF)_{ik} = D^i F_k, i = 1, \dots, n, k = 1, \dots, d$  (transposed to the convention for the Jacobi matrix). We shall use the space

$$\mathbb{D} = \bigcap_{p>1} \mathbb{D}_{p,1}.$$

The following theorem proved in [65] is based on Theorem 4.14 in [60] where it is used to give a new proof of the generalization of Clark's formula from Brownian motion to Itô processes found in [36]. For complete proof of Proposition G.2 we refer to Proposition 8.3 in [65].

**Proposition G.2.** For  $d \in \mathbb{N}$  we consider the  $d$ -dimensional SDE

$$dX_t = f^\nu(t, X_t)dt + f^\sigma(t, X_t)dW_t, \quad t \in [0, T], X_0 = x_0, \quad (\text{G.9})$$

assuming that  $x_0 \in \mathbb{R}^d$ ,  $f^\nu$  and  $f^\sigma$  are measurable  $\mathbb{R}^d$  and  $\mathbb{R}^{d \times n}$ -valued functions which are continuously differentiable and satisfy

$$\sup_{t \in [0, T], x \in \mathbb{R}^d} \left( \left| \frac{\partial}{\partial x_k} f_i^\nu(t, x) + \frac{\partial}{\partial x_k} f_{ij}^\sigma(t, x) \right| \right) < \infty, \quad \sup_{t \in [0, T]} (|f_i^\nu(t, 0) + f_{ij}^\sigma(t, 0)|) < \infty$$

for  $i, k = 1, \dots, d, j = 1, \dots, n$ . Then (G.9) has a unique continuous solution  $(X_t)_{t \in [0, T]}$  which satisfies  $X_s^k \in \mathbb{D}, k = 1, \dots, d$ ,

$$\begin{aligned} D_t X_s &= (f^\sigma(t, X_t))^\top + \int_t^s D_t X_u (\partial_x f^\nu(u, X_u))^\top du \\ &+ \int_t^s D_t X_u \sum_{j=1}^n (\partial_x f_{.j}^\sigma(u, X_u))^\top dW_u^j \end{aligned}$$

for  $t \in [0, s]$ , and  $D_t X_s = 0$  for  $t \in (s, T]$ . Here  $\partial_x$  denotes the Jacobi matrix, i.e.,  $(\partial_x f_i^\nu)_{ij} = \frac{\partial}{\partial x_j} f_i^\nu$ , and  $f_{.j}^\sigma$  is the  $j$ th column of  $f^\sigma$ .

The optimal strategy we are looking for is given in terms of Malliavin derivative of the optimal terminal wealth provided this terminal wealth belongs to  $\mathbb{D}_{1,1}$ . The chain rule is crucial to show that our terminal wealth which is a functional of the conditional state density  $\zeta$  lies in  $\mathbb{D}_{1,1}$ . In our model it is easier to work in  $\mathbb{D}$  since the required integrability conditions we need to apply the chain rules follow directly from Hölder's inequality. Moreover, the chain rules in [60] and [61] are not adequate for our situation. So we state the following chain rules proved in [65]

**Proposition G.3.** Suppose that  $F \in \mathbb{D}$  with values in some open interval  $J$ ,  $g \in C^1(J, \mathbb{R})$ , and  $g(F), g'(F) \in \bigcap_{p>1} L^p$ . Then  $g(F) \in \mathbb{D}$  and  $Dg(F) = g'(F)DF$ .

**Proof.** Suppose  $n = 1$  and  $p > 1$ . Then for every  $q > p$  Hölder's inequality implies

$$\| \|g'(F)DF\|_{L^2} \|g'(F)\| \|DF\|_{L^2} \|g'(F)\|_q \|DF\|_{L^2}^{\frac{pq}{q-p}} < \infty,$$

hence  $\| \|g'(F)DF\|_{L^2} \|g'(F)\| \|DF\|_{L^2} \|g'(F)\|_q \|DF\|_{L^2}^{\frac{pq}{q-p}} < \infty$ . The rest of the proof is similar to the proof of Lemma A.1 in [61]. ■

**Proposition G.4.** Suppose  $F = (F_1, \dots, F_d) \in (\mathbb{D})^d$  with values  $\mathbb{R}^d$ ,  $g \in C^1(\mathbb{R}^d, \mathbb{R})$ , and  $g(F) \in L^1$ . If  $g'(F) \in L^q$  for some  $q > 1$ . Then  $g(F) \in \mathbb{D}_{1,1}$  and

$$Dg(F) = \sum_{k=1}^d \left( \frac{\partial}{\partial x_k} g(F_1, \dots, F_d) \right) D(F_k).$$

**Proof.** The assumptions imply using again Hölder's inequality

$$\begin{aligned} \|\|g'(F)DF\|_{L^2}\|_1 &= \| |g'(F)| \|DF\|_{L^2} \|1\|_1 \\ &\leq \|g'(F)\|_q \| \|DF\|_{L^2} \|_{\frac{q}{q-1}} < \infty, \end{aligned}$$

hence  $\|\|g'(F)DF\|_{L^2}\|_{1,1} < \infty$  and similarly to the proof of Lemma A.1 in [61] one can obtain the claim.  $\blacksquare$



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