Optimization with set relations

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Introduction

Since the middle of the 80ies, *set–valued optimization problems*, i.e., optimization problems with a set–valued objective map, have been investigated. This field of research is called *set–valued optimization* or *set optimization*. The most results in the literature are based on the following (or a similar) solution concept:

One considers a set–valued objective map $F : X \rightrightarrows Y$ and a set of feasible points $S \subseteq X$, where X and Y are linear spaces and Y is partially ordered by a convex pointed cone. A pair $(\bar{x}, \bar{y})$ with $\bar{x} \in S$ and $\bar{y} \in F(\bar{x})$ is called a *minimizer* of the set–valued optimization problem if $\bar{y}$ is a minimal element of the set $F(S) := \bigcup_{x \in S} F(x)$, where the minimality notion is understood with respect to the partial ordering in the space Y. Therefore, (in this work) set optimization based on this solution concept is called *set optimization with point relations*. In this field, there are papers on optimality conditions, e.g. Corley [21], Luc [58], Jahn and Rauh [43], Götz and Jahn [32], Chen and Jahn [16], Crespi, Ginchev and Rocca [22], as well as papers on duality assertions, e.g. Tanino and Sawaragi [79], Corley [19], where only the dual problems are set–valued, and Kawasaki [44], Postolică [66], Corley [20], Luc and Jahn [59], Song [73], where both the primal and the dual problem are set–valued. A good survey as well as further references can be found in the book of Jahn [42]. Set–valued optimization problems based on this solution concept naturally occur in vector optimization, for instance, as dual problems (see e.g. Tanino and Sawaragi [79], Corley [19]). Of course, vector optimization problems provide a very important special case of set optimization with numerous applications. Moreover, it turned out that the answer to certain problems in vector optimization can be found, if the vector optimization problem is considered in a set–valued framework. For instance, by Hamel, Heyde, Löhne, Tammer and Winkler [34] a set–valued approach is used to solve the problem of the duality gap in linear vector optimization in the case that the right hand side of the inequality constraints is zero.

Optimization with set relations (actually "set optimization with set relations") provides quite a different approach to set optimization. The basic idea is to understand the set–valued objective map as a function $f : X \to \mathcal{P}(Y)$ into the space $\mathcal{P}(Y)$ of all subsets of Y. This space is provided with an appropriate ordering relation. Such ordering relations have been investigated, for instance, by Young [83], Nishnianidze [64], Brink [13], Kuroiwa [49] and Hamel [33]. In the special case that Y is a linear space, $K \subseteq Y$ a convex pointed cone and $A, B \subseteq Y$, these relations can be expressed by

$$A \preccurlyeq_K B \;:\Leftrightarrow\; B \subseteq A + K \qquad \text{and} \qquad A \curlyeqprec_K B \;:\Leftrightarrow\; A \subseteq B - K.$$

In a sequence of very similar papers of Kuroiwa, e.g. [49], [50], [51], [52], [53], corresponding optimization problems are investigated and some first steps towards a duality theory are taken. These optimization problems are based on the following solution concept: A point $x \in S$ is called a $\preccurlyeq_K$–minimal solution of the set–valued optimization problem if

$$\big(f(x) \preccurlyeq_K f(\bar{x}),\ x \in S\big) \Rightarrow f(\bar{x}) \preccurlyeq_K f(x).$$

Some other results in this field are generalizations of Ekeland's variational principle (Truong

[80]) as well as Phelps type minimal point theorems (Hamel and Löhne [35]) to the framework of these ordering relations.

This work is concerned with optimization with set relations, however, our investigations are not based on the solution concept introduced above. Instead we use the lattice structure of the image space $\mathcal{P}(Y)$ and ask for the infimum and supremum of the objective function over the set S. Our theory is modeled on scalar optimization rather than vector optimization. Nevertheless we show that this approach is beneficial for a better understanding of the structures in vector optimization and set optimization with point relations, because problems of this kind are hidden in every vector optimization and set optimization problem. Let us proceed with a more detailed discussion.

In this work we investigate optimization problems based on set relations where we restrict the image space of the objective function f to the family of closed convex subsets of $\mathbb{R}^p$, in the following denoted by $\hat{\mathcal{C}}$. The space $\hat{\mathcal{C}}$ is equipped with an addition and a multiplication by nonnegative reals, more precisely, it is a conlinear space (see Appendix C). The reason for supposing convex sets is that many assertions, even in the classical theory, require convexity of the objective function. The convexity notion in our framework, however, implies that the values are convex sets. The convexity of the sets also ensures that the "second distributive law" is valid in our image space $\hat{\mathcal{C}}$, which is beneficial in some cases. The reason for the closedness is that we only use convergences that do not distinguish between a set and its closure. The space $\hat{\mathcal{C}}$ is ordered by set inclusion. In fact, $(\hat{\mathcal{C}}, \supseteq)$ is an ordered conlinear space (see Appendix C). In contrast to that, $\hat{\mathcal{C}}$ being equipped with the orderings $\preccurlyeq_K$ or $\curlyeqprec_K$ is just a quasi–ordered conlinear space, because $\preccurlyeq_K$ and $\curlyeqprec_K$ are not antisymmetric. Of course, one can switch over to equivalence classes in order to obtain antisymmetry. Then, however, $\preccurlyeq_K$ and $\curlyeqprec_K$ reduce to the usual set inclusion $\supseteq$ and $\subseteq$, respectively. We next observe that our image space $(\hat{\mathcal{C}}, \supseteq)$ is order complete (a complete lattice). Infimum and supremum of a nonempty subset $\mathcal{A} \subseteq \hat{\mathcal{C}}$ can be expressed by

$$\inf \mathcal{A} = \operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathcal{A}} A \qquad \text{and} \qquad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A.$$

This allows us to proceed analogously to the classical scalar optimization theory. We consider a $\hat{\mathcal{C}}$–valued objective function and a set $S \subseteq X$ of feasible points. Our goal is to determine the sets $\inf_{x \in S} f(x)$ and $\sup_{x \in S} f(x)$ or, equivalently, the sets $\operatorname{cl}\operatorname{conv} \bigcup_{x \in S} f(x)$ and $\bigcap_{x \in S} f(x)$. It is also interesting to ask for a corresponding solution concept (in the space X), however, this is beyond the scope of this work.

Optimization based on set inclusion subsumes the classical scalar optimization theory. This can be seen by the following simple reformulation. Consider an extended real–valued objective function $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$. Then the function $\bar{f}$, defined by $\bar{f}(\cdot) := \{f(\cdot)\} + \mathbb{R}_+$ (where we set $\{+\infty\} + \mathbb{R}_+ = \emptyset$ and $\{-\infty\} + \mathbb{R}_+ = \mathbb{R}$), is a $\hat{\mathcal{C}}$–valued function and it holds

$$\left\{ \inf_{x \in S} f(x) \right\} + \mathbb{R}_+ = \inf_{x \in S} \bar{f}(x) \quad \text{and} \quad \left\{ \sup_{x \in S} f(x) \right\} + \mathbb{R}_+ = \sup_{x \in S} \bar{f}(x).$$

On the other hand, in case we know $\inf_{x \in S} \bar{f}(x) \in \hat{\mathcal{C}}$ and $\sup_{x \in S} \bar{f}(x) \in \hat{\mathcal{C}}$ we obtain the real counterparts by

$$\inf_{x\in S} f(x) = \inf\left\{\inf_{x\in S} \bar{f}(x)\right\} \qquad \text{and} \qquad \sup_{x\in S} f(x) = \inf\left\{\sup_{x\in S} \bar{f}(x)\right\}.$$

We observe that a scalar optimization problem can be decomposed into two components, namely,

1. an optimization problem based on set inclusion, where the involved sets have the special structure of $A_r := \{s \in \mathbb{R}|\ s \geq r\}$, $r \in \mathbb{R} \cup \{-\infty, +\infty\}$.
2. the (trivial) problem of determining the infimum of such sets.

In this manner it is interesting to compare the construction of real numbers by Dedekind cuts.

Optimization problems based on set inclusion are also hidden in other problems such as vector optimization problems and set optimization problems based on point relations. Similar to the scalar case, set optimization problems can be decomposed into two components, namely,

1. an optimization problem based on set inclusion, in fact, the problem to determine the set
$$F(S) := \bigcup_{x\in S} F(x),$$
2. the problem of determining the set $\operatorname{Inf} F(S)$ of infimal or the set $\operatorname{Min} F(S)$ of minimal (efficient) points with respect to the partial ordering in Y.

We show in this work that a separate investigation of both parts leads to deeper insights into the structure of set optimization problems based on point relations. Many assertions that are well–known for scalar problems can be generalized to optimization problems based on set inclusion. For instance, we will generalize a biconjugation theorem and duality assertions of Fenchel as well as Lagrange type to this framework. In doing so, we essentially maintain the structures of the scalar case. Putting both components together, certain parts of these structures are lost. This is due to the fact that the second component, which is "nontrivial" in contrast to the scalar case, destroys the lattice structure in the sense that the set of minimal points cannot be described with an infimum (in the sense of lattice theory) in the space Y.

The investigations on optimization problems with set relations leads to some results, which could be of independent interest. Embedding of convex sets into a linear space has been investigated by many authors, beginning with the paper of Rådström [67], but only for spaces of compact or bounded sets. We extend some results to nonbounded sets. Secondly, we obtain some results on convergence in the space of closed convex subsets of $\mathbb{R}^p$. We introduce a convergence class and investigate the relationsship to well–known convergences.

This work is organized as follows. Chapter 1 is devoted to the study of algebraic and order theoretic properties of the ordered conlinear space $(\hat{\mathcal{C}}, \oplus, \supseteq)$ of closed convex subsets of $\mathbb{R}^p$

(where $\oplus$ is the closure of the Minkowski addition). This involves an investigation of the dual description of a closed convex set by its support function, which leads to an assertion about the possibility of embedding certain subsets of $\hat{\mathcal{C}}$ into a linear space. Some results in this context are extensively used in the subsequent chapters.

The second chapter deals with topological properties of $\hat{\mathcal{C}}$. We introduce a convergence class in $\hat{\mathcal{C}}$, which seems to be new. We define this convergence by appropriate concepts of upper and lower limits. A characterization by (a certain type of) convergence of support functions is indicated. We compare our convergence with well–established concepts such as Painlevé–Kuratowski convergence and scalar convergence.

In Chapter 3, we investigate functions with values in $\hat{\mathcal{C}}$. Based on the results of the second chapter, we develop semi–continuity concepts. Moreover, we introduce the notion of a conjugate of a $\hat{\mathcal{C}}$–valued function. The main result of this chapter is a biconjugation theorem for $\hat{\mathcal{C}}$–valued functions.

Chapter 4 is devoted to optimization problems with $\hat{\mathcal{C}}$–valued objective function and with respect to the set relation approach. In particular, we draw our attention to duality theory. We prove weak as well as strong duality assertions based on Fenchel as well as Lagrange approach. The biconjugation theorem, established in Chapter 3, provides a main tool for the proof of the strong duality assertions.

The last chapter contains a comparison of the duality results for optimization with set inclusion and duality assertions in vector optimization (and set optimization with point relations). The closest relationship between both types of problems can be shown for vector optimization problems based on weakly minimal points. In this context, a strong duality result for vector optimization problems is easily obtained from a duality result for a problem based on set inclusion and vise versa. Moreover, we continue to point out that optimization problems based on set relations are hidden in every vector optimization problem.

In the appendix we summarize some well–known facts, which could be useful for reading this work. Part A is a collection of calculus rules for sets. In Part B we recall some fundamental notions with respect to partially ordered sets and Part C gives the definition of an *ordered conlinear space*, which is the underlying structure of the image space of an optimization problem based on set inclusion.

Notation

Up to a few exceptions, we frequently use the notation of Rockafellar's "Convex Analysis" [1]. The following notations are not in accordance with this book. We write $\mathbb{R}$ (instead of R) for the set of real numbers and $\mathbb{N}$ for the set of positive integers. We set $\mathbb{R}^p_+ := \{y \in \mathbb{R}^p |\ \forall i \in \{1,...,p\} : y_i \geq 0\}$ and $\mathbb{R}_+ := \mathbb{R}^1_+$. The symbol $\oplus$ does not mean the direct sum, instead it will be used for the "closed Minkowski addition", i.e., for $A, B \in \mathbb{R}^p$ we set $A \oplus B := \operatorname{cl}(A+B)$. The Euclidian norm in $\mathbb{R}^p$ is denoted by $\|\cdot\|$ (instead of $|\cdot|$). Furthermore we write $\operatorname{rg} T := \bigcup_{x \in X} T(x)$ for the range of a function $T : X \to Y$. By $\operatorname{rb} A$ we denote the relative boundary of a set $A \subseteq \mathbb{R}^p$, i.e., $\operatorname{rb} A := \operatorname{cl} A \setminus \operatorname{ri} A$. A cone $C \subseteq \mathbb{R}^p$ is said to be *pointed* if its lineality space is zero, i.e., $C \cap -C = \{0\}$. The Euclidian (closed) unit ball is denoted by $\mathbb{B}$ (instead of B). If M is a matrix, M^T denotes the transposed matrix. Further notations are defined when they first occur. A short explanation can be found at page 95 f.

Chapter 1

The closed convex sets in $\mathbb{R}^p$

In this chapter, we investigate algebraic and order theoretic properties of the space of closed convex subsets of $\mathbb{R}^p$. In the following, this space is denoted by $\hat{\mathcal{C}}$. Equipped with the partial ordering "set inclusion" this space is an example for an ordered conlinear space, see Appendix C. In particular, $\hat{\mathcal{C}}$ is not a linear space. However, the structure of a conlinear space is rich enough for a meaningful definition of a cone as well as a convex set.

This chapter is organized as follows. In Section 1.1, we briefly summarize some properties of the space of closed subsets of $\mathbb{R}^p$. Section 1.2 is devoted to corresponding properties of the space of closed convex subsets of $\mathbb{R}^p$. In contrast to the closed subsets, the second distributive law is satisfied. Another very important aspect is that closed convex sets can be equivalently described by its support functions. Rockafellar [1] introduced the concept of *oriented* closed convex sets. This concept plays a crucial role in this work. In the third section of this chapter, we investigate the possibility of embedding subsets of $\hat{\mathcal{C}}$ into a linear space. We divide this space into classes that can be embedded. These classes depend on the recession cone of the sets. With the aid of the orientation we can re-interpret the inverse elements of the embedded elements as sets (with opposite orientation).

1.1 The space $\hat{\mathcal{F}}$ of closed subsets of $\mathbb{R}^p$

In this section we summarize some simple facts about the space of closed subsets of $\mathbb{R}^p$. Note that all the assertions of this section remain valid if $\mathbb{R}^p$ is replaced by an arbitrary linear topological space. The space of all *nonempty closed subsets* of $\mathbb{R}^p$ is denoted by $\mathcal{F}(\mathbb{R}^p)$ and the space of all *closed subsets* of $\mathbb{R}^p$ is denoted by $\hat{\mathcal{F}}(\mathbb{R}^p)$. For simplicity of notation, we just write $\mathcal{F}$ and $\hat{\mathcal{F}}$, respectively.

In $\hat{\mathcal{F}}$ we introduce an addition $\oplus: \hat{\mathcal{F}} \times \hat{\mathcal{F}} \to \hat{\mathcal{F}}$ and a multiplication by nonnegative real numbers $\cdot: \mathbb{R}_+ \times \hat{\mathcal{F}} \to \hat{\mathcal{F}}$, defined by

$$\forall A, B \in \hat{\mathcal{F}}: \qquad A \oplus B := \mathrm{cl}\,(A + B) = \mathrm{cl}\,\{y \in \mathbb{R}^p |\ \exists a \in A, \exists b \in B : y = a + b\},$$

$$\forall A \in \hat{\mathcal{F}},\ \alpha \geq 0: \qquad \alpha \cdot A := \begin{cases} \{y \in \mathbb{R}^p |\ \exists a \in A : y = \alpha\, a\} & \text{if} \quad \alpha > 0 \\ \{0\} & \text{if} \quad \alpha = 0. \end{cases}$$

The latter distinction of cases ensures that $0 \cdot \emptyset = \{0\}$. As usual, we sometimes write αA instead of $\alpha \cdot A$. For all $A, B, C \in \hat{\mathcal{F}}$ (including the empty set) and all $\alpha, \beta \in \mathbb{R}_+$ the following calculus rules hold true:

(C1) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,
(C2) $\{0\} \oplus A = A$,
(C3) $A \oplus B = B \oplus A$,
(C4) $\alpha\,(\beta\, A) = (\alpha\beta)\, A$,
(C5) $1 \cdot A = A$,
(C6) $\alpha\,(A \oplus B) = \alpha\, A \oplus \alpha\, B$,
(C7) $0 \cdot A = \{0\}$.

This means that $(\hat{\mathcal{F}}, \oplus, \cdot)$ is a conlinear space, see Appendix C. For simplicity, we say $\hat{\mathcal{F}}$ is a conlinear space. The rule (C1) (in particular, if $\mathbb{R}^p$ is replaced by an arbitrary linear topological space) can be shown by Proposition A.4 (vii). The other rules are obvious.

The second distributive law is not valid in $\hat{\mathcal{F}}$. This is clear because the second distributive law would imply that each member of $\hat{\mathcal{F}}$ is a convex subset of $\mathbb{R}^p$.

The set inclusion provides a partial ordering in $\hat{\mathcal{F}}$ such that $(\hat{\mathcal{F}}, \subseteq)$ and $(\hat{\mathcal{F}}, \supseteq)$ are ordered conlinear spaces. Both spaces are order complete, see Appendix B. For instance, for a nonempty subset $\mathcal{A} \subseteq (\hat{\mathcal{F}}, \subseteq)$, the infimum and supremum can be expressed by

$$\operatorname{INF} \mathcal{A} = \bigcap_{A \in \mathcal{A}} A, \qquad \operatorname{SUP} \mathcal{A} = \operatorname{cl} \bigcup_{A \in \mathcal{A}} A.$$

1.2 The space $\hat{\mathcal{C}}$ of closed convex subsets of $\mathbb{R}^p$

This section deals with the space of closed convex subsets of $\mathbb{R}^p$, which plays an important role in the whole work. Many assertions in this section are also valid in a more general setting, i.e., $\mathbb{R}^p$ can be replaced by a more general space.

The space of all *nonempty closed convex subsets* of $\mathbb{R}^p$ is denoted by $\mathcal{C}(\mathbb{R}^p)$ and the space of all *closed convex subsets* of $\mathbb{R}^p$ is denoted by $\hat{\mathcal{C}}(\mathbb{R}^p)$. For abbreviation, we continue to write $\mathcal{C}$ and $\hat{\mathcal{C}}$ instead of $\mathcal{C}(\mathbb{R}^p)$ and $\hat{\mathcal{C}}(\mathbb{R}^p)$, respectively.

In $\hat{\mathcal{C}}$ we introduce the same addition and the same multiplication by nonnegative real numbers as in $\hat{\mathcal{F}}$. Of course, these operations are closed in $\hat{\mathcal{C}}$, i.e., $\hat{\mathcal{C}}$ is a conlinear subspace of $\hat{\mathcal{F}}$. Hence the rules (C1) – (C7) are valid. Additionally, the second distributive law is satisfied, i.e.,

(C8) $\forall A \in \hat{\mathcal{C}}, \forall \alpha, \beta \geq 0: \qquad \alpha\, A \oplus \beta\, A = (\alpha + \beta)\, A.$

Conversely, $\hat{\mathcal{C}}$ is the set of all members of $\hat{\mathcal{F}}$ satisfying (C8). It is easy to see that neither $\hat{\mathcal{C}}$ nor $\mathcal{C}$ is a linear space, since the axiom of existence of an inverse element is violated. Moreover, it is *not possible to embed* $\mathcal{C}$ *into a linear space*. Indeed, assuming there is an injective homomorphism j (an embedding) from $\mathcal{C}$ into a linear space L. For $K := \mathbb{R}^p_+ \in \mathcal{C}$ we have $K = K \oplus K$. Hence $j(K) = j(K) + j(K)$ and $0_L = j(\{0\}) \neq j(K)$. Then there must be an inverse element $l \in L$ of $j(K)$, i.e., $j(K) + l = 0_L$. It follows $0_L = j(K) + l = j(K) + l + j(K) = j(K)$, a contradiction.

Let $K \subseteq \mathbb{R}^p$ be a nonempty closed convex cone. The set $\mathcal{C}_K \subseteq \hat{\mathcal{C}}$ is defined to be the set of all elements $A \in \mathcal{C}$ with $0^+A = K$. The following assertions tells us that $\mathcal{C}_K \subseteq \hat{\mathcal{C}}$ has the structure of a convex cone, even though $\hat{\mathcal{C}}$ is not a linear space. The concept of a cone and a convex set can be analogously defined as in the framework of linear spaces, for the details see Appendix C.

Proposition 1.2.1 *$\mathcal{C}_K$ is a convex cone in $\hat{\mathcal{C}}$.*

Proof. Let $A, B \in \mathcal{C}_K$. It remains to show $0^+(A \oplus B) = K$. This is a consequence of [1, Corollary 9.1.1] if we can verify the following condition: If $z_1 \in 0^+A$ and $z_2 \in 0^+B$ such that $z_1 + z_2 = 0$, then z_1 belongs to the lineality space of A and z_2 belongs to the lineality space of B. Indeed, we have $0^+A = 0^+B = K$ and the lineality spaces of A and B are equal, namely $0^+A \cap (-0^+A) = 0^+B \cap (-0^+B) = K \cap (-K)$. Hence the mentioned condition is satisfied. □

Note that [1, Corollary 9.1.1] also implies that the addition $\oplus$ in $\mathcal{C}_K \subseteq \mathcal{C}$ reduces to the usual Minkowski addition $+$, i.e., the closure operation is superfluous.

Rockafellar [1, Section 39] introduced the concept of *orientation* of convex sets in $\mathbb{R}^p$. A convex set $A \subseteq \mathbb{R}^p$ that is identified with its convex indicator function $\delta(\,\cdot\,|A)$ is said to be *supremum oriented* and a convex set A that is identified with the concave function $-\delta(\,\cdot\,|A)$ is called *infimum oriented*. This concept plays a crucial role in our theory. Thus we introduce the following notation: The space $\hat{\mathcal{C}}^\star$ is defined to be the space $\hat{\mathcal{C}}$ having supremum oriented members. By $\hat{\mathcal{C}}^\diamond$ we denote the same space, but with infimum oriented members. Analogously, we define $\mathcal{C}^\star$, $\mathcal{C}^\diamond$, $\mathcal{C}^\star_K$ and $\mathcal{C}^\diamond_K$. If not stated otherwise, the orientation is not changed while manipulating sets. For instance, this means that $\hat{\mathcal{C}}^\star$ and $\hat{\mathcal{C}}^\diamond$ are conlinear spaces, the recession cone of a supremum (infimum) oriented set is supremum (infimum) oriented, and so on.

The space $\hat{\mathcal{C}}$ is now equipped with one of the reflexive, transitive and antisymmetric relations $\supseteq$ and $\subseteq$. We establish standard relations in dependence on the orientation of the members of the space. In fact, let the standard relation be $\supseteq$ in the space $\hat{\mathcal{C}}^\star$ and $\subseteq$ in $\hat{\mathcal{C}}^\diamond$. Both these standard relations have the meaning of "less or equal". Since sets with opposite orientation will be used in order to define a substitute for inverse elements in the framework of the conlinear space $\hat{\mathcal{C}}$, this identification is useful to obtain the well–known formulas of the context of linear spaces. Further motivation is given later on.

Convention 1.2.2 *Throughout this work we use the following conventions.*

(i) *For $A_1, A_2 \in \mathcal{A} \subseteq \hat{\mathcal{C}}^\star$ we write $A_1 \leq A_2$ instead of $A_1 \supseteq A_2$.*

(ii) *For $A_1, A_2 \in \mathcal{A} \subseteq \hat{\mathcal{C}}^\diamond$ we write $A_1 \leq A_2$ instead of $A_1 \subseteq A_2$.*

(iii) *We write $\mathcal{A} \subseteq \hat{\mathcal{C}}$ if the corresponding assertion is valid for both $\mathcal{A} \subseteq \hat{\mathcal{C}}^\star$ or $\mathcal{A} \subseteq \hat{\mathcal{C}}^\diamond$.*

For the notation $\mathcal{A} \subseteq \mathcal{C}$ *and* $\mathcal{A} \subseteq \mathcal{C}_K$ *we use a convention analogous to* (iii)*.*

This convention means that certain assertions for subsets of $\hat{\mathcal{C}}$ have to be interpreted differently, where the interpretation depends on the orientation, and they are valid for both interpretations. For instance, for $A_1, A_2 \in \mathcal{A} \subseteq \hat{\mathcal{C}}$ we write

$$A_1 \leq A_2 \quad \text{if} \quad \begin{cases} A_1 \supseteq A_2 \text{ in case that } \mathcal{A} \subseteq \hat{\mathcal{C}}^\star \\ A_1 \subseteq A_2 \text{ in case that } \mathcal{A} \subseteq \hat{\mathcal{C}}^\diamond. \end{cases}$$

Of course, $\hat{\mathcal{C}}^\star$ and $\hat{\mathcal{C}}^\diamond$ (equipped with its standard relations) are ordered conlinear spaces.

Proposition 1.2.3 *The spaces* $\hat{\mathcal{C}}^\star$ *and* $\hat{\mathcal{C}}^\diamond$ *are order complete and the infimum and supremum of nonempty sets* $\mathcal{A} \subseteq \hat{\mathcal{C}}^\star$ *and* $\mathcal{B} \subseteq \hat{\mathcal{C}}^\diamond$ *can be expressed as follows:*

$$\begin{aligned} \sup \mathcal{A} &= \textstyle\bigcap_{A \in \mathcal{A}} A, & \inf \mathcal{A} &= \operatorname{cl}\operatorname{conv} \textstyle\bigcup_{A \in \mathcal{A}} A, \\ \sup \mathcal{B} &= \operatorname{cl}\operatorname{conv} \textstyle\bigcup_{B \in \mathcal{B}} B, & \inf \mathcal{B} &= \textstyle\bigcap_{B \in \mathcal{B}} B. \end{aligned}$$

Proof. Follows from the definition. □

In every order complete ordered conlinear space $(\mathcal{Y}, \oplus, \preceq)$ it is evident that

$$\inf \mathcal{A} \oplus \inf \mathcal{B} \preceq \inf(\mathcal{A} + \mathcal{B}) \qquad \text{and} \qquad \sup(\mathcal{A} + \mathcal{B}) \preceq \sup \mathcal{A} \oplus \sup \mathcal{B}, \tag{1.1}$$

where $\mathcal{A} + \mathcal{B} := \{A \oplus B |\ A \in \mathcal{A}, B \in \mathcal{B}\}$. In particular, (1.1) is valid in $\hat{\mathcal{C}}^\star$ and $\hat{\mathcal{C}}^\diamond$. In general, (1.1) does not hold with equality, see Example 1.2.5 below. We next show that, in dependence on the orientation, one inequality in (1.1) is even satisfied with equality. This equality is essential in duality theory, see Remark 3.5.4 below.

Proposition 1.2.4 *For nonempty sets* $\mathcal{A}, \mathcal{B} \subseteq \hat{\mathcal{C}}^\star$ *and* $\bar{\mathcal{A}}, \bar{\mathcal{B}} \subseteq \hat{\mathcal{C}}^\diamond$ *it holds*

$$\inf(\mathcal{A} + \mathcal{B}) = \inf \mathcal{A} \oplus \inf \mathcal{B} \qquad \textit{and} \qquad \sup(\bar{\mathcal{A}} + \bar{\mathcal{B}}) = \sup \bar{\mathcal{A}} \oplus \sup \bar{\mathcal{B}}.$$

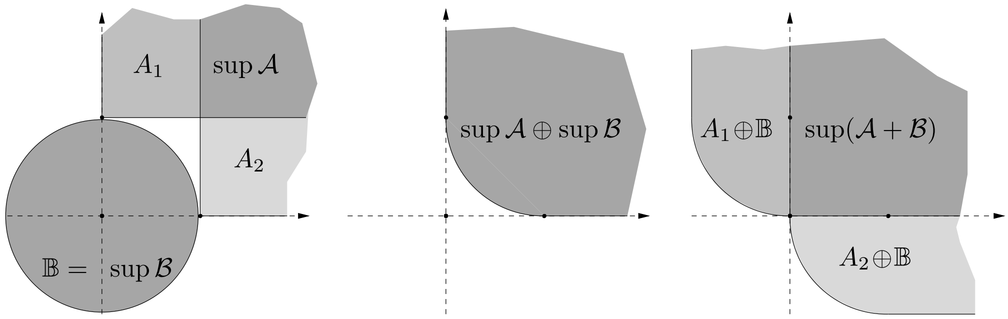
Proof. It holds

$$\begin{aligned} \inf(\mathcal{A} + \mathcal{B}) &\overset{\text{Pr. 1.2.3}}{=} \operatorname{cl}\operatorname{conv} \bigcup_{C \in \mathcal{A} + \mathcal{B}} C = \operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathcal{A},\, B \in \mathcal{B}} (A \oplus B) \\ &\leq \operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathcal{A},\, B \in \mathcal{B}} (A + B) = \operatorname{cl}\operatorname{conv} \left(\bigcup_{A \in \mathcal{A}} A + \bigcup_{B \in \mathcal{B}} B \right) \\ &\overset{\text{Pr. A.4 (ix)}}{=} \operatorname{cl} \left(\operatorname{cl}\operatorname{conv} \bigcup_{A \in \mathcal{A}} A + \operatorname{cl}\operatorname{conv} \bigcup_{B \in \mathcal{B}} B \right) \overset{\text{Pr. 1.2.3}}{=} \inf \mathcal{A} \oplus \inf \mathcal{B}. \end{aligned}$$

By (1.1) (or directly) we deduce equality. The second part is completely analogous. □

In general, the latter assertion is not true for the supremum in $\hat{\mathcal{C}}^\star$ and the infimum in $\hat{\mathcal{C}}^\diamond$, as the following example shows.

Example 1.2.5 Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}^\star(\mathbb{R}^2)$, $\mathcal{A} := \{A_1, A_2\}$, where $A_1 := \{(0,1)^T\} + \mathbb{R}^2_+$, $A_2 := \{(1,0)^T\} + \mathbb{R}^2_+$, and $\mathcal{B} := \{\mathbb{B}\}$. Then, $\sup \mathcal{A} \oplus \sup \mathcal{B} = \big(A_1 \cap A_2\big) \oplus \mathbb{B} = \big\{(1,1)^T\big\} + \mathbb{B} + \mathbb{R}^2_+$. But, $\sup(\mathcal{A} + \mathcal{B}) = \sup\big\{A_1 \oplus \mathbb{B}, A_2 \oplus \mathbb{B}\big\} = \mathbb{R}^2_+$, i.e., $\sup(\mathcal{A}+\mathcal{B}) \neq \sup \mathcal{A} \oplus \sup \mathcal{B}$.



Up to now, all the operations used did not influence the orientation of a set. We express the change of the orientation of a set as follows: Given an oriented set A we denote by $\boxplus A$ the same set, but with opposite orientation. As usual, the negative of a convex set A is defined by

$$-A := \{y \in \mathbb{R}^p |\ -y \in A\}\,.$$

By convention, if A is an oriented set, this operation does not manipulate the orientation of A. In contrast to this, we introduce a second concept of a negative of a convex set which does so. Given an oriented set A, we define $\boxminus A$ being the set $-A$, but with the opposite orientation. Instead of two signs, we now have four signs, namely $+, -, \boxplus, \boxminus$. Obviously, the following assertions hold true:

$$\begin{aligned} A &= \boxplus \boxplus A = \boxminus \boxminus A, & -A &= \boxplus \boxminus A = \boxminus \boxplus A, \\ \boxplus A &= + \boxplus A = - \boxminus A, & \boxminus A &= + \boxminus A = - \boxplus A. \end{aligned}$$

Clearly, an expression is independent of the order of the signs. Note that $\boxplus$ and $\boxminus$ are signs but not operations. This means, the addition of contrarily oriented elements is not defined. Nevertheless, we write $A \boxplus B := A + (\boxplus B)$ and $A \boxminus B := A + (\boxminus B)$, if these expressions are defined, i.e., A and B are contrarily oriented. For a given set $\mathcal{A} \subseteq \hat{\mathcal{C}}$ we set $\boxminus \mathcal{A} := \{\boxminus A |\ A \in \mathcal{A}\}$. It can be easily shown (Proposition 1.2.3) that

$$\boxminus \inf \mathcal{A} = \sup \boxminus \mathcal{A} \quad \text{and} \quad \boxminus \sup \mathcal{A} = \inf \boxminus \mathcal{A}. \tag{1.2}$$

Further motivation for the usage of the sign $\boxminus$ is given later on. For instance, we define convex and concave functions with values in $\hat{\mathcal{C}}$ and we obtain the well–known convexity–concavity dualism for such functions, i.e., a function $f : X \to \hat{\mathcal{C}}$ is convex if and only if $\boxminus f : X \to \hat{\mathcal{C}}$ is concave. In the next section we embed certain subsets of $\mathcal{C}^\star$ and $\mathcal{C}^\diamond$ in a (the same) linear space. There, $\boxminus A$ gets the meaning of an "inverse element" of A with respect to this embedding.

1.3 Embedding subsets of $\hat{\mathcal{C}}$ into a linear space

Embedding of spaces of convex sets into linear spaces was already investigated by Rådström [67], and as remarked there, the idea seems to go back to investigations of Brunn [15] in 1889. A nice overview over the literature in this field can be found in [3]. In the literature, compactness or boundedness assumptions to the sets are usually supposed. In this section, we investigate spaces of unbounded convex sets, too.

The aim of this section is to embed the convex cone $\mathcal{C}_K \subseteq \mathcal{C}$ into a partially ordered linear space. In dependence on the orientation of the members of $\mathcal{C}_K$ we use different embedding maps. This procedure allows us to re–interpret the inverse element of the embedding map's image of a member of $\mathcal{C}_K$ as an element of $\mathcal{C}_{-K}$ having opposite orientation.

The following lemma is a refinement of [1, Theorem 13.1]. It is shown that only the set $\operatorname{ri}(0^+A)^\circ := \operatorname{ri}((0^+A)^\circ)$ (instead of the whole space $\mathbb{R}^p$) is essential for the description of a nonempty closed convex set by its support function.

Lemma 1.3.1 *Let $A, B \subseteq \mathbb{R}^p$ be a nonempty closed and convex. Then,*

(i) $\operatorname{ri}(0^+A)^\circ \subseteq \operatorname{dom}\delta^*(\,\cdot\,|\,A\,) \subseteq (0^+A)^\circ$,

(ii) $A = \displaystyle\bigcap_{y^*\in \operatorname{ri}(0^+A)^\circ} \{y \in \mathbb{R}^p|\ \langle y^*, y\rangle \le \delta^*(y^*|\,A\,)\}$,

(iii) $A \subseteq B \quad \Leftrightarrow \quad \forall y^* \in \operatorname{ri}(0^+B)^\circ:\ \delta^*(y^*|A) \le \delta^*(y^*|B)$,

(iv) *If 0^+B is pointed, then*

$$A \subseteq B \quad \Leftrightarrow \quad \forall y^* \in \mathbb{R}^p \setminus \operatorname{rb}(0^+A)^\circ:\ \delta^*(y^*|A) \le \delta^*(y^*|B).$$

Proof. (i) As a consequence of [1, Theorem 14.2] we have $\operatorname{cl}\operatorname{dom}\delta^*(\,\cdot\,|\,A\,) = (0^+A)^\circ$, compare [36, Theorem 2.2.4], too. Together with [1, Theorem 6.3] this yields $\operatorname{ri}(0^+A)^\circ = \operatorname{ri}\operatorname{cl}\operatorname{dom}\delta^*(\,\cdot\,|\,A\,) \subseteq \operatorname{dom}\delta^*(\,\cdot\,|\,A\,) \subseteq (0^+A)^\circ$.

(ii) From [1, Theorem 13.1] and (i) we obtain

$$A = \bigcap_{y^*\in\mathbb{R}^p} \left\{y \in \mathbb{R}^p|\ \langle y^*, y\rangle \le \delta^*(y^*|\,A\,)\right\} = \bigcap_{y^*\in(0^+A)^\circ} \left\{y \in \mathbb{R}^p|\ \langle y^*, y\rangle \le \delta^*(y^*|\,A\,)\right\}.$$

It remains to show

$$Y_1 := \bigcap_{y^*\in(0^+A)^\circ} \left\{y \in \mathbb{R}^p|\ \langle y^*, y\rangle \le \delta^*(y^*|\,A\,)\right\} = \bigcap_{y^*\in\operatorname{ri}(0^+A)^\circ} \left\{y \in \mathbb{R}^p|\ \langle y^*, y\rangle \le \delta^*(y^*|\,A\,)\right\} =: Y_2.$$

The inclusion $Y_1 \subseteq Y_2$ is obvious. In order to show $Y_2 \subseteq Y_1$ let $y \in Y_2$ be arbitrarily chosen. It holds $\langle y^*, y\rangle \le \delta^*(y^*|\,A\,)$ for all $y^* \in \operatorname{ri}(0^+A)^\circ$. Let $\bar{y}^* \in (0^+A)^\circ$ and $y^* \in \operatorname{ri}(0^+A)^\circ$, then $\lambda\bar{y}^* + (1-\lambda)y^* \in \operatorname{ri}(0^+A)^\circ$ for all $\lambda \in [0,1)$ (compare [1, Theorem 6.1]). It follows

$$\langle \lambda\bar{y}^* + (1-\lambda)y^*, y\rangle \le \delta^*(\lambda\bar{y}^* + (1-\lambda)y^*|\,A\,) \le \lambda\,\delta^*(\bar{y}^*|\,A\,) + (1-\lambda)\delta^*(y^*|\,A\,).$$

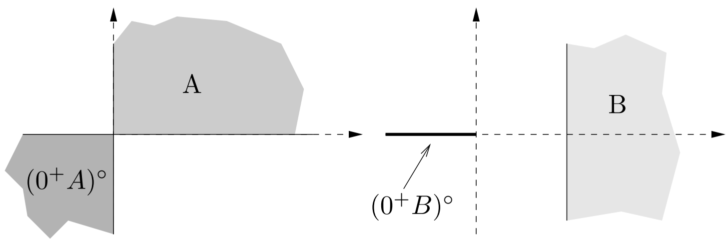
By virtue of (i) we deduce that $\delta^*(y^*|\,A) < +\infty$. Letting $\lambda \to 1$ we obtain $\langle \bar{y}^*, y\rangle \le \delta^*(\bar{y}^*|\,A)$, i.e., $y \in Y_1$.

(iii) It remains to show "$\Leftarrow$". Let $y \in A$ be given. Then we have $\langle y^*, y\rangle \le \delta^*(y^*|A)$ for all $y^* \in \mathbb{R}^p$ and so $\langle y^*, y\rangle \le \delta^*(y^*|B)$ for all $y^* \in \operatorname{ri}(0^+B)^\circ$. From (ii) we obtain $y \in B$.

(iv) Again, it remains to show "$\Leftarrow$". Let $y^* \in \mathbb{R}^p \setminus (0^+A)^\circ$. By (i) we obtain $+\infty = \delta^*(y^*|A) \le \delta^*(y^*|B)$ and hence $y^* \notin \operatorname{ri}(0^+B)^\circ$. This means we have $\operatorname{ri}(0^+B)^\circ \subseteq (0^+A)^\circ$. Since 0^+B is pointed, we deduce that $\operatorname{ri}(0^+B)^\circ = \operatorname{int}(0^+B)^\circ \subseteq \operatorname{int}(0^+A)^\circ = \operatorname{ri}(0^+A)^\circ \subseteq \mathbb{R}^p \setminus \operatorname{rb}(0^+A)^\circ$. From (iii) we deduce $A \subseteq B$. □

The following example shows that the pointedness assumption in statement (iv) of the preceding lemma cannot be omitted.

Example 1.3.2 Let $A = \mathbb{R}^2_+$ and $B = \left\{y \in \mathbb{R}^2 |\; y_1 \ge 1\right\}$. Then we have $(0^+B)^\circ \subseteq \operatorname{rb}(0^+A)^\circ$ and, by (i), the right–hand side of (iv) is satisfied. But $A \not\subseteq B$.



The preceeding lemma is an essential tool for the proof of duality assertions for $\hat{\mathcal{C}}$–valued functions. Furthermore, in the next chapter it is used to characterize a new type of convergence of convex sets by convergence of support functions. As a byproduct, we obtain the statement that $\mathcal{C}^\star_K$ and $\mathcal{C}^\diamond_K$ can be embedded into a linear space. We first give an equivalent characterization of the ordered conlinear spaces $\mathcal{C}^\star_K$ and $\mathcal{C}^\diamond_K$ (where K is the neutral element, i.e., they are cones but not conlinear subspaces of $\mathcal{C}^\star$ and $\mathcal{C}^\diamond$) by certain spaces of real–valued functions having the same domain (which depends on K). Then, the possibility of embedding into a linear space is obvious.

Let $\Gamma^\star_K$ be the space of all positively homogeneous concave functions from $\operatorname{ri} K^\circ$ into $\mathbb{R}$ and let $\Gamma^\diamond_K$ be the space of all positively homogeneous convex functions from $\operatorname{ri} K^\circ$ into $\mathbb{R}$. The spaces $\Gamma^\star_K$ and $\Gamma^\diamond_K$ are conlinear spaces with respect to the addition and multiplication by nonnegative real numbers, being defined pointwise using the corresponding operation in $\mathbb{R}$. Moreover, $\Gamma^\star_K$ and $\Gamma^\diamond_K$ equipped with the ordering relation $\le$, which is defined pointwise using the usual $\le$ relation in $\mathbb{R}$, are ordered conlinear spaces.

Theorem 1.3.3 *Let $K \subseteq \mathbb{R}^p$ be a nonempty closed convex cone. Then,*

(i) *There exists a bijective map $j^\star : \mathcal{C}^\star_K \to \Gamma^\star_K$ such that for all $A, B \in \mathcal{C}^\star_K$ and all positive real numbers $\alpha > 0$ it holds*

(a) $j^\star(A + B) = j^\star(A) + j^\star(B)$, (b) $j^\star(\alpha A) = \alpha j^\star(A)$,

(c) $j^\star(K) = 0_{\Gamma^\star_K}$, (d) $A \supseteq B \;\Leftrightarrow\; j^\star(A) \le j^\star(B)$.

(ii) *There exists a bijective map* $j^\diamond : \mathcal{C}^\diamond_{-K} \to \Gamma^\diamond_K$ *such that for all* $A, B \in \mathcal{C}^\diamond_{-K}$ *and all positive real numbers* $\alpha > 0$ *it is true that*

(a) $j^\diamond(A + B) = j^\diamond(A) + j^\diamond(B)$, (b) $j^\diamond(\alpha A) = \alpha j^\diamond(A)$,

(c) $j^\diamond(-K) = 0_{\Gamma^\diamond_K}$, (d) $A \subseteq B \;\Leftrightarrow\; j^\diamond(A) \le j^\diamond(B)$.

Proof. (i) Consider the map $j^\star$, assigning to every $A \in \mathcal{C}^\star_K$ the negative support function of the set $A \subseteq \mathbb{R}^p$, restricted to the set $\operatorname{ri} K^\circ \subseteq \mathbb{R}^p$. More precisely, $\gamma_A = j^\star(A)$ is defined by $\gamma_A : \operatorname{ri} K^\circ \to \mathbb{R} \cup \{-\infty, +\infty\}$, $\gamma_A(y^*) := -\delta^*(y^* |\, A\,)$. The map $j^\star$ is a function from $\mathcal{C}^\star_K$ into $\Gamma^\star_K$. Indeed, let $A \in \mathcal{C}^\star_K$. Since A is nonempty, we have $\delta^*(y^* |\, A\,) > -\infty$ for all $y^* \in \mathbb{R}^p$. With the aid of Lemma 1.3.1 (i) we obtain $\delta^*(y^* |\, A\,) < +\infty$ for all $y^* \in \operatorname{ri} K^\circ$. Hence $\gamma_A = j^\star(A)$ only attains values in $\mathbb{R}$. Since support functions are sublinear and $\operatorname{ri} K^\circ$ is a convex cone, $\gamma_A = j^\star(A)$ is positively homogeneous and concave on $\operatorname{ri} K^\circ$.

Lemma 1.3.1 (ii) yields that $j^\star : \mathcal{C}^\star_K \to \Gamma^\star_K$ is injective.

We next show that $j^\star : \mathcal{C}^\star_K \to \Gamma^\star_K$ is surjective. Given some $\gamma \in \Gamma^\star_K$ we define a function $d : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ by

$$d(y^*) := \begin{cases} -\gamma(y^*) & \text{if} \quad y^* \in \operatorname{ri} K^\circ \\ +\infty & \text{else.} \end{cases}$$

It is easy to see that d is convex, positively homogeneous, not identically $+\infty$ and never $-\infty$. With the aid of [1, Corollary 13.2.1] we conclude that $\operatorname{cl} d$ is the support function of the nonempty closed convex set

$$A_\gamma := \bigcap_{y^* \in \operatorname{ri} K^\circ} \{y \in \mathbb{R}^p |\; \langle y^*, y\rangle \le d(y^*)\} = \bigcap_{y^* \in \operatorname{ri} K^\circ} \{y \in \mathbb{R}^p |\; \langle y^*, y\rangle \le -\gamma(y^*)\}.$$

Applying [1, Corollary 8.3.3], taking into account the considerations in [1, page 62] and applying Lemma 1.3.1 (ii) we obtain

$$0^+ A_\gamma = \bigcap_{y^* \in \operatorname{ri} K^\circ} 0^+ \{y \in \mathbb{R}^p |\; \langle y^*, y\rangle \le d(y^*)\} = \bigcap_{y^* \in \operatorname{ri} K^\circ} \{y \in \mathbb{R}^p |\; \langle y^*, y\rangle \le 0\} = K.$$

By definition, we have $j^\star(A_\gamma)(y^*) = -\operatorname{cl} d(y^*)$ for all $y^* \in \operatorname{ri} K^\circ$. With the aid of [1, Theorem 7.4] we have $\operatorname{cl} d(y^*) = d(y^*)$ for all $y^* \in \operatorname{ri} K^\circ$. Hence $j^\star(A_\gamma) = \gamma$.

(i)(a) and (i)(b) follow from elementary properties of the supremum in $\mathbb{R}$, compare [1, page 113], too. (i)(c) follows from the definition of the polar cone and of the support function. (i)(d) is a consequence of [1, Corollary 13.1.1].

(ii) Define $j^\diamond(A) := -j^\star(\boxminus A)$ and use (i). □

Let $\hat{\mathcal{C}}^\star_K := \mathcal{C}^\star_K \cup \{\emptyset\} \subseteq \hat{\mathcal{C}}^\star$ and let $\hat{\Gamma}^\star_K := \Gamma^\star_K \cup \{+\infty_K\}$, where $+\infty_K$ is an abbreviation for the function defined on $\operatorname{ri} K^\circ$ and being identically $+\infty$. Likewise, we define $\hat{\mathcal{C}}^\diamond_K := \mathcal{C}^\diamond_K \cup \{\emptyset\} \subseteq \hat{\mathcal{C}}^\diamond$ and $\hat{\Gamma}^\diamond_K := \Gamma^\diamond_K \cup \{-\infty_K\}$.

Corollary 1.3.4 *Theorem 1.3.3 remains valid if $\mathcal{C}_K^\star$, $\mathcal{C}_{-K}^\diamond$, $\Gamma_K^\star$ and $\Gamma_K^\diamond$ are replaced by $\hat{\mathcal{C}}_K^\star$, $\hat{\mathcal{C}}_{-K}^\diamond$, $\hat{\Gamma}_K^\star$ and $\hat{\Gamma}_K^\diamond$, respectively.*

Proof. Extend the isomorphisms $j^\star$ and $j^\diamond$ of Theorem 1.3.3 using the conventions $j^\star(\emptyset) = +\infty_K$ and $j^\diamond(\emptyset) = -\infty_K$, respectively. □

Let Γ_K be the space of all positively homogeneous (and not necessarily convex or concave) functions $\gamma : \operatorname{ri} K^\circ \to \mathbb{R}$. Let Γ_K be equipped with an addition and a scalar multiplication, defined pointwise using the corresponding operation in $\mathbb{R}$, and with an ordering relation $\leq$, defined pointwise using the usual $\leq$ relation in $\mathbb{R}$. Then, the space Γ_K is a partially ordered linear space. Theorem 1.3.3 yields that $(\mathcal{C}_K^\star, \supseteq)$ and $(\mathcal{C}_{-K}^\diamond, \subseteq)$ are isomorphic to convex cones in the partially ordered linear space $(\Gamma_K, \leq)$. Let $j^\star : \mathcal{C}_K^\star \to \Gamma_K$ be the injective homomorphism which embeds $\mathcal{C}_K^\star$ into Γ_K and let $j^\diamond : \mathcal{C}_{-K}^\diamond \to \Gamma_K$ be analogously defined. Then it easily follows that

$$\forall A \in \mathcal{C}_K^\star : \;\; j^\star(A) + j^\diamond(\boxminus A) = 0, \qquad \forall A \in \mathcal{C}_{-K}^\diamond : \;\; j^\diamond(A) + j^\star(\boxminus A) = 0. \tag{1.3}$$

In this sense, $\boxminus A$ can be regarded as the "inverse element" of a nonempty closed convex set A. However, this does not imply that $\mathcal{C}_K^\star \cup \mathcal{C}_{-K}^\diamond$ is a linear space, because it is not a conlinear space.

Of course, $\hat{\mathcal{C}}_K^\star$ and $\hat{\mathcal{C}}_K^\diamond$ can also be embedded into a smaller linear space than Γ_K, for instance in the space of real–valued positively homogeneous DC-functions (e.g. [81]) being defined on $\operatorname{ri} K^\circ$. However, the main advantage of Γ_K is that infimum and supremum in Γ_K can be described pointwise by the infimum and supremum in $\mathbb{R}$. This means that the space $(\Gamma_K, \leq)$ is Dedekind complete and for nonempty subsets $\mathcal{A} \subseteq \Gamma_K$ it holds

$$\begin{aligned}
\mathcal{A} \text{ bounded above} \quad &\Rightarrow \quad \forall y^* \in \operatorname{ri} K^\circ : \;\; \big(\sup_{\gamma \in \mathcal{A}} \gamma\big)(y^*) = \sup_{\gamma \in \mathcal{A}} \big(\gamma(y^*)\big), \\
\mathcal{A} \text{ bounded below} \quad &\Rightarrow \quad \forall y^* \in \operatorname{ri} K^\circ : \;\; \big(\inf_{\gamma \in \mathcal{A}} \gamma\big)(y^*) = \inf_{\gamma \in \mathcal{A}} \big(\gamma(y^*)\big).
\end{aligned}$$

Let $\mathcal{A}, \mathcal{B}$ be nonempty subsets of $\mathcal{C}_K$ such that $\mathcal{A}$ is bounded below and $\mathcal{B}$ bounded above (see Convention 1.2.2). Further, let j be the map which embeds $\mathcal{C}_K$ into the linear space Γ_K (where j stands for $j^\star$ or $j^\diamond$ as defined above). Then, from Proposition B.1 we conclude that

$$j(\inf \mathcal{A}) \leq \inf_{A \in \mathcal{A}} j(A) \qquad \text{and} \qquad j(\sup \mathcal{B}) \geq \sup_{B \in \mathcal{B}} j(B). \tag{1.4}$$

The following proposition shows that, in dependence of the orientation, one assertion in (1.4) even holds with equality.

Proposition 1.3.5 *Let $\mathcal{A} \subseteq \mathcal{C}_K^\star$ be nonempty and bounded below and let $\mathcal{B} \subseteq \mathcal{C}_K^\diamond$ be nonempty and bounded above. Then,*

$$j^\star(\inf \mathcal{A}) = \inf_{A \in \mathcal{A}} j^\star(A) \qquad \text{and} \qquad j^\diamond(\sup \mathcal{B}) = \sup_{B \in \mathcal{B}} j^\diamond(B).$$

Proof. Taking into account Proposition B.1, it remains to show that $\inf_{A\in\mathcal{A}} j^\star(A) \in \Gamma^\star_K$ and $\sup_{B\in\mathcal{B}} j^\diamond(B) \in \Gamma^\diamond_K$. This is true because the pointwise infimum (supremum) over a set of concave (convex) functions is concave (convex), too. □

The following example shows that, in general, (1.4) is not satisfied with equality.

Example 1.3.6 Let $\mathcal{A} = \{A_1, A_2\} \subseteq \hat{\mathcal{C}}^\star(\mathbb{R}^2)$ as in Example 1.2.5. Then, $j^\star(\sup\mathcal{A})$, $j^\star(A_1)$ and $j^\star(A_2)$ are functions from $-\operatorname{int}\mathbb{R}^2_+$ into $\mathbb{R}$, given by

$$j^\star(\sup\mathcal{A})(y^*) = -\delta^*\big(y^*\big|A_1 \cap A_2\big) = -(y_1^* + y_2^*),$$

$$j^\star(A_1)(y^*) = -\delta^*\big(y^*\big|A_1\big) = -y_2^*, \qquad j^\star(A_2)(y^*) = -\delta^*\big(y^*\big|A_2\big) = -y_1^*.$$

Hence, $\sup_{A\in\mathcal{A}} j^\star(A) \neq j^\star(\sup\mathcal{A})$.

The boundedness assumptions in (1.4) are very restrictive, because the infimum with respect to $\hat{\mathcal{C}}^\star$ over a subset $\mathcal{A} \subseteq \hat{\mathcal{C}}^\star_K \subseteq \hat{\mathcal{C}}^\star$ often has a recession cone larger than K. The following result is valid without boundedness assumptions. It is used to prove duality assertions for $\hat{\mathcal{C}}$–valued functions.

Proposition 1.3.7 *Let $\mathcal{A} := \{A_i \in \hat{\mathcal{C}}^\star |\; i \in I\} \subseteq \hat{\mathcal{C}}^\star$, where I is an arbitrary index set. Then it holds*

$$\forall y^* \in \mathbb{R}^p:\; -\delta^*\big(y^*|\inf_{i\in I} A_i\big) = \inf_{i\in I}\big\{-\delta^*\big(y^*|A_i\big)\big\},$$

$$\forall y^* \in \mathbb{R}^p:\; -\delta^*\big(y^*|\sup_{i\in I} A_i\big) \geq \sup_{i\in I}\big\{-\delta^*\big(y^*|A_i\big)\big\}.$$

Proof. Without loss of generality we can assume $\mathcal{A} \subseteq \mathcal{C}^\star$. We have $\inf_{i\in I} A_i = \operatorname{cl}\operatorname{conv}\bigcup_{i\in I} A_i$. Hence, the first assertion follows from the first part of [1, Corollary 16.5.1].

Since $\sup_{i\in I} A_i = \bigcap_{i\in I} A_i$, the second part of [1, Corollary 16.5.1] yields $\delta^*\big(\cdot\,|\sup_{i\in I} A_i\big) = \operatorname{cl}\operatorname{conv}\big\{\delta^*\big(\cdot\,|A_i\big)\,|\, i \in I\big\}$, where the convex hull of a collection of functions is defined as the convex hull of the pointwise infimum of the collection, i.e., $\operatorname{cl}\operatorname{conv}\big\{\delta^*\big(\cdot\,|A_i\big)\,|\, i \in I\big\} = \operatorname{cl}\operatorname{conv}\inf_{i\in I}\delta^*\big(\cdot\,|A_i\big)$, compare [1, page 37]. It follows that $\delta^*\big(\cdot\,|\sup_{i\in I} A_i\big) \leq \inf_{i\in I}\delta^*\big(\cdot\,|A_i\big)$ which proves the second assertion. □

Chapter 2

Convergence of closed convex sets

In this chapter, we introduce a convergence concept for closed convex subsets of $\mathbb{R}^p$, which seems to be new. This convergence is called $\mathcal{C}$*–convergence*. It is defined by appropriate notions of upper and lower limits. We compare this convergence with the well–known Painlevé–Kuratowski convergence [47], [4], [68] as well as with scalar convergence (i.e., the pointwise convergence of support functions) of convex sets [82], [69], [70], [74]. In Chapter 3, $\mathcal{C}$–convergence is used to define a meaningful concept of "lower semi–continuous hull" of a $\hat{\mathcal{C}}$–valued functions. Although $\mathcal{C}$–convergence does not coincide with scalar convergence, it can be equivalently described by convergence of support functions. In fact, we show that a sequence $\{A_n\}_{n\in\mathbb{N}}$ $\mathcal{C}$–converges to $\bar{A}$ if and only if the corresponding support functions converge pointwise, except at relative boundary points of the domain the support function of $\bar{A}$, to the support function of $\bar{A}$. This characterization of $\mathcal{C}$–convergence is used in the next chapter in order to prove a biconjugation theorem for $\hat{\mathcal{C}}$–valued functions.

This chapter is organized as follows. In Section 2.1 we recall some basic properties of the well–known Painlevé–Kuratowski convergence. As mentioned in [68, page 111], the natural setting for the study of Painlevé–Kuratowski convergence is the space $\hat{\mathcal{F}}$ of closed subsets of $\mathbb{R}^p$. Section 2.2 is devoted to $\mathcal{C}$–convergence. We proceed analogously to the previous section by introducing upper and lower limits. However, in contrast to Painlevé–Kuratowski convergence, $\mathcal{C}$–convergence is adapted to the space $\hat{\mathcal{C}}$ of closed convex subsets of $\mathbb{R}^p$. We show that, under certain assumptions, our new concepts of upper and lower limits coincide with the well–known concepts related to Painlevé–Kuratowski convergence. In Section 2.3 we investigate the relationship between $\mathcal{C}$–convergence and scalar convergence.

2.1 Painlevé–Kuratowski convergence

In this section, we summarize some results about Painlevé–Kuratowski convergence in the space $\hat{\mathcal{F}}$ of closed subsets of $\mathbb{R}^p$. Our main reference is the book of Rockafellar/Wets [68]. We frequently use the following notation of [68]:

$$\mathcal{N}_\infty := \{N \subseteq \mathbb{N} |\ \mathbb{N} \setminus N \text{ finite}\} \quad \text{and} \quad \mathcal{N}_\infty^\# := \{N \subseteq \mathbb{N} |\ N \text{ infinite}\}.$$

Let $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^p$ be a sequence. Sometimes we write $\bar{y} = \lim_{n\in\mathbb{N}} y_n$ or $y_n \to \bar{y}$ instead of $\bar{y} = \lim_{n\to\infty} y_n$. Moreover, we write $\bar{y} = \lim_{n\in N} y_n$ or $y_n \xrightarrow{N} \bar{y}$ in the case of convergence of a subsequence designated by an index set $N \in \mathcal{N}_\infty^\#$ or $N \in \mathcal{N}_\infty$. It is clear that every subsequence of $\{y_n\}_{n\in\mathbb{N}}$ can be expressed by $\{y_n\}_{n\in N}$, where N belongs to $\mathcal{N}_\infty^\#$. In case of $N \in \mathcal{N}_\infty$, $\{y_n\}_{n\in N}$ denotes a subsequence of $\{y_n\}_{n\in\mathbb{N}}$ that arises by omitting finitely many members. For example, a subsequence of a subsequence $\{y_n\}_{n\in N}$ ($N \in \mathcal{N}_\infty^\#$) can be expressed by some $\bar{N} \in \mathcal{N}_\infty^\#$ with $\bar{N} \subseteq N$ as $\{y_n\}_{n\in\bar{N}}$. An analogous notation is used for sequences in other spaces, where the limit has to be defined appropriately.

Definition 2.1.1 ([68]) *For a sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{F}}$ the* outer limit *is the set*

$$\operatorname*{LIM\,SUP}_{n\to\infty} A_n := \left\{ y \in \mathbb{R}^p |\ \exists N \in \mathcal{N}_\infty^\#,\ \forall n \in N,\ \exists y_n \in A_n : y_n \xrightarrow{N} y \right\}.$$

The inner limit *of a sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{F}}$ is the set*

$$\operatorname*{LIM\,INF}_{n\to\infty} A_n := \left\{ y \in \mathbb{R}^p |\ \exists N \in \mathcal{N}_\infty,\ \forall n \in N,\ \exists y_n \in A_n : y_n \xrightarrow{N} y \right\}.$$

The limit of the sequence exists if the outer and inner limits coincide. Then we write

$$\operatorname*{LIM}_{n\to\infty} A_n = \operatorname*{LIM\,SUP}_{n\to\infty} A_n = \operatorname*{LIM\,INF}_{n\to\infty} A_n.$$

In [68], the closedness of the members of the sequence is not supposed a priori. However, as it can be seen in [68, Proposition 4.4], the outer and inner limits only depend on the closure of the sequence's members.

In contrast to [68], we use capital letters in the notation of the (outer and inner) limit. This is because the notation with small letters is reserved for the (upper and lower) limit in the space $\hat{\mathcal{C}}$ to be defined later on. The following characterization of outer and inner limits is very important for the considerations in the next section.

Proposition 2.1.2 ([68]) *For a sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{F}}$ it holds*

$$\operatorname*{LIM\,SUP}_{n\to\infty} A_n = \bigcap_{N\in\mathcal{N}_\infty} \operatorname{cl} \bigcup_{n\in N} A_n, \qquad \operatorname*{LIM\,INF}_{n\to\infty} A_n = \bigcap_{N\in\mathcal{N}_\infty^\#} \operatorname{cl} \bigcup_{n\in N} A_n.$$

Proof. See [68, Exercise 4.2.(b)]. □

We observe that the characterization in the preceeding proposition can be expressed by the supremum and infimum in the space $\hat{\mathcal{F}}$. According to the definition of outer and inner limits of [68] the relation $\subseteq$ has the meaning of "less or equal", i.e., the infimum is related to the intersection and the supremum is related to the closure of the union. Hence, we can write

$$\operatorname*{LIM\,SUP}_{n\to\infty} A_n = \operatorname*{INF}_{N\in\mathcal{N}_\infty} \operatorname*{SUP}_{n\in N} A_n, \qquad \operatorname*{LIM\,INF}_{n\to\infty} A_n = \operatorname*{INF}_{N\in\mathcal{N}_\infty^\#} \operatorname*{SUP}_{n\in N} A_n. \tag{2.1}$$

This characterization is the starting point in the next section.

2.2 $\mathcal{C}$–Convergence

In the preceeding section, we have seen that Painlevé–Kuratowski convergence can be expressed with the aid of the supremum and infimum in the space $\hat{\mathcal{F}}$. Likewise, we introduce corresponding concepts in the space $\hat{\mathcal{C}}$. This means, the supremum and infimum notions in the following definition are based on the considerations in Section 1.2.

Definition 2.2.1 *The* upper limit *and the* lower limit *of a sequence* $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star$ *are defined, respectively, by*

$$\limsup_{n\to\infty} A_n := \sup_{N\in\mathcal{N}_\infty^\#} \inf_{n\in N} A_n \quad \text{and} \quad \liminf_{n\to\infty} A_n := \sup_{N\in\mathcal{N}_\infty} \inf_{n\in N} A_n.$$

The upper limit *and the* lower limit *of a sequence* $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\diamond$ *are defined, respectively, by*

$$\limsup_{n\to\infty} A_n := \inf_{N\in\mathcal{N}_\infty} \sup_{n\in N} A_n \quad \text{and} \quad \liminf_{n\to\infty} A_n := \inf_{N\in\mathcal{N}_\infty^\#} \sup_{n\in N} A_n.$$

In the following, we frequently use Convention 1.2.2. For a sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}$ it is evident that

$$\limsup_{n\to\infty} -A_n = -\limsup_{n\to\infty} A_n, \qquad \liminf_{n\to\infty} -A_n = -\liminf_{n\to\infty} A_n,$$
$$\limsup_{n\to\infty} \boxplus A_n = \boxplus \liminf_{n\to\infty} A_n, \qquad \limsup_{n\to\infty} \boxminus A_n = \boxminus \liminf_{n\to\infty} A_n.$$

In the space $\mathbb{R}\cup\{-\infty,+\infty\}$, we only know the latter equality (with the usual "$-$" instead of "$\boxminus$"), where the first two equalities are also valid in the framework of Painlevé–Kuratowski convergence.

Proposition 2.2.2 *Let* $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}$ *be a sequence. Then,*

$$\liminf_{n\to\infty} A_n \le \limsup_{n\to\infty} A_n.$$

Proof. Since $\mathcal{N}_\infty \subseteq \mathcal{N}_\infty^\#$, this follows from the definition. □

The upper and lower limits can be used to introduce a convergence concept in $\hat{\mathcal{C}}$.

Definition 2.2.3 *A sequence* $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}$ *is said to be* $\mathcal{C}$*–convergent to some* $A \in \hat{\mathcal{C}}$ *(with the same orientation as the sequence) if*

$$\liminf_{n\in\mathbb{N}} A_n = \limsup_{n\in\mathbb{N}} A_n = A.$$

Then, the limit A *is denoted by* $\lim_{n\in\mathbb{N}} A_n$ *and we write* $A_n \to A$ *or* $A_n \xrightarrow{\mathcal{C}} A$.

Proposition 2.2.4 (upper vs. outer limit) *Let* $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star$ *and* $\{B_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\diamond$*, where* $\hat{\mathcal{C}}^\star$ *and* $\hat{\mathcal{C}}^\diamond$ *are considered to be subsets of* $\hat{\mathcal{F}}$. *Then it holds*

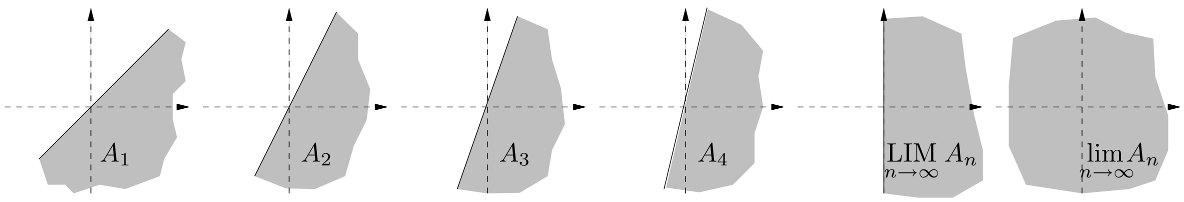
$$\begin{aligned} \operatorname*{LIM\,SUP}_{n\to\infty} A_n &\subseteq \liminf_{n\to\infty} A_n, & \operatorname*{LIM\,SUP}_{n\to\infty} B_n &\subseteq \limsup_{n\to\infty} B_n,\\ \operatorname*{LIM\,INF}_{n\to\infty} A_n &\subseteq \limsup_{n\to\infty} A_n, & \operatorname*{LIM\,INF}_{n\to\infty} B_n &\subseteq \liminf_{n\to\infty} B_n. \end{aligned}$$

Proof. Rely on the definition of the upper and lower limits and (2.1). □

The following examples show that (in case of existence) the limit with respect to Painlevé–Kuratowski convergence can be different from the limit with respect to $\mathcal{C}$–convergence. It can be seen that neither $\mathcal{C}$–convergence implies Painlevé–Kuratowski convergence nor vice versa.

Example 2.2.5 Painlevé–Kuratowski convergence does not coincide with $\mathcal{C}$–convergence: (i) Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star(\mathbb{R}^2)$ be defined by $A_n := \{(y_1, y_2) \in \mathbb{R}^2 |\; y_2 \leq n y_1\}$. By an easy calculation it can be seen that

$$\{(y_1, y_2) \in \mathbb{R}^2 |\; 0 \leq y_1\} = \operatorname*{LIM}_{n\to\infty} A_n \neq \lim_{n\to\infty} A_n = \mathbb{R}^2.$$



(ii) Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star(\mathbb{R}^2)$ be defined by

$$A_n := \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 |\; y_2 \leq n y_1\} & \text{if} \quad n \text{ is odd} \\ \mathbb{R}^2 & \text{if} \quad n \text{ is even.} \end{cases}$$

In view of (i), it can be easily seen that $\lim_{n\to\infty} A_n = \mathbb{R}^2$, but $\operatorname{LIM}_{n\to\infty} A_n$ does not exist. In fact, we have $\operatorname{LIM\,SUP}_{n\to\infty} A_n = \mathbb{R}^2$, but $\operatorname{LIM\,INF}_{n\to\infty} A_n = \{(y_1, y_2) \in \mathbb{R}^2 |\; 0 \leq y_1\}$. (iii) Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star(\mathbb{R}^2)$ be defined by

$$A_n := \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 |\; y_2 \leq n y_1\} & \text{if} \quad n \text{ is odd} \\ \{(y_1, y_2) \in \mathbb{R}^2 |\; 0 \leq y_1\} & \text{if} \quad n \text{ is even.} \end{cases}$$

By (i), it can be easily seen that $\operatorname{LIM}_{n\to\infty} A_n = \{(y_1, y_2) \in \mathbb{R}^2 |\; 0 \leq y_1\}$, but $\lim_{n\to\infty} A_n$ does not exist. In fact, we have $\limsup_{n\to\infty} A_n = \{(y_1, y_2) \in \mathbb{R}^2 |\; 0 \leq y_1\}$, but $\liminf_{n\to\infty} A_n = \mathbb{R}^2$.

Proposition 2.2.6 (Convergence of subsequences) *Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}$ be a sequence. Then the following statements hold true:*

(i) $\forall \hat{N} \in \mathcal{N}_\infty^\# : \qquad \liminf_{n\in\hat{N}} A_n \geq \liminf_{n\to\infty} A_n,$

(ii) $\forall \hat{N} \in \mathcal{N}_\infty^\# : \qquad \limsup_{n\in\hat{N}} A_n \leq \limsup_{n\to\infty} A_n,$

(iii) *If $A_n \to \bar{A}$, then every subsequence of $\{A_n\}_{n\in\mathbb{N}}$ converges to the same limit.*

Proof. (i), (ii) In case of $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star$ we have

$$\begin{aligned}
\liminf_{n\in\hat{N}} A_n &= \sup_{N\in\mathcal{N}_\infty} \inf_{n\in N\cap\hat{N}} A_n \geq \sup_{N\in\mathcal{N}_\infty} \inf_{n\in N} A_n = \liminf_{n\to\infty} A_n, \\
\limsup_{n\in\hat{N}} A_n &= \sup_{N\in\mathcal{N}_\infty^\#, N\subseteq\hat{N}} \inf_{n\in N} A_n \leq \sup_{N\in\mathcal{N}_\infty^\#} \inf_{n\in N} A_n = \limsup_{n\to\infty} A_n.
\end{aligned}$$

The case $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\diamond$ is similar.

(iii) For all $N \in \mathcal{N}_\infty^\#$ it holds

$$\limsup_{n\in N} A_n \overset{\text{(ii)}}{\leq} \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n \overset{\text{(i)}}{\leq} \liminf_{n\in N} A_n.$$

Proposition 2.2.2 yields equality. □

Proposition 2.2.7 (Monotonicity of upper and lower limits) *Let be given two sequences $\{A_n\}_{n\in\mathbb{N}}, \{B_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}$ (with the same orientation). Then,*

(i) $\left(\exists \hat{N} \in \mathcal{N}_\infty,\ \forall n \in \hat{N} : A_n \leq B_n\right) \quad\Rightarrow\quad \liminf_{n\to\infty} A_n \leq \liminf_{n\to\infty} B_n,$

(ii) $\left(\exists \hat{N} \in \mathcal{N}_\infty,\ \forall n \in \hat{N} : A_n \leq B_n\right) \quad\Rightarrow\quad \limsup_{n\to\infty} A_n \leq \limsup_{n\to\infty} B_n.$

Proof. For example, let $\{A_n\}_{n\in\mathbb{N}}, \{B_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star$. Let $\hat{N} \in \mathcal{N}_\infty$ such that $A_n \leq B_n$ for all $n \in \hat{N}$. Hence, for all $N \in \mathcal{N}_\infty$ with $N \subseteq \hat{N}$ we have $\inf_{n\in N} A_n \leq \inf_{n\in N} B_n$.

(i) Given some $y \in \liminf_{n\to\infty} B_n$, we conclude that

$$\forall N \in \mathcal{N}_\infty \text{ with } N \subseteq \hat{N} : \quad y \in \inf_{n\in N} B_n \subseteq \inf_{n\in N} A_n.$$

By the definition of $\mathcal{N}_\infty$, we have $y \in \inf_{n\in N} A_n$ even for all $N \in \mathcal{N}_\infty$, i.e., $y \in \liminf_{n\to\infty} A_n$.

(ii) Let be given some $y \in \limsup_{n\to\infty} B_n$. Note that $N \cap \hat{N} \in \mathcal{N}_\infty^\#$ for all $N \in \mathcal{N}_\infty^\#$. Hence

$$\forall N \in \mathcal{N}_\infty^\# : \quad y \in \inf_{n\in N\cap\hat{N}} B_n \subseteq \inf_{n\in N\cap\hat{N}} A_n \subseteq \inf_{n\in N} A_n,$$

i.e., $y \in \limsup_{n\to\infty} A_n$. □

Proposition 2.2.8 *Let be given a set of sequences $\{A_n^{(i)}\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}$, where i belongs to an arbitrary index set I. Then it holds*

(i) $\liminf_{n\to\infty} \inf_{i\in I} A_n^{(i)} \leq \inf_{i\in I} \liminf_{n\to\infty} A_n^{(i)}$, (ii) $\limsup_{n\to\infty} \inf_{i\in I} A_n^{(i)} \leq \inf_{i\in I} \limsup_{n\to\infty} A_n^{(i)}$,

(iii) $\liminf_{n\to\infty} \sup_{i\in I} A_n^{(i)} \geq \sup_{i\in I} \liminf_{n\to\infty} A_n^{(i)}$, (iv) $\limsup_{n\to\infty} \sup_{i\in I} A_n^{(i)} \geq \sup_{i\in I} \limsup_{n\to\infty} A_n^{(i)}$.

Proof. For all $n \in \mathbb{N}$ and all $i \in I$ it holds $\inf_{i\in I} A_n^{(i)} \leq A_n^{(i)}$. Proposition 2.2.7 yields that $\liminf_{n\to\infty} \inf_{i\in I} A_n^{(i)} \leq \liminf_{n\to\infty} A_n^{(i)}$ for all $i \in I$. Hence (i) is true. The proof of (ii) – (iv) is similar. □

Remark 2.2.9 (limit of sum $\neq$ sum of limits) In the space $\hat{\mathcal{F}}$, the supremum–oriented version of (i) (i.e., we have to replace "liminf" by "LIMSUP" and "inf" by "SUP") even holds with equality if I is finite. This is not true in the present case, see Example 2.2.10 below. In Kuratowski [47], the union of sets is understood as a "sum". This means, for Painlevé–Kuratowski convergence, but not for $\mathcal{C}$–convergence, the limit of the "sum" of two sequences is equal to the "sum" of its limits. In our framework, however, the sum is understood as the closure of the Minkowski–addition. Nevertheless, also in this sense, limit and sum cannot be interchanged, see Example 2.2.11 below. But, even in the space $\mathcal{F}$, this is only possible by an extension of the convergence concept to so–called "total convergence" and by additional assumptions, see [68, Exercise 4.29]. It remains open if similar extensions are successful for $\mathcal{C}$–convergence, too. A detailed discussion about topologies for which the basic operations in $\mathcal{C}(X)$ (namely $(A,B) \to A \oplus B$, $(A,B) \to \inf\{A,B\}$ and $(\alpha, A) \to \alpha \cdot A$) in $\mathcal{C}(X)$ are jointly continuous can be found in [4, Section 4.3]. These operations are jointly continuous with respect to the scalar topology (which is related to the scalar convergence, see Section 2.3), but this topology is admissible (i.e., it extends the topology of the underlying space X) if and only if X is finite dimensional, see [4, Exercise 4.3.1 (c)]. If X is infinite dimensional, the *linear topology* [4] suffices all requirements. In Theorem 2.3.6 below, we give an equivalent characterization of $\mathcal{C}$–convergence by convergence of support functions. This shows that we can interchange limit and sum at least in many special cases.

Example 2.2.10 Proposition 2.2.8 (i) – (iv) does not hold with equality (for instance, (i) and (ii) in case of supremum orientation): Let $\{A_n\}_{n\in\mathbb{N}}, \{B_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star(\mathbb{R})$, $A_n = \{n\}$, $B_n = \{-n\}$. Then $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \emptyset$, hence $\inf\{\lim_{n\to\infty} A_n, \lim_{n\to\infty} B_n\} = \emptyset$. But, $\inf\{A_n, B_n\} = [-n, n]$ and consequently, $\lim_{n\to\infty} \inf\{A_n, B_n\} = \mathbb{R}$.

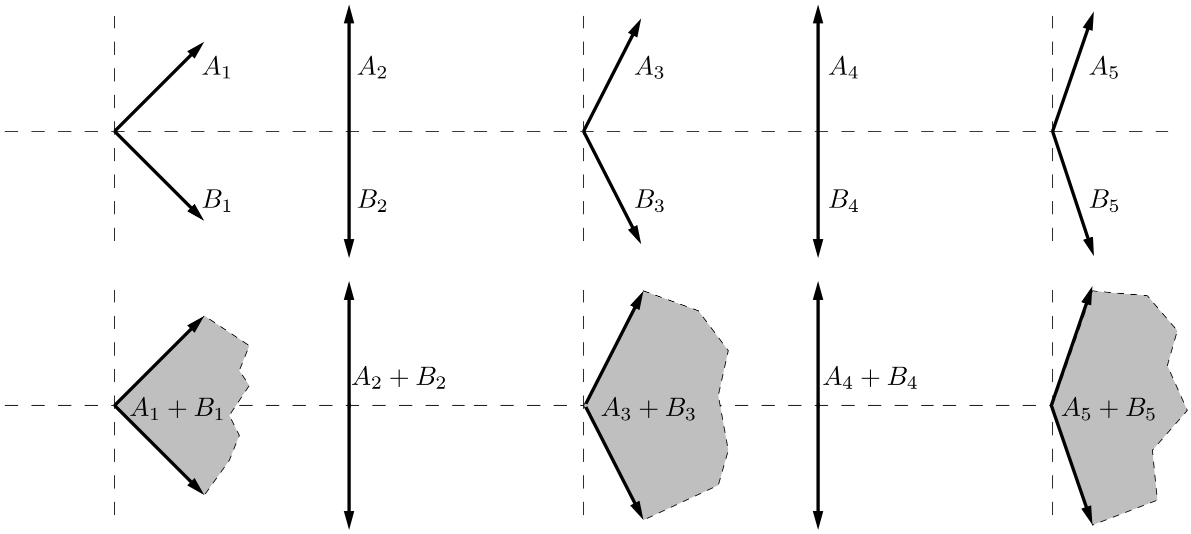
Example 2.2.11 The sum of limits is different from the limit of the sum: Let the sequences $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ as in Example 2.2.10. Then $\emptyset = \lim_{n\to\infty} A_n \oplus \lim_{n\to\infty} B_n \neq \lim_{n\to\infty}(A_n \oplus B_n) = \{0\}$.

Example 2.2.12 The sum of convergent sequences is not convergent: Consider two sequences $\{A_n\}_{n\in\mathbb{N}}, \{B_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star(\mathbb{R}^2)$ being defined by

$$A_n := \begin{cases} \left\{(y_1, y_2) \in \mathbb{R}^2 |\ y_2 = n y_1, y_2 \geq 0\right\} & \text{if} \quad n \text{ is odd} \\ \left\{(y_1, y_2) \in \mathbb{R}^2 |\ y_1 = 0, y_2 \geq 0\right\} & \text{if} \quad n \text{ is even,} \end{cases}$$

$$B_n := \begin{cases} \left\{(y_1, y_2) \in \mathbb{R}^2 |\ y_2 = -n y_1, y_2 \leq 0\right\} & \text{if} \quad n \text{ is odd} \\ \left\{(y_1, y_2) \in \mathbb{R}^2 |\ y_1 = 0, y_2 \leq 0\right\} & \text{if} \quad n \text{ is even.} \end{cases}$$

Then, $\{A_n\}_{n\in\mathbb{N}}$ converges to $\bar{A} = \left\{(y_1, y_2) \in \mathbb{R}^2 |\ y_1 = 0, y_2 \geq 0\right\}$ and $\{B_n\}_{n\in\mathbb{N}}$ converges to $\bar{B} = \left\{(y_1, y_2) \in \mathbb{R}^2 |\ y_1 = 0, y_2 \leq 0\right\}$. But, the subsequence $\{A_{2n} \oplus B_{2n}\}_{n\in\mathbb{N}}$ converges to $\left\{(y_1, y_2) \in \mathbb{R}^2 |\ y_1 = 0\right\}$ and $\{A_{2n+1} \oplus B_{2n+1}\}_{n\in\mathbb{N}}$ converges to $\left\{(y_1, y_2) \in \mathbb{R}^2 |\ y_1 \geq 0\right\}$, i.e., $\{A_n \oplus B_n\}_{n\in\mathbb{N}}$ is not convergent.



Proposition 2.2.13 (Sequences of singletons) *Let $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^p$ be a sequence.*

(i) *If $y_n \to \bar{y}$, then $\{y_n\} \xrightarrow{\mathcal{C}} \{\bar{y}\}$.*

(ii) *For $\big\{\{y_n\}\big\}_{n\in\mathbb{N}} \subseteq \mathcal{C}^\star$, $\|y_n\| \to +\infty$ implies that $\limsup\limits_{n\to\infty} \{y_n\} = \emptyset$.*

(iii) *Conversely, if $\{y_n\} \xrightarrow{\mathcal{C}} \bar{A} \in \hat{\mathcal{C}}^\star$, then $\bar{A} = \emptyset$ or $\bar{A} = \{\bar{y}\}$ for some $\bar{y} \in \mathbb{R}^p$. In the latter case we have $y_n \to \bar{y}$.*

Proof. (i) For all $N \in \mathcal{N}_\infty^\#$ it holds $\bar{y} \in \operatorname{cl} \bigcup_{n\in N} \{y_n\} \subseteq \operatorname{cl}\operatorname{conv} \bigcup_{n\in N} \{y_n\}$. Hence $\bar{y} \in \limsup_{n\to\infty} \{y_n\}$. Since $y_n \to \bar{y}$, we can write

$$\forall \varepsilon > 0,\ \exists n_0(\varepsilon) :\ \operatorname{cl}\operatorname{conv} \bigcup_{n\geq n_0(\varepsilon)} \{y_n\} \subseteq \mathbb{B}_\varepsilon(\bar{y}).$$

Hence

$$\liminf_{n\to\infty} \{y_n\} = \bigcap_{n_0\in\mathbb{N}} \operatorname{cl}\operatorname{conv} \bigcup_{n\geq n_0} \{y_n\} \subseteq \bigcap_{\varepsilon>0} \operatorname{cl}\operatorname{conv} \bigcup_{n\geq n_0(\varepsilon)} \{y_n\} \subseteq \bigcap_{\varepsilon>0} \mathbb{B}_\varepsilon(\bar{y}) = \{\bar{y}\}\,.$$

This yields $\{\bar{y}\} \geq \limsup_{n\to\infty} \{y_n\} \geq \liminf_{n\to\infty} \{y_n\} \geq \{\bar{y}\}$.

(ii) If $\|y_n\| \to +\infty$, there exists $N \in \mathcal{N}_\infty^\#$ such that $y_n/\|y_n\| \xrightarrow{N} y \in \mathbb{R}^p$. Using Proposition 2.2.6 (ii) we obtain $\emptyset = \liminf_{n\in N} \{y_n\} \leq \limsup_{n\in N} \{y_n\} \leq \limsup_{n\to\infty} \{y_n\}$.

(iii) The sequence $\{y_n\}_{n\in\mathbb{N}}$ has a subsequence $\{y_n\}_{n\in\hat{N}}$ ($\hat{N} \in \mathcal{N}_\infty^\#$) such that either $y_n \xrightarrow{\hat{N}} \bar{y}$ for some $\bar{y} \in \mathbb{R}^p$ or $\|y_n\| \xrightarrow{\hat{N}} +\infty$. By (i) and (ii) we conclude that, respectively, either $\lim_{n\in\hat{N}} \{y_n\} = \{\bar{y}\}$ or $\limsup_{n\in N} \{y_n\} = \emptyset$ (without loss of generality we assume $\big\{\{y_n\}\big\}_{n\in\mathbb{N}} \subseteq \mathcal{C}^\star$). Since $\big\{\{y_n\}\big\}_{n\in\mathbb{N}}$ is supposed to be convergent, the result follows from Proposition 2.2.6 (iii). $\square$

Sometimes, a slight generalization of part (i) of the previous assertions is useful. Taking into account Example 2.2.11 and Example 2.2.12 this is not a direct consequence of the previous assertion.

Proposition 2.2.14 *Let $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^p$ be a sequence and $A \in \hat{\mathcal{C}}$. Then, $y_n \to \bar{y}$ implies $\big\{\{y_n\} + A\big\} \xrightarrow{\mathcal{C}} \{\bar{y} + A\}$.*

Proof. A direct proof can be given similarly to the proof of Proposition 2.2.13 (i). Furthermore, this statement is a direct consequence of Proposition 2.3.1 below. □

In the following we restrict ourselves to supremum–oriented sets. Of course, analogous assertions can be shown for infimum oriented sets if we simultaneously switch over from "liminf" to "limsup", "inf" to "sup" and so on.

The following characterization of the lower limit is useful in order to show further properties of the lower and upper limits.

Proposition 2.2.15 *Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star$ be a sequence. Then, $y \in \liminf_{n\to\infty} A_n$ if and only if the following assertion holds true:*

$$\exists \left\{(\lambda_0^{(m)}, ..., \lambda_p^{(m)}), (k_0^{(m)}, ..., k_p^{(m)}), (z_0^{(m)}, ..., z_p^{(m)})\right\}_{m\in\mathbb{N}} \subseteq [0,1]^{p+1} \times \mathbb{N}^{p+1} \times (\mathbb{R}^p)^{p+1} :$$

$$y = \lim_{m\to\infty} \sum_{j=0}^{p} \lambda_j^{(m)} z_j^{(m)},$$

$$\forall j \in \{0,1,...,p\}, \forall m \in \mathbb{N}: \quad z_j^{(m)} \in A_{k_j^{(m)}},$$

$$\forall j \in \{0,1,...,p\}, \forall m \in \mathbb{N}: \quad k_j^{(m)} \geq m,$$

$$\forall m \in \mathbb{N}: \quad \sum_{j=0}^{p} \lambda_j^{(m)} = 1.$$

Proof. (i) Let $y \in A := \liminf_{n\to\infty} A_n$. By definition we have $A = \bigcap_{m\in\mathbb{N}} \operatorname{cl} V_m$, where $V_m := \operatorname{conv} \bigcup_{k\geq m} A_k$. Hence $y \in \operatorname{cl} V_m$ for all $m \in \mathbb{N}$. This yields

$$\forall m \in \mathbb{N},\ \forall \varepsilon > 0\ \ \exists v_{m,\varepsilon} \in V_m : \quad \|y - v_{m,\varepsilon}\| < \varepsilon.$$

Choosing $\varepsilon := 1/m$ we obtain a sequence $\{v^{(m)}\}_{m\in\mathbb{N}}$, defined by $v^{(m)} := v_{m,1/m}$, which is convergent to $y \in A$. The sequence $\{v^{(m)}\}_{m\in\mathbb{N}}$ is a sequence of convex combinations $v^{(m)}$ of elements in $Z_m := \bigcup_{k\geq m} A_k$. Since $Z_m \subseteq \mathbb{R}^p$, Carathéodory's theorem allows us to write

$$y = \lim_{m\to\infty} v^{(m)}, \qquad v^{(m)} = \sum_{j=0}^{p} \lambda_j^{(m)} z_j^{(m)},$$

where $\lambda_j^{(m)} \in [0,1]$, $z_j^{(m)} \in Z_m$ $(j \in \{0,...,p\})$ and $\sum_{j=0}^{p} \lambda_j^{(m)} = 1$. Since $z_j^{(m)} \in Z_m = \bigcup_{k\geq m} A_k$, there exists $k_j^{(m)} \geq m$ such that $z_j^{(m)} \in A_{k_j^{(m)}}$.

(ii) The opposite direction can be seen as follows. For arbitrarily given $n \in \mathbb{N}$, we have

$$y = \lim_{m\in\mathbb{N}} \sum_{j=0}^{p} \lambda_j^{(m)} z_j^{(m)} = \lim_{m\geq n} \sum_{j=0}^{p} \lambda_j^{(m)} z_j^{(m)}.$$

Hence $y \in \operatorname{cl}\operatorname{conv} \bigcup_{k\geq n} A_k$ for all $n \in \mathbb{N}$. This yields $y \in \liminf_{n\to\infty} A_n$. □

Remark 2.2.16 Of course, the previous proposition remains true if the condition is formulated for a sequence in $[0,1]^{\bar{p}+1} \times \mathbb{N}^{\bar{p}+1} \times (\mathbb{R}^p)^{\bar{p}+1}$, where $\bar{p} \geq p$.

The following assertion is used in the following several times.

Proposition 2.2.17 *Let L be a linear subspace of $\mathbb{R}^p$. The map $T_L : \hat{\mathcal{C}}^\star \to \hat{\mathcal{C}}^\star$, defined by $T_L(A) = (A \oplus L) \cap L^\perp$ has the following properties:*

(i) *For $y \in \mathbb{R}^p$, $T_L(\{y\})$ is a singleton set,*

(ii) *$\bar{T}_L : \mathbb{R}^p \to \mathbb{R}^p$, $\bar{T}_L(y) \in T_L(\{y\})$ is a well–defined linear continuous operator.*

(iii) *$T_L(A) = \operatorname{cl} \bar{T}_L(A)$, where $\bar{T}_L(A) := \bigcup_{y\in A} \bar{T}_L(y)$.*

(iv) *If L is the lineality space of $A \subseteq \mathbb{R}^p$, then $A = T_L(A) + L$.*

(v) *If L is the lineality space of $A \subseteq \mathbb{R}^p$, then the lineality space of $T_L(A) \subseteq \mathbb{R}^p$ is $\{0\}$, i.e., $T_L(A)$ has a pointed recession cone.*

Proof. By [1, Corollary 1.6.1], there is a one–to–one linear transformation of $\mathbb{R}^p$ onto itself, which carries L onto the subspace $\bar{L} := \{y \in \mathbb{R}^p |\ y_{m+1} = 0, y_{m+2} = 0, ..., y_p = 0\}$, where $m = \dim L$. Hence, without loss of generality we can assume that $L = \bar{L}$. Thus, the assertions (i) – (iv) are elementary. By [1, Corollary 8.3.3], we have $0^+T_L(A) \cap -0^+T_L(A) = 0^+(A \oplus L) \cap 0^+(-A \oplus L) \cap L^\perp = 0^+A \cap -0^+A \cap L^\perp = L \cap L^\perp = \{0\}$, which proves (v). □

The next two theorems tell us that, under additional assumptions, Painlevé–Kuratowski convergence and $\mathcal{C}$–convergence coincide. For the special case $K = \{0\}$, the statement of the next theorem can be found in [2, Lemma 1.1.9]. A noticeable simplification of part (A) of the proof is due to C. Zălinescu[1].

Theorem 2.2.18 *Let $K \subseteq \mathbb{R}^p$ be a nonempty closed convex cone and let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}_K^\star$ be a sequence such that $\inf_{n\in\mathbb{N}} A_n \in \mathcal{C}_K^\star$. Then,*

$$\liminf_{n\to\infty} A_n = \operatorname{cl}\operatorname{conv} \operatorname*{LIM\,SUP}_{n\to\infty} A_n.$$

[1] e-mail conversation, October 2004

Proof. Of course, we have $\liminf_{n\to\infty} A_n \supseteq \mathrm{LIM\,SUP}_{n\to\infty} A_n$ and hence

$$\liminf_{n\to\infty} A_n \supseteq \mathrm{cl\,conv}\ \underset{n\to\infty}{\mathrm{LIM\,SUP}}\, A_n. \tag{2.2}$$

In the following we can assume that $\liminf_{n\to\infty} A_n \neq \emptyset$, because otherwise (2.2) yields the desired assertion.

(A) First, we prove the opposite inclusion for the special case that K is pointed, i.e., $K \cap -K = \{0\}$. Let $y \in A := \liminf_{n\to\infty} A_n$ be given. By Proposition 2.2.15 we have

$$\exists \left\{(\lambda_0^{(m)}, ..., \lambda_p^{(m)}), (k_0^{(m)}, ..., k_p^{(m)}), (z_0^{(m)}, ..., z_p^{(m)})\right\}_{m\in\mathbb{N}} \subseteq [0,1]^{p+1} \times \mathbb{N}^{p+1} \times (\mathbb{R}^p)^{p+1} :$$

$$y = \lim_{m\to\infty} v^{(m)}, \quad v^{(m)} = \sum_{j=0}^{p} \lambda_j^{(m)} z_j^{(m)}, \tag{2.3}$$

$$\forall j \in \{0,1,...,p\}, \forall m \in \mathbb{N}: \quad z_j^{(m)} \in A_{k_j^{(m)}},$$

$$\forall j \in \{0,1,...,p\}, \forall m \in \mathbb{N}: \quad k_j^{(m)} \geq m,$$

$$\forall m \in \mathbb{N}: \quad \sum_{j=0}^{p} \lambda_j^{(m)} = 1.$$

Without loss of generality we can assume that

$$\forall m \in \mathbb{N}: \qquad \left\|\lambda_0^{(m)} z_0^{(m)}\right\| \leq \left\|\lambda_1^{(m)} z_1^{(m)}\right\| \leq ... \leq \left\|\lambda_p^{(m)} z_p^{(m)}\right\|, \tag{2.4}$$

and $\|\lambda_p^{(m)} z_p^{(m)}\| \neq 0$ for all $m \in \mathbb{N}$. We can successively switch over to subsequences, again indexed by m, such that

$$\forall j \in \{0,1,...,p\}: \quad \lambda_j^{(m)} \xrightarrow{m} \lambda_j \in [0,1] \qquad \text{and} \qquad \frac{\lambda_j^{(m)} z_j^{(m)}}{\left\|\lambda_p^{(m)} z_p^{(m)}\right\|} \xrightarrow{m} y_j \in \mathbb{R}^p. \tag{2.5}$$

Assume that the sequence $\{\lambda_j^{(m)} z_j^{(m)}\}_{m\in\mathbb{N}}$ is unbounded for some $j \in \{0,1,...,p\}$, then it is unbounded for $j = p$. Hence, there is a subsequence, again indexed by m, such that $\|\lambda_p^{(m)} z_p^{(m)}\| \xrightarrow{m} +\infty$. It follows $\lambda_j^{(m)} / \left\|\lambda_p^{(m)} z_p^{(m)}\right\| \xrightarrow{m} 0$. By the characterization of recession cones of [1, Theorem 8.2], applied to the set $\inf_{n\in\mathbb{N}} A_n$, we deduce that $y_j \in K$ for all $j \in \{0,...,p\}$. From (2.3) we deduce that

$$\frac{v^{(m)}}{\|\lambda_p^{(m)} z_p^{(m)}\|} = \sum_{j=0}^{p} \frac{\lambda_j^{(m)} z_j^{(m)}}{\left\|\lambda_p^{(m)} z_p^{(m)}\right\|}.$$

Taking the limit we obtain $0 = \sum_{j=0}^{p} y_j$. Hence $y_p \in K \cap -K = \{0\}$. This contradicts (2.5), which yields that $\|y_p\| = 1$. Hence the sequence $\{\lambda_j^{(m)} z_j^{(m)}\}_{m\in\mathbb{N}}$ is bounded for each $j \in \{0,...,p\}$. Therefore, we can successively extract subsequences (again denoted by) $\{\lambda_j^{(m)} z_j^{(m)}\}_{m\in\mathbb{N}}$ being convergent to $z_j \in \mathbb{R}^p$.

If $\lambda_j = 0$, [1, Theorem 8.2] yields that $z_j \in K$. In case of $\lambda_j \neq 0$, we can assume that $\lambda_j^{(m)} \neq 0$ for all $m \in \mathbb{N}$. Hence we have

$$\left\| z_j^{(m)} - \frac{z_j}{\lambda_j} \right\| \leq \left\| z_j^{(m)} - \frac{z_j}{\lambda_j^{(m)}} \right\| + \left\| \frac{z_j}{\lambda_j^{(m)}} - \frac{z_j}{\lambda_j} \right\| \xrightarrow{m} 0,$$

i.e., $\{z_j^{(m)}\}_{m\in\mathbb{N}}$, where $z_j^{(m)} \in \bigcup_{n\geq m} A_n$, converges to $c_j := z_j/\lambda_j \in \mathbb{R}^p$. Thus we find a sequence $\{a_j^{(n)}\}_{n\in\mathbb{N}}$ with $a_j^{(n)} \in A_n$ having a subsequence converging to c_j. This means $c_j = z_j/\lambda_j \in \mathrm{LIM\,SUP}_{n\to\infty} A_n$. Let us assume that $\lambda_j \neq 0$ for $j \in \{0,...,r\}$ and $\lambda_j = 0$ for $j \in \{r+1,...,p\}$. Setting $k := \sum_{j=r+1}^{p} z_j$ we obtain

$$y = \sum_{j=0}^{r} \lambda_j c_j + k \in \operatorname{conv} \mathop{\mathrm{LIM\,SUP}}_{n\to\infty} A_n + K \subseteq \operatorname{cl}\operatorname{conv} \mathop{\mathrm{LIM\,SUP}}_{n\to\infty} A_n + K.$$

From Proposition A.4 (viii), A.1 (iv), A.4 (vi), A.1 (iii) we obtain

$$y \in \operatorname{cl}\operatorname{conv} \mathop{\mathrm{LIM\,SUP}}_{n\to\infty} A_n + K \subseteq \operatorname{cl}\operatorname{conv} \mathop{\mathrm{LIM\,SUP}}_{n\to\infty} (A_n + K) \subseteq \operatorname{cl}\operatorname{conv} \mathop{\mathrm{LIM\,SUP}}_{n\to\infty} A_n. \tag{2.6}$$

(B) We now turn to the general case, i.e., the lineality space $L := K \cap -K$ of $\inf_{n\in\mathbb{N}} A_n$ is not necessarily $\{0\}$. Let $T_L : \hat{\mathcal{C}}^\star \to \hat{\mathcal{C}}^\star$ and $\bar{T}_L : \mathbb{R}^p \to \mathbb{R}^p$ be defined as in Proposition 2.2.17. The following quantities hold true:

$$\begin{aligned} T_L(\inf_{n\to\infty} A_n) &\overset{\text{Pr. 2.2.17 (iii)}}{=} \operatorname{cl} \bar{T}_L(\operatorname{cl}\operatorname{conv} \bigcup_{n\in\mathbb{N}} A_n) \overset{\text{Pr. A.5 (i), A.7 (iii)}}{=} \operatorname{cl}\operatorname{conv} \bigcup_{n\in\mathbb{N}} \bar{T}_L(A_n) \\ &\overset{\text{Pr. 2.2.17 (iii)}}{=} \inf_{n\to\infty} T_L(A_n), \end{aligned} \tag{2.7}$$

$$\begin{aligned} T_L(\liminf_{n\to\infty} A_n) &\overset{\text{Pr. 2.2.17 (iii)}}{=} \operatorname{cl} \bar{T}_L(\liminf_{n\to\infty} A_n) \overset{\text{Pr. A.5 (ii)}}{\subseteq} \operatorname{cl} \bigcap_{N\in\mathcal{N}_\infty} \bar{T}_L(\inf_{n\in N} A_n) \\ &\overset{\text{Pr. A.2 (i)}}{\subseteq} \bigcap_{N\in\mathcal{N}_\infty} T_L(\inf_{n\in N} A_n) \overset{(2.7)}{=} \liminf_{n\to\infty} T_L(A_n). \end{aligned} \tag{2.8}$$

By (2.7), we have $0^+ \inf_{n\to\infty} T_L(A_n) = T_L(K)$. Proposition 2.2.17 (v) yields that $T_L(K)$ is pointed. Applying part (A) to the sequence $\{T_L(A_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{C}^\star_{T_L(K)}$ we obtain

$$T_L(\liminf_{n\to\infty} A_n) \overset{(2.8)}{\subseteq} \liminf_{n\to\infty} T_L(A_n) \overset{\text{part (A)}}{\subseteq} \operatorname{cl}\operatorname{conv} \mathop{\mathrm{LIM\,SUP}}_{n\to\infty} T_L(A_n). \tag{2.9}$$

Of course, we have $A_n + L = A_n$ for all $n \in \mathbb{N}$. This yields

$$\operatorname{cl}\operatorname{conv} \mathop{\mathrm{LIM\,SUP}}_{n\to\infty} T_L(A_n) = \operatorname{cl}\operatorname{conv} \mathop{\mathrm{LIM\,SUP}}_{n\to\infty} (A_n \cap L^\perp) \subseteq \operatorname{cl}\operatorname{conv} \mathop{\mathrm{LIM\,SUP}}_{n\to\infty} A_n, \tag{2.10}$$

By [1, Corollary 8.3.3], we have $0^+ \liminf_{n\to\infty} A_n = \bigcap_{N\in\mathcal{N}_\infty} 0^+ \inf_{n\in N} A_n = K$. Thus,

$$\liminf_{n\to\infty} A_n \overset{\text{Pr. 2.2.17 (iv)}}{=} T_L(\liminf_{n\to\infty} A_n) + L \overset{(2.9),\,(2.10)}{\subseteq} \operatorname{cl}\operatorname{conv} \mathop{\mathrm{LIM\,SUP}}_{n\to\infty} A_n + L.$$

As in (2.6) we obtain $\operatorname{cl}\operatorname{conv} \mathrm{LIM\,SUP}_{n\to\infty} A_n + L \subseteq \operatorname{cl}\operatorname{conv} \mathrm{LIM\,SUP}_{n\to\infty} A_n$. □

An analogous result for the upper and inner limits can be obtained even by weaker assumptions. The result is proven using the previous theorem.

Theorem 2.2.19 *Let* $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star$ *be a sequence such that for all* $\bar{N} \in \mathcal{N}_\infty^\#$ *there exists some* $\widetilde{N} \in \mathcal{N}_\infty^\#$ *with* $\widetilde{N} \subseteq \bar{N}$ *and some nonempty closed convex cone* $K \subseteq \mathbb{R}^p$ *such that*

$$\forall n \in \widetilde{N}: \; A_n \in \mathcal{C}_K^\star \qquad \textit{and} \qquad \inf_{n\in\widetilde{N}} A_n \in \mathcal{C}_K^\star.$$

Then it holds

$$\limsup_{n\to\infty} A_n = \operatorname*{LIM\,INF}_{n\to\infty} A_n.$$

Proof. Clearly, we have $A := \limsup_{n\to\infty} \supseteq \mathrm{LIM\,INF}_{n\to\infty} A_n$. To show the opposite inclusion let $\widehat{N} \in \mathcal{N}_\infty^\#$ be arbitrarily given. By the cluster point description of outer limits [68, Proposition 4.19] there exists some $\bar{N} \in \mathcal{N}_\infty^\#$ with $\bar{N} \subseteq \widehat{N}$ such that $\{A_n\}_{n\in\bar{N}}$ is convergent (with respect to Painlevé–Kuratowski convergence). By assumption, there exists $\widetilde{N} \in \mathcal{N}_\infty^\#$ with $\widetilde{N} \subseteq \bar{N}$ such that $0^+\left(\inf_{n\in\widetilde{N}} A_n\right) = K$. Of course, $\{A_n\}_{n\in\widetilde{N}}$ is convergent, too. By Theorem 2.2.18 it follows

$$A = \limsup_{n\to\infty} A_n \overset{\text{Pr. 2.2.6}}{\subseteq} \limsup_{n\in\widetilde{N}} A_n \overset{\text{Pr. 2.2.2}}{\subseteq} \liminf_{n\in\widetilde{N}} A_n = \operatorname{cl}\operatorname{conv} \operatorname*{LIM\,SUP}_{n\in\widetilde{N}} A_n. \tag{2.11}$$

The convergence of $\{A_n\}_{n\in\widetilde{N}}$ implies the convexity of the set $\mathrm{LIM\,SUP}_{n\in\widetilde{N}} A_n$ (because $\mathrm{LIM\,INF}_{n\in\widetilde{N}} A_n$ is a convex set, see [68, Proposition 4.15]). Hence we obtain

$$A \subseteq \operatorname*{LIM\,SUP}_{n\in\widetilde{N}} A_n \subseteq \operatorname{cl} \bigcup_{n\in\widetilde{N}} A_n \subseteq \operatorname{cl} \bigcup_{n\in\widehat{N}} A_n.$$

Since $\widehat{N} \in \mathcal{N}_\infty^\#$ was chosen arbitrarily, it follows $A \subseteq \mathrm{LIM\,INF}_{n\to\infty} A_n$. □

Corollary 2.2.20 *Let* $K \subseteq \mathbb{R}^p$ *be a nonempty closed convex cone and let* $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}_K^\star$ *a sequence such that* $\inf_{n\in\mathbb{N}} A_n \in \mathcal{C}_K^\star$. *Then,* $\{A_n\}_{n\in\mathbb{N}}$ *is* $\mathcal{C}$*–convergent if and only if it is Painlevé–Kuratowski–convergent. In case of convergence, we have*

$$\lim_{n\to\infty} A_n = \operatorname*{LIM}_{n\to\infty} A_n.$$

Proof. Follows from Theorem 2.2.18 and Theorem 2.2.19. □

2.3 Scalar convergence vs. $\mathcal{C}$–convergence

Convergence of convex closed sets that is determined by the pointwise convergence of the support functions associated to the sets was investigated by many authors such as Wijsman [82], Salinetti/Wets [70] and Sonntag/Zălinescu [74]. An overview and further references can be found in Beer [4]. Following the article by Sonntag and Zălinescu we call this convergence *scalar convergence.* A sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}$ is said to be S–convergent to some $\bar{A} \in \mathcal{C}$ if

$$\forall y^* \in \mathbb{R}^p: \qquad \delta^*(y^*|A_n) \to \delta^*(y^*|\bar{A}).$$

Then we write $A_n \xrightarrow{S} \bar{A}$. In this section, we investigate the relationship between scalar convergence and $\mathcal{C}$–convergence. We start with a result of Sonntag and Zălinescu [74]. The following extension of this result is due to C. Zălinescu[2].

Proposition 2.3.1 ([74]) *Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}$. Then $A_n \xrightarrow{S} \bar{A}$ implies $A_n \xrightarrow{\mathcal{C}} \bar{A}$.*

Proof. For example, let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}^\star$. In [74, Proposition 1] it is shown that $A_n \xrightarrow{S} \bar{A}$ implies that $\bar{A} = \liminf_{n\to\infty} A_n$.

To show that $\bar{A} = \limsup_{n\to\infty} A_n$, let $N \in \mathcal{N}_\infty^\#$. Then, $\bar{A} = S - \lim_{n\in N} A_n$, and so $\bar{A} = \liminf_{n\in N} A_n \geq \inf_{n\in N} A_n$ for all $N \in \mathcal{N}_\infty^\#$. Hence we have $\bar{A} = \liminf_{n\in N} A_n \geq \sup_{N\in\mathcal{N}_\infty^\#} \inf_{n\in N} A_n = \limsup_{n\to\infty} A_n$. It follows $\bar{A} = \liminf_{n\to\infty} A_n \leq \limsup_{n\to\infty} A_n \leq \bar{A}$, i.e., $A_n \xrightarrow{\mathcal{C}} \bar{A}$. □

We next want to investigate the opposite inclusion. Assuming that $\mathcal{C}$–convergence implies scalar convergence we can deduce that the sum of two $\mathcal{C}$–convergent sequences is convergent as well. This, however, contradicts Example 2.2.12. Indeed, the sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}$ in Example 2.2.12 $\mathcal{C}$–converges to $\bar{A} := \left\{(y_1, y_2) \in \mathbb{R}^2 |\ y_1 = 0, y_2 \geq 0\right\}$, but for $y^* = (1,0)$ the sequence $\{\delta^*(y^*|A_n)\}_{n\in\mathbb{N}}$ is not convergent. However, we can show that $\mathcal{C}$–convergence implies the pointwise convergence of the support function, except at relative boundary points of the domain of the support function of the $\mathcal{C}$–limit. Moreover, $\mathcal{C}$–convergence is equivalent to this property. We start with some auxiliary assertions.

Proposition 2.3.2 *Let $K \subseteq \mathbb{R}^p$ be a nonempty pointed closed convex cone. Then, for $y^* \in \operatorname{ri} K^\circ$ and $k \in K \setminus \{0\}$ it holds $\langle y^*, k\rangle < 0$.*

Proof. We have $\operatorname{int} K^\circ \neq \emptyset$ and consequently $y^* \in \operatorname{int} K^\circ$. Hence there exists some $\varepsilon > 0$ such that $y^* + \varepsilon k \in K^\circ$. This yields $\langle y^* + \varepsilon k, k\rangle = \langle y^*, k\rangle + \varepsilon \|k\|^2 \leq 0$, hence $\langle y^*, k\rangle < 0$. □

Proposition 2.3.3 *Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star$ such that $\liminf_{n\to\infty} A_n \neq \emptyset$ and let $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^p \setminus \{0\}$ be a sequence such that $y_n \in A_n$ for all $n \in \mathbb{N}$, $\|y_n\| \to \infty$ and $y_n / \|y_n\| \to k$. Then, $k \in 0^+ \liminf_{n\to\infty} A_n$.*

Proof. We show that $y + \mu k \in \liminf_{n\to\infty} A_n$ for arbitrarily given $y \in \liminf_{n\to\infty} A_n$ and $\mu > 0$. Using the characterization of Proposition 2.2.15, we obtain

$$y + \mu k = \lim_{m\to\infty} \sum_{j=0}^{p} \lambda_j^{(m)} z_j^{(m)} + \lim_{m\to\infty} \mu \frac{y_m}{\|y_m\|}.$$

Setting $\lambda_{p+1}^{(m)} := \frac{\mu}{\|y_m\|}$, $z_{p+1}^{(m)} := y_m$, $k_{p+1}^{(m)} := m$ and

$$\forall j \in \{0, ..., p+1\}: \qquad \tilde{\lambda}_j^{(m)} := \frac{\lambda_j^{(m)}}{(1 + \lambda_{p+1}^{(m)})},$$

[2] e-mail conversation, October 2004

we obtain

$$y + \mu k = \lim_{m\to\infty} (1 + \lambda_{p+1}^{(m)}) \sum_{j=0}^{p+1} \tilde{\lambda}_j^{(m)} z_j^{(m)} = \lim_{m\to\infty} \sum_{j=0}^{p+1} \tilde{\lambda}_j^{(m)} z_j^{(m)}.$$

By Proposition 2.2.15 and taking into account Remark 2.2.16 we obtain that $y + \mu k \in \liminf_{n\to\infty} A_n$. □

Proposition 2.3.4 *Let* $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star$ *such that* $\liminf_{n\to\infty} A_n \neq \emptyset$. *Denoting by* L *the lineality space of* $\liminf_{n\to\infty} A_n$, *we define* $T_L : \hat{\mathcal{C}}^\star \to \hat{\mathcal{C}}^\star$ *as in Proposition 2.2.17. Then,*

$$T_L(\liminf_{n\to\infty} A_n) = \liminf_{n\to\infty} T_L(A_n).$$

Proof. Set $\bar{A} := \liminf_{n\to\infty} A_n$. As in (2.8) we obtain $T_L(\bar{A}) \geq \liminf_{n\to\infty} T_L(A_n)$. It follows that $T_L(\bar{A}) \geq \liminf_{n\to\infty}\big(A_n \oplus L\big)$. For all $N \in \mathcal{N}_\infty$ we have $\inf_{n\in N} A_n \leq \bar{A}$ and hence $0^+\big(\inf_{n\in N} A_n\big) \supseteq 0^+\bar{A} \supseteq L$. With the aid of Proposition 1.2.4 we obtain $\inf_{n\in N}\big(A_n \oplus L\big) = \big(\inf_{n\in N} A_n\big) \oplus L = \inf_{n\in N} A_n$ for all $N \in \mathcal{N}_\infty$. Hence $\liminf_{n\to\infty}\big(A_n \oplus L\big) = \liminf_{n\to\infty} A_n = \bar{A}$. It follows that $T_L(\bar{A}) \geq \liminf_{n\to\infty} T_L(A_n) \geq \bar{A}$. Of course, $\liminf_{n\to\infty} T_L(A_n) \geq L^\perp$ and so $T_L(\bar{A}) \geq \liminf_{n\to\infty} T_L(A_n) \geq \bar{A} \cap L^\perp = T_L(\bar{A})$. □

Lemma 2.3.5 *Let* $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star$ *such that* $\liminf_{n\to\infty} A_n \neq \emptyset$. *Then,*

$$\forall y^* \in \operatorname{ri}(0^+ \liminf_{n\to\infty} A_n)^\circ : \qquad \liminf_{n\to\infty} -\delta^*(y^*|A_n) = -\delta^*(y^*|\liminf_{n\to\infty} A_n).$$

Proof. Set $\bar{A} := \liminf_{n\to\infty} A_n$. From Proposition 1.3.7 we easily obtain

$$\forall y^* \in \mathbb{R}^p : \qquad \liminf_{n\to\infty} -\delta^*(y^*|A_n) \leq -\delta^*(y^*|\bar{A}).$$

It remains to show that $\limsup_{n\to\infty} \delta^*(y^*|A_n) \leq \delta^*(y^*|\bar{A})$ for all $y^* \in \operatorname{ri}(0^+\bar{A})^\circ$.

(A) We first prove the case that $0^+\bar{A}$ is pointed. Assume the assertion is not true. This means there exists some $y^* \in \operatorname{ri}(0^+\bar{A})^\circ$ such that $\limsup_{n\to\infty} \delta^*(y^*|A_n) > \delta^*(y^*|\bar{A})$. Hence, there is some $\varepsilon > 0$ and some $N \in \mathcal{N}_\infty^\#$ such that $\delta^*(y^*|A_n) > \delta^*(y^*|\bar{A}) + \varepsilon$ for all $n \in N$. It follows

$$\forall n \in N, \quad \exists y_n \in A_n : \quad \langle y^*, y_n\rangle > \delta^*(y^*|\bar{A}) + \frac{\varepsilon}{2}. \tag{2.12}$$

We distinguish between two cases.

(i) If there exists some $\tilde{N} \in \mathcal{N}_\infty^\#$ with $\tilde{N} \subseteq N$ such that $y_n \to \bar{y} \in \mathbb{R}^p$ we conclude that $\bar{y} \in \operatorname{LIM\,SUP}_{n\to\infty} A_n \subseteq \liminf_{n\to\infty} A_n = \bar{A}$. This contradicts (2.12).

(ii) Otherwise there is some $\tilde{N} \in \mathcal{N}_\infty^\#$ with $\tilde{N} \subseteq N$ such that $\|y_n\| \neq 0$, $\|y_n\| \to \infty$ and $y_n/\|y_n\| \to k \in \mathbb{R}^p$. From Proposition 2.3.3 we deduce that $k \in 0^+\bar{A}$. Proposition 2.3.2 yields $\langle y^*, k\rangle < 0$. But, from (2.12) we deduce that $\langle y^*, k\rangle \geq 0$, a contradiction.

(B) If $0^+\bar{A}$ is not pointed, consider its lineality space $L := 0^+\bar{A} \cap (-0^+\bar{A}) \neq \{0\}$. Let $T_L : \hat{\mathcal{C}}^\star \to \hat{\mathcal{C}}^\star$ be defined as in Proposition 2.2.17. Proposition 2.3.4 yields that $T_L(\bar{A}) =$

$\liminf_{n\to\infty} T_L(A_n)$. From [1, Corollary 8.3.3] and [1, Corollary 16.4.2] we deduce that $\big(0^+T_L(\bar A)\big)^\circ = (0^+\bar A)^\circ \oplus L$. By [1, Theorem 6.3] and [1, Corollary 6.6.2] we conclude that $\operatorname{ri}\big(0^+T_L(\bar A)\big)^\circ = \operatorname{ri}(0^+\bar A)^\circ + L$. This implies that $\operatorname{ri}\big(0^+T_L(\bar A)\big)^\circ \supseteq \operatorname{ri}(0^+\bar A)^\circ$. Moreover, from $L \subseteq 0^+\bar A$ we conclude that $L^\perp \supseteq (0^+\bar A)^\circ$. For arbitrary $A \in \mathcal{C}$ it is true that $\delta^*\big(y^*|T_L(A)\big) = \delta^*(y^*|A)$ for all $y^* \in L^\perp$. Since $0^+T_L(\bar A)$ is pointed, for all $y^* \in \operatorname{ri}(0^+\bar A)^\circ$ ($\subseteq \operatorname{ri}\big(0^+T_L(\bar A)\big)^\circ \cap L^\perp$) we have

$$\limsup_{n\to\infty} \delta^*(y^*|A_n) = \limsup_{n\to\infty} \delta^*(y^*|T_L(A_n)) \overset{\text{part (A)}}{\leq} \delta^*(y^*|T_L(\bar A)) = \delta^*(y^*|\bar A).$$

This completes the proof. □

It follows the main result of this section, an equivalent characterization of $\mathcal{C}$–convergence. For $A \in \mathcal{C}$ we denote by $\operatorname{rb} A := A \setminus \operatorname{ri} A$ the relative boundary of A.

Theorem 2.3.6 *If $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}$ and $\bar A \in \mathcal{C}$, the following statements are equivalent:*

(i) $A_n \xrightarrow{\mathcal{C}} \bar A$,

(ii) $\forall y^* \in \mathbb{R}^p \setminus \operatorname{rb}(0^+\bar A)^\circ : \qquad \delta^*(y^*|A_n) \to \delta^*(y^*|\bar A)$.

Proof. We adopt our notation to the case $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star$.

(i) ⇒ (ii). By Proposition 1.3.7 we easily deduce that

$$\forall y^* \in \mathbb{R}^p : \qquad \delta^*(y^*|\bar A) = \delta^*(y^*|\limsup_{n\to\infty} A_n) \leq \liminf_{n\to\infty} \delta^*(y^*|A_n). \tag{2.13}$$

By Lemma 1.3.1 (i) we obtain $\delta^*(y^*|A_n) \to \delta^*(y^*|\bar A) = +\infty$ for all $y^* \in \mathbb{R}^p \setminus (0^+\bar A)^\circ$. It remains to show that $\delta^*(y^*|A_n) \to \delta^*(y^*|\bar A)$ for all $y^* \in \operatorname{ri}(0^+\bar A)^\circ$. This follows from Lemma 2.3.5 and (2.13).

(ii) ⇒ (i). Let $B_n := \inf_{k\geq n} A_k$. Then, $\{B_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star$ and $\{-\delta^*(\,\cdot\,|B_n)\}_{n\in\mathbb{N}}$ are increasing. It follows

$$\begin{aligned} \forall y^* \in \mathbb{R}^p \setminus \operatorname{rb}(0^+\bar A)^\circ : \quad -\delta^*(y^*|\bar A) \quad &= \quad \liminf_{n\to\infty} -\delta^*(y^*|A_n) \\ &= \quad \sup_{n\in\mathbb{N}} \inf_{k\geq n} -\delta^*(y^*|A_n) \\ &\overset{\text{Pr. 1.3.7}}{=} \quad \sup_{n\in\mathbb{N}} -\delta^*(y^*|B_n) = \lim_{n\to\infty} -\delta^*(y^*|B_n). \end{aligned} \tag{2.14}$$

Hence, for all $n \in \mathbb{N}$ we obtain

$$\forall y^* \in \mathbb{R}^p \setminus \operatorname{rb}(0^+\bar A)^\circ : \qquad \delta^*(y^*|\bar A) \leq \delta^*(y^*|B_n). \tag{2.15}$$

Since $B_{n+1} \subseteq B_n$ and $B_n \neq \emptyset$ for all $n \in \mathbb{N}$, there exists some $n_0 \in \mathbb{N}$ such that the lineality space of B_n is constant for all $n \geq n_0$. Without loss of generality we can set $L := 0^+B_n \cap -0^+B_n$ for all $n \in \mathbb{N}$. We next show that $0^+\bar A \cap -0^+\bar A = L$. For $y^* \in \mathbb{R}^p \setminus L^\perp$ we have $\delta^*(y^*|B_n) = +\infty$ for all $n \in \mathbb{N}$. By (2.14) we have either $\delta^*(y^*|\bar A) = +\infty$ or

$y^* \in \operatorname{rb}(0^+\bar{A})^\circ$. Taking into account Lemma 1.3.1 (i) we obtain $\operatorname{ri}(0^+\bar{A})^\circ \subseteq L^\perp$, hence $L \subseteq 0^+A$. It follows that $L \subseteq 0^+\bar{A} \cap -0^+\bar{A}$. Further, from (2.15) and Lemma 1.3.1 (i) we deduce that $\operatorname{ri}(0^+B_n)^\circ \subseteq (0^+\bar{A})^\circ$ and hence $0^+\bar{A} \subseteq 0^+B_n$ for all $n \in \mathbb{N}$. This yields that $0^+\bar{A} \cap -0^+\bar{A} \subseteq L$.

Proceeding as in the proof of Lemma 2.3.5 we have $\big(0^+T_L(\bar{A})\big)^\circ = (0^+\bar{A})^\circ \oplus L$. Since $(0^+\bar{A})^\circ \subseteq L^\perp$, it follows $\big(0^+T_L(\bar{A})\big)^\circ \cap L^\perp = (0^+\bar{A})^\circ$. From [1, Theorem 6.5] we conclude that $\operatorname{ri}\big(0^+T_L(\bar{A})\big)^\circ \cap L^\perp = \operatorname{ri}(0^+\bar{A})^\circ$. Moreover, we have $\delta^*(y^*|\bar{A}) = \delta^*(y^*|T_L(\bar{A}))$ for all $y^* \in L^\perp$. For all $y^* \in \mathbb{R}^p$ and all $l \in L$ it holds $\delta^*(y^*|T_L(\bar{A})) = \delta^*(y^* + l\,|\,T_L(\bar{A}))$. Analogous assertions are valid for B_n (instead of $\bar{A}$). Together with (2.15) we obtain $\delta^*(y^*|T_L(\bar{A})) \leq \delta^*(y^*|T_L(B_n))$ for all $y^* \in \mathbb{R}^p \setminus \operatorname{rb}(0^+T_L(\bar{A}))^\circ$. Lemma 1.3.1 (iv) yields that $T_L(\bar{A}) \subseteq T_L(B_n)$ for all $n \in \mathbb{N}$. With the aid of Proposition 2.2.17 (iv) we conclude that $\bar{A} \subseteq B_n$ for all $n \in \mathbb{N}$ and so $\bar{A} \subseteq \bigcap_{n\in\mathbb{N}} B_n = \liminf_{n\to\infty} A_n$.

On the other hand, for all $y^* \in \mathbb{R}^p \setminus \operatorname{rb}(0^+\bar{A})^\circ$ we have

$$-\delta^*(y^*|\liminf_{n\to\infty} A_n) = -\delta^*(y^*|\sup_{n\in\mathbb{N}} B_n) \overset{\text{Pr. 1.3.7}}{\geq} \sup_{n\in\mathbb{N}} -\delta^*(y^*|B_n) \overset{(2.14)}{=} -\delta^*(y^*|\bar{A}).$$

From Lemma 1.3.1 (iii) we deduce that $\liminf_{n\to\infty} A_n \subseteq \bar{A}$. Hence $\liminf_{n\to\infty} A_n = \bar{A}$. As in the second part of the proof of Proposition 2.3.1 we obtain $\bar{A} = \limsup_{n\to\infty} A_n$ and hence $A_n \xrightarrow{\mathcal{C}} \bar{A}$. □

Chapter 3

Functions with values in $\hat{\mathcal{C}}$

In this chapter, we investigate $\hat{\mathcal{C}}$–valued functions, i.e., functions with values in the space $\hat{\mathcal{C}}$ of closed convex subsets of $\mathbb{R}^p$. In our optimization problems, based on set relations, the objective functions are of this type. Moreover, $\hat{\mathcal{C}}$–valued functions frequently occur in other fields of Optimization and Nonlinear Analysis. For instance, the sublevel map of a (epi–)closed quasi–convex vector–valued function (compare Luc [57, Theorem 5.8 and Proposition 6.3 (i)]) and many kinds of subdifferential maps $\partial f(\cdot)$ (e.g. the classical one [1] and that of Clarke [18]) are $\hat{\mathcal{C}}$–valued functions.

The aim of our investigations is to generalize some important concepts and assertions from Convex Analysis to the case of $\hat{\mathcal{C}}$–valued functions (instead of extended real–valued functions). In doing so, we try to point out the similar structures. It turns out that certain well–known concepts for set–valued maps, such as semi–continuity notions, are not appropriate in this context. Therefore, we introduce some new concepts. Our semi–continuity notions are compared with the usual notions of upper, lower, outer and inner semi–continuity of a set–valued map.

This chapter is organized as follows. In the first section, we summarize some well–known semi–continuity notions of set–valued maps. We point out that these concepts are adapted to the framework of $\hat{\mathcal{F}}$–valued functions. In the next section, we introduce a semi–continuity concept for $\hat{\mathcal{C}}$–valued functions. This concept is based on the results of Chapter 2 and seems to be new. In Section 3.3, we present a sufficient condition, called *local boundedness*, such that semi–continuity for $\hat{\mathcal{C}}$–valued functions (see Section 3.2) coincides with the classical concepts based on the outer and inner limits (see Section 3.1). Section 3.4 is devoted to convex functions. It can be seen that many assertions, which are known for extended real–valued convex functions, can be generalized to the context of this work. So we introduce conjugates for $\hat{\mathcal{C}}$–valued functions and, finally, in Section 3.6 we prove a biconjugation theorem, which is completely analogous to the classical variant.

3.1 Semi–continuity of $\hat{\mathcal{F}}$–valued functions

In this section, we summarize some well–known facts on semi–continuity of set–valued maps. Our main reference is the book by Rockafellar and Wets [68]. Even though in [68] arbitrary set–valued maps $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ are considered, we formulate these results only for $\hat{\mathcal{F}}$–valued functions, i.e., we suppose the closedness of the values. This is no loss of generality, because semi–continuity properties are not influenced by the closure operation to the values of the map f, see [68, Proposition 4.4, Definition 5.4].

Although in [68] the terms *outer* and *inner limit* (instead of upper and lower limit) are used, it might be beneficial to identify the relation $\subseteq$ by "less or equal". This is suggestive by the usage of the notation "limsup" and "liminf" in [68] and, furthermore, we use concepts like supremum, infimum, epigraph and hypograph, which are based on this identification. Let us recall the concepts of outer and inner limits as well as some of their basic properties. As in Section 2.1, we use capital letters in order to distinguish these concepts from the new concepts to be introduced in the next section.

Throughout this section (and this chapter) let $X = \mathbb{R}^n$, although many assertions are also valid in a more general context. The notation $\bigcup_{x_n \to \bar{x}}$, for instance, means the union over all sequences converging to $\bar{x}$.

Definition 3.1.1 ([68]) *Let $f : X \to \hat{\mathcal{F}}$. The* outer limit *of f at $\bar{x} \in X$ is defined by*

$$\operatorname*{LIM\,SUP}_{x \to \bar{x}} f(x) := \bigcup_{x_n \to \bar{x}} \operatorname*{LIM\,SUP}_{n \to \infty} f(x_n),$$

and the inner limit *of f at $\bar{x} \in X$ is defined by*

$$\operatorname*{LIM\,INF}_{x \to \bar{x}} f(x) := \bigcap_{x_n \to \bar{x}} \operatorname*{LIM\,INF}_{n \to \infty} f(x_n).$$

Note that the outer limit (and obviously also the inner limit) is always a closed subset of $\mathbb{R}^p$, i.e., it belongs to $\hat{\mathcal{F}}$, see [68, Proposition 4.4]. Together with (2.1) this yields the following description of outer and inner limits by the supremum and infimum in $(\hat{\mathcal{F}}, \subseteq)$.

$$\operatorname*{LIM\,SUP}_{x \to \bar{x}} f(x) = \operatorname*{SUP}_{x_n \to \bar{x}} \operatorname*{INF}_{N \in \mathcal{N}_\infty} \operatorname*{SUP}_{n \in N} f(x_n), \tag{3.1}$$

$$\operatorname*{LIM\,INF}_{x \to \bar{x}} f(x) = \operatorname*{INF}_{x_n \to \bar{x}} \operatorname*{INF}_{N \in \mathcal{N}_\infty^\#} \operatorname*{SUP}_{n \in N} f(x_n).$$

Definition 3.1.2 ([68]) *A function $f : X \to \hat{\mathcal{F}}$ is said to be* outer semi–continuous (osc) *at $\bar{x} \in X$ if*

$$f(\bar{x}) \supseteq \operatorname*{LIM\,SUP}_{x \to \bar{x}} f(x).$$

A function $f : X \to \hat{\mathcal{F}}$ is said to be inner semi–continuous (isc) *at $\bar{x} \in X$ if*

$$f(\bar{x}) \subseteq \operatorname*{LIM\,INF}_{x \to \bar{x}} f(x).$$

If f is osc (isc) at every $\bar{x} \in X$ we simply say f is osc (isc).

By the identification of $\subseteq$ with "less or equal" it is clear that outer and inner semi–continuity can be considered as generalizations of upper and lower semi–continuity of extended real–valued functions to the set–valued case. As noticed in [68], Choquet [17] already used the term *upper semi–continuous* instead of *outer semi–continuous*. But, what is often called *upper semi–continuous* in the literature differs from this concept, see [68, p.193 and 5.7(b)]. The inner semi–continuity, however, is mostly called lower semi–continuity in the literature. It is quite natural to ask for an equivalent description of semi–continuity by the "epigraph" and "hypograph" as it is well–known for extended real–valued functions.

Definition 3.1.3 *The* epigraph *of a function $f : X \to \hat{\mathcal{F}}$ is defined to be the set*

$$\operatorname{epi} f := \left\{(x, A) \in X \times \hat{\mathcal{F}} | \ A \supseteq f(x)\right\},$$

and the hypograph *of a function $f : X \to \hat{\mathcal{F}}$ is the set*

$$\operatorname{hyp} f := \left\{(x, A) \in X \times \hat{\mathcal{F}} | \ A \subseteq f(x)\right\}.$$

Note that, for all $x \in X$, we have $(x, \emptyset) \in \operatorname{hyp} f$ and $(x, \mathbb{R}^p) \in \operatorname{epi} f$. For a characterization of semi–continuity we need to know what is meant by closedness of the epigraph and hypograph.

Definition 3.1.4 *A subset $\mathcal{A} \subseteq X \times \hat{\mathcal{F}}$ is closed if for every sequence $\{(x_n, A_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ with $x_n \to \bar{x} \in X$ and $A_n \to \bar{A} \in \hat{\mathcal{F}}$ (with respect to Painlevé–Kuratowski convergence) it is true that $(\bar{x}, \bar{A}) \in \mathcal{A}$. The* closure *of a set $\mathcal{A} \subseteq X \times \hat{\mathcal{F}}$ is defined to be the set of all $(\bar{x}, \bar{A}) \in X \times \hat{\mathcal{F}}$ such that there is a sequence $\{(x_n, A_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ with $x_n \to \bar{x}$ and $A_n \to \bar{A}$. The closure of $\mathcal{A}$ is denoted by $\operatorname{cl} \mathcal{A}$.*

It follows the characterization of outer semi–continuity by the hypograph (and the graph) of the function.

Proposition 3.1.5 *For all functions $f : X \to \hat{\mathcal{F}}$ the following statements are equivalent:*

(i) $\operatorname{hyp} f$ *is closed,*

(ii) f *is osc,*

(iii) $\operatorname{gr} f \subseteq X \times \mathbb{R}^p$ *is closed.*

Proof. (i) $\Leftrightarrow$ (ii). Follows from [68, Exercise 5.6 (c)].
(ii) $\Leftrightarrow$ (iii). See [68, Theorem 5.7 (a)]. □

Likewise, inner semi–continuity can be characterized by the closedness of the epigraph. Note that the description by the graph fails in this case, i.e., a function $f : X \to \hat{\mathcal{F}}$ that is isc does not necessarily have a closed graph, see [68, Fig. 5–3. (b)].

Proposition 3.1.6 *For all functions $f : X \to \hat{\mathcal{F}}$ the following statements are equivalent:*

(i) $\operatorname{epi} f$ *is closed,*

(ii) f *is isc.*

Proof. Follows from [68, Exercise 5.6 (d)]. □

We next consider the outer semi–continuous hull of a $\hat{\mathcal{F}}$–valued function. We observe that the outer semi–continuous hull is defined analogously to the upper semi–continuous hull of extended real–valued functions and it has comparable properties.

Definition 3.1.7 *Let* $f : X \to \hat{\mathcal{F}}$. *The* outer semi–continuous hull *of* f *is the function* $(\operatorname{osc} f) : X \to \hat{\mathcal{F}}$ *defined by*

$$(\operatorname{osc} f)(x) := \operatorname*{LIM\,SUP}_{x' \to x} f(x').$$

Let us collect some basic properties of the outer semi–continuous hull.

Proposition 3.1.8 *Let* $f : X \to \hat{\mathcal{F}}$. *Then it holds*

(i) $\operatorname{gr}(\operatorname{osc} f) = \operatorname{cl}(\operatorname{gr} f)$,

(ii) $\operatorname{hyp}(\operatorname{osc} f) \supseteq \operatorname{cl}(\operatorname{hyp} f)$,

(iii) $(\operatorname{osc} f)$ *is osc,*

(iv) $\forall x \in X : \quad (\operatorname{osc} f)(x) \supseteq f(x)$,

(v) f *is osc at* $\bar{x} \in X \quad \Leftrightarrow \quad (\operatorname{osc} f)(\bar{x}) = f(\bar{x})$,

(vi) $\operatorname{gr} f$ *convex* $\Rightarrow$ $\operatorname{gr}(\operatorname{osc} f)$ *convex.*

Proof. (i) See [68, page 154, 5(2) and 5(3)].

(ii) Let $(\bar{x}, \bar{A}) \in \operatorname{cl}(\operatorname{hyp} f)$. Then, there exist $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ and $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{F}}$ such that $\bar{x} = \lim_{n\to\infty} x_n$, $\bar{A} = \operatorname{LIM}_{n\to\infty} A_n$ and $A_n \subseteq f(x_n)$ for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned}(\operatorname{osc} f)(\bar{x}) &= \operatorname*{LIM\,SUP}_{x\to\bar{x}} f(x) \supseteq \operatorname*{LIM\,SUP}_{n\to\infty} f(x_n)\\ &\supseteq \operatorname*{LIM\,SUP}_{n\to\infty} A_n = \operatorname*{LIM}_{n\to\infty} A_n = \bar{A},\end{aligned}$$

i.e., $(\bar{x}, \bar{A}) \in \operatorname{hyp}(\operatorname{osc} f)$.

(iii) By (i), $\operatorname{gr}(\operatorname{osc} f)$ is closed. Hence, $(\operatorname{osc} f)$ is osc, by Proposition 3.1.5.

(iv) Choosing the special sequence $x_n \equiv x$, we obtain

$$(\operatorname{osc} f)(x) = \operatorname*{LIM\,SUP}_{x'\to x} f(x') \supseteq \operatorname*{LIM\,SUP}_{n\to\infty} f(x_n) = \operatorname*{LIM\,SUP}_{n\to\infty} f(x) = f(x).$$

(v) By definition, f is osc at $\bar{x}$ if and only if $f(\bar{x}) \supseteq (\operatorname{osc} f)(\bar{x})$. By (iv), this equivalent to $f(\bar{x}) = (\operatorname{osc} f)(\bar{x})$.

(vi) Since $\operatorname{gr} f$ is convex, $\operatorname{cl}(\operatorname{gr} f)$ is convex, too. Hence, the convexity of $\operatorname{gr}(\operatorname{osc} f)$ follows from (i). □

The next example shows that the opposite inclusion in assertion (ii) of the previous proposition does not hold true, in general.

Example 3.1.9 Let $f : \mathbb{R} \to \hat{\mathcal{F}}(\mathbb{R})$ be defined by

$$f(x) = \begin{cases} \{-1\} & \text{if} \quad x < 0 \\ \{1\} & \text{if} \quad x > 0 \\ \emptyset & \text{if} \quad x = 0. \end{cases}$$

Then, $(0, \{-1, 1\})$ belongs to $\operatorname{hyp}(\operatorname{osc} f)$ but it does not belong to $\operatorname{cl}(\operatorname{hyp} f)$.

Remark 3.1.10 As noticed in [68], an analogous definition of the inner semi–continuous hull, namely by $(\operatorname{isc} f)(x) := \operatorname{LIM\,INF}_{x' \to x} f(x')$, is not constructive in the sense that $(\operatorname{isc} f)$ is not necessarily isc. In the framework of $\hat{\mathcal{C}}$–valued functions we will have similar problems. An example is given there.

Remark 3.1.11 In Proposition 3.1.8 (vi) we consider the convexity of $\operatorname{gr} f$. Recall that $\operatorname{gr} f$ is convex if and only if for all $\lambda \in [0, 1]$ and all $x_1, x_2 \in X$ is holds $f\big(\lambda x_1 + (1 - \lambda)x_2\big) \supseteq \lambda f(x_1) \oplus (1 - \lambda) f(x_2)$. Having in mind that $\subseteq$ has the meaning of "less or equal", this can be interpreted as concavity of f although in the literature this property is often called convexity of f. In view of the analogy to extended real–valued functions, Proposition 3.1.8 (vi) should be expressed as follows: f concave $\Rightarrow$ $(\operatorname{osc} f)$ concave.

3.2 Semi–continuity

Based on the considerations of Section 2.2 we introduce the concepts of upper and lower limits for functions with values in $\hat{\mathcal{C}}$. We first show that the concept of outer limit is not appropriate to define semi–continuity in the framework of $\hat{\mathcal{C}}$–valued functions. As in the previous section, we set $X = \mathbb{R}^n$, although many assertions are also valid in a more general context.

Example 3.2.1 Let $f : \mathbb{R} \to \mathcal{C}(\mathbb{R})$,

$$f(x) := \begin{cases} \{-1\} & \text{if} \quad x < 0 \\ \{0\} & \text{if} \quad x = 0 \\ \{1\} & \text{if} \quad x > 0. \end{cases}$$

Then the outer semi–continuous hull of f, namely

$$(\operatorname{osc} f)(x) = \begin{cases} \{-1\} & \text{if} \quad x < 0 \\ \{-1, 0, 1\} & \text{if} \quad x = 0 \\ \{1\} & \text{if} \quad x > 0, \end{cases}$$

is not convex–valued.

This might suggest to redefine the outer semi–continuous hull as follows:

$$(\widetilde{\mathrm{osc}}\, f)(x) := \mathrm{cl}\,\mathrm{conv} \operatorname*{LIM\,SUP}_{x' \to x} f(x').$$

However, $(\widetilde{\mathrm{osc}}\, f)$ has not necessarily a closed graph (and hence it is not osc) as the following example shows.

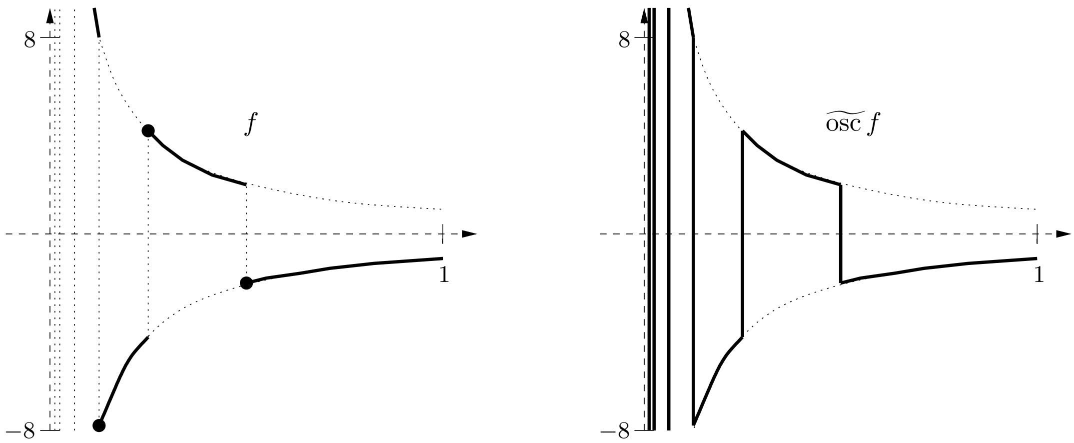
Example 3.2.2 Let $f : \mathbb{R} \to \hat{\mathcal{C}}(\mathbb{R})$,

$$f(x) := \begin{cases} \left\{\frac{1}{x}\right\} & \text{if} \quad \exists n \in \mathbb{N} : x \in \left[2^{-2n}, 2^{-2n+1}\right) \\ \left\{-\frac{1}{x}\right\} & \text{if} \quad \exists n \in \mathbb{N} : x \in \left[2^{-2n+1}, 2^{-2n+2}\right) \\ \emptyset & \text{else.} \end{cases}$$

Then the modified outer semi–continuous hull $(\widetilde{\mathrm{osc}}\, f)$ of f is obtained as

$$(\widetilde{\mathrm{osc}}\, f)(x) = \begin{cases} \left\{\frac{1}{x}\right\} & \text{if} \quad \exists n \in \mathbb{N} : x \in \left(2^{-2n}, 2^{-2n+1}\right) \\ \left\{-\frac{1}{x}\right\} & \text{if} \quad \exists n \in \mathbb{N} : x \in \left(2^{-2n+1}, 2^{-2n+2}\right) \\ \left[-\frac{1}{x}, \frac{1}{x}\right] & \text{if} \quad \exists n \in \mathbb{N} : x = 2^{-n} \\ \emptyset & \text{else.} \end{cases}$$

It is easily seen that $\mathrm{gr}\,(\widetilde{\mathrm{osc}}\, f)$ is not closed. Indeed, the sequence $\{(2^{-n}, 0)\}_{n\in\mathbb{N}}$ belongs to the graph of $(\widetilde{\mathrm{osc}}\, f)$, but its limit $(0, 0)$ does not.



We next introduce a new notion of upper and lower limits, which is adapted to the framework of $\hat{\mathcal{C}}$–valued functions. In the following, we frequently use Convention 1.2.2.

Definition 3.2.3 *Let $f : X \to \hat{\mathcal{C}}$ and let $\bar{x} \in X$. The* upper limit *of f at $\bar{x}$ is defined by*

$$\limsup_{x \to \bar{x}} f(x) := \sup_{x_n \to \bar{x}} \limsup_{n \to \infty} f(x_n),$$

and the lower limit *of f at $\bar{x}$ is defined by*

$$\liminf_{x \to \bar{x}} f(x) := \inf_{x_n \to \bar{x}} \liminf_{n \to \infty} f(x_n).$$

The limit of f at $\bar{x}$ exists if the upper and lower limits coincide. Then we write

$$\lim_{x \to \bar{x}} f(x) = \limsup_{x \to \bar{x}} f(x) = \liminf_{x \to \bar{x}} f(x).$$

In case $\hat{\mathcal{C}} = \hat{\mathcal{C}}(\mathbb{R})$, these concepts coincide with the classical upper and lower limits of extended real-valued functions $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ if we identify $f(\cdot)$ with the function $(\{f(\cdot)\} + \mathbb{R}_+) : X \to \hat{\mathcal{C}}^\star$ (where we set $\{+\infty\} + \mathbb{R}_+ = \emptyset$ and $\{-\infty\} + \mathbb{R}_+ = \mathbb{R}$).

As in the case of $\hat{\mathcal{F}}$-valued functions, the upper and lower limits can be expressed by the supremum and infimum.

Proposition 3.2.4 *Let $f : X \to \hat{\mathcal{C}}^\star$ and $\bar{x} \in X$. Then,*

$$\limsup_{x \to \bar{x}} f(x) = \sup_{x_n \to \bar{x}} \sup_{N \in \mathcal{N}_\infty^\#} \inf_{n \in N} f(x_n),$$

$$\liminf_{x \to \bar{x}} f(x) = \inf_{x_n \to \bar{x}} \sup_{N \in \mathcal{N}_\infty} \inf_{n \in N} f(x_n).$$

Let $f : X \to \hat{\mathcal{C}}^\diamond$ and $\bar{x} \in X$. Then,

$$\limsup_{x \to \bar{x}} f(x) = \sup_{x_n \to \bar{x}} \inf_{N \in \mathcal{N}_\infty} \sup_{n \in N} f(x_n),$$

$$\liminf_{x \to \bar{x}} f(x) = \inf_{x_n \to \bar{x}} \inf_{N \in \mathcal{N}_\infty^\#} \sup_{n \in N} f(x_n).$$

Proof. Follows from Definition 2.2.1 and Definition 3.2.3. □

In case of outer limits for $\hat{\mathcal{F}}$–valued functions (see Definition 3.1.1), we observe that the set $\bigcup_{x_n \to \bar{x}} \operatorname{LIM\,SUP}_{n\to\infty} f(x_n)$ is always closed, i.e., the closure operation, which is implicitly contained in the infimum in formula (3.1), is superfluous. An analogous result is valid for $\hat{\mathcal{C}}$–valued functions.

Proposition 3.2.5 *Let $f : X \to \hat{\mathcal{C}}^\star$ and $\bar{x} \in X$. Then it holds*

$$\liminf_{x \to \bar{x}} f(x) = \bigcup_{x_n \to \bar{x}} \liminf_{n \to \infty} f(x_n).$$

Proof. We have to show that $\bigcup_{x_n \to \bar{x}} \liminf_{n\to\infty} f(x_n)$ is convex and closed.

(i) *Convexity.* Let $y_1, y_2 \in \bigcup_{x_n \to \bar{x}} \liminf_{n\to\infty} f(x_n)$ and $\lambda \in [0,1]$ be given. Hence there exist sequences $\{x_n^{(i)}\}_{n\in\mathbb{N}} \subseteq X$, $(i = 1,2)$ with $x_n^{(i)} \to \bar{x}$ such that $y_i \in \liminf_{n\to\infty} f(x_n^{(i)})$. We define a sequence $\{x_n^{(3)}\}_{n\in\mathbb{N}} \subseteq X$ by

$$\{x_n^{(3)}\}_{n\in\mathbb{N}} := \left\{x_1^{(1)}, x_1^{(2)}, x_2^{(1)}, x_2^{(2)}, x_3^{(1)}, x_3^{(2)}, ...\right\}.$$

Since $\{x_n^{(i)}\}_{n\in\mathbb{N}}$, $(i = 1,2)$ are subsequences of $\{x_n^{(3)}\}_{n\in\mathbb{N}}$, Proposition 2.2.6 yields that $\liminf_{n\to\infty} f(x_n^{(i)}) \subseteq \liminf_{n\to\infty} f(x_n^{(3)})$, $(i = 1,2)$. Hence we obtain $\lambda y_1 + (1-\lambda) y_2 \in \liminf_{n\to\infty} f(x_n^{(3)})$. From $x_n^{(3)} \to \bar{x}$ it follows that $\lambda y_1 + (1-\lambda)y_2 \in \bigcup_{x_n \to \bar{x}} \liminf_{n\to\infty} f(x_n)$.

(ii) *Closedness.* Let $\{y_m\}_{m\in\mathbb{N}} \subseteq \bigcup_{x_n \to \bar{x}} \liminf_{n\to\infty} f(x_n)$ be convergent to some $\bar{y} \in \mathbb{R}^p$. For all $m \in \mathbb{N}$ there exists a sequence $\{x_n^{(m)}\}_{n\in\mathbb{N}} \subseteq X$ such that $\bar{x} = \lim_{n\to\infty} x_n^{(m)}$ and

$y_m \in \liminf_{n\to\infty} f(x_n^{(m)})$. Hence, we can construct a strictly increasing function $n_0 : \mathbb{N} \to \mathbb{N}$ by

$$\forall m \in \mathbb{N},\ \exists n_0(m) \in \mathbb{N},\ \forall n \geq n_0(m),\ \forall k \in \{1, ..., m\} : \ \left\| x_n^{(k)} - \bar{x} \right\| < \frac{1}{m}.$$

An "inverse function" of $n_0 : \mathbb{N} \to \mathbb{N}$ can be defined by $m_0(n) := \sup \{m \in \mathbb{N} |\ n \geq n_0(m)\}$. Of course, $m_0 : \mathbb{N} \to \mathbb{N} \cup \{-\infty\}$ is (not necessarily strictly) increasing and we have $m_0(n) \to \infty$ for $n \to \infty$. Consider the sequence $\{\bar{x}_n\}_{n\in\mathbb{N}} \subseteq X$ defined by

$$\{\bar{x}_n\}_{n\in\mathbb{N}} := \left\{ x_1^{(1)}, x_1^{(2)}, ..., x_1^{(m_0(1))}, x_2^{(1)}, x_2^{(2)}, ..., x_2^{(m_0(2))}, \ldots, x_n^{(1)}, x_n^{(2)}, ..., x_n^{(m_0(n))}, \ldots \right\}.$$

where, without loss of generality, it can be assumed that $m_0(n) \neq -\infty$ for all $n \in \mathbb{N}$. Clearly, the sequence $\{\bar{x}_n\}_{n\in\mathbb{N}}$ converges to $\bar{x}$ and it has the following property:

$$\forall m \in \mathbb{N} : \{x_n^{(m)}\}_{n \geq n_0(m)} \subseteq \{\bar{x}_n\}_{n\in\mathbb{N}}.$$

By Proposition 2.2.6 (i), for all $m \in \mathbb{N}$ we have $\liminf_{n\to\infty} f(x_n^{(m)}) \subseteq \liminf_{n\to\infty} f(\bar{x}_n)$ and, consequently, the sequence $\{y_m\}_{m\in\mathbb{N}}$ is a subset of $\liminf_{n\to\infty} f(\bar{x}_n)$. Since $\liminf_{n\to\infty} f(\bar{x}_n)$ is a closed subset of $\mathbb{R}^p$ it follows that $\bar{y} \in \liminf_{n\to\infty} f(\bar{x}_n) \subseteq \bigcup_{x_n \to \bar{x}} \liminf_{n\to\infty} f(x_n)$. □

The following relationship between the outer and inner limits in $\hat{\mathcal{F}}$ and the upper and lower limits in $\hat{\mathcal{C}}$ is an easy consequence of the definition. For $f : X \to \hat{\mathcal{C}}^\star$ we have

$$\liminf_{x\to\bar{x}} f(x) \supseteq \operatorname*{LIM\,SUP}_{x\to\bar{x}} f(x) \qquad \text{and} \qquad \limsup_{x\to\bar{x}} f(x) \supseteq \operatorname*{LIM\,INF}_{x\to\bar{x}} f(x). \tag{3.2}$$

In case of $f : X \to \hat{\mathcal{C}}^\diamond$, it always holds

$$\liminf_{x\to\bar{x}} f(x) \supseteq \operatorname*{LIM\,INF}_{x\to\bar{x}} f(x) \qquad \text{and} \qquad \limsup_{x\to\bar{x}} f(x) \supseteq \operatorname*{LIM\,SUP}_{x\to\bar{x}} f(x).$$

We next define semi–continuity concepts for functions $\hat{\mathcal{C}}$–valued functions.

Definition 3.2.6 *A function $f : X \to \hat{\mathcal{C}}$ is* lower semi–continuous (lsc) *at $\bar{x} \in X$ if*

$$f(\bar{x}) \leq \liminf_{x\to\bar{x}} f(x).$$

A function $f : X \to \hat{\mathcal{C}}$ is said to be upper semi–continuous (usc) *at $\bar{x} \in X$ if*

$$f(\bar{x}) \geq \limsup_{x\to\bar{x}} f(x).$$

A function $f : X \to \hat{\mathcal{C}}$ is said to be continuous *at $\bar{x} \in X$ if f is simultaneously lsc and usc at $\bar{x}$. If f is lsc (usc, continuous) at every $\bar{x} \in X$ we say f is lsc (usc, continuous).*

Of course, f is continuous at $\bar{x} \in X$ if and only if $f(\bar{x}) = \lim_{x\to\bar{x}} f(x)$.

Proposition 3.2.7 *Let $f : X \to \hat{\mathcal{C}}^\star$ and $\bar{x} \in X$. If f is isc (lsc) at $\bar{x}$, then f is usc (osc) at the same point.*

Proof. Follows from (3.2). □

Another benefit of our semi–continuity concept is the following characterization, which will be used in the proof of the biconjugation theorem in Section 3.6 below. The proof is based on the assertions of Section 2.3.

Theorem 3.2.8 *Let $f : X \to \hat{\mathcal{C}}^\star$ and let $\bar{x} \in \operatorname{dom} f$. Then the following statements are equivalent:*

(i) *f is lsc at $\bar{x}$,*

(ii) *For all $y^* \in \operatorname{ri}\left(0^+ f(\bar{x})\right)^\circ$ the function*

$$\bar{f}_{y^*} : X \to \mathbb{R} \cup \{-\infty, +\infty\}, \quad \bar{f}_{y^*}(x) := -\delta^*\big(y^* | f(x)\big)$$

is lsc at $\bar{x}$.

Proof. Let be given an arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \to \bar{x}$.

(i) $\Rightarrow$ (ii). Let the sequence $\{\tilde{x}_n\}_{n\in\mathbb{N}}$ be defined by $\tilde{x}_{2n} := x_n$ and $\tilde{x}_{2n+1} := \bar{x}$. From (i) we deduce that $f(\bar{x}) = \liminf_{n\to\infty} f(\tilde{x}_n)$. Lemma 2.3.5 implies that

$$\forall y^* \in \operatorname{ri}\left(0^+ f(\bar{x})\right)^\circ : \quad -\delta^*\big(y^*|f(\bar{x})\big) = \liminf_{n\to\infty} -\delta^*\big(y^*|f(\tilde{x}_n)\big) \leq \liminf_{n\to\infty} -\delta^*\big(y^*|f(x_n)\big).$$

(ii) $\Rightarrow$ (i). With aid of Proposition 1.3.7 we obtain

$$\forall y^* \in \operatorname{ri}\left(0^+ f(\bar{x})\right)^\circ : \quad -\delta^*\big(y^*|f(\bar{x})\big) \leq \liminf_{n\to\infty} -\delta^*\big(y^*|f(x_n)\big) \leq -\delta^*\big(y^*| \liminf_{n\to\infty} f(x_n)\big).$$

From Lemma 1.3.1 (iii) we deduce that $f(\bar{x}) \leq \liminf_{n\to\infty} f(x_n)$. □

The next assertion about nested lower limits is essential for an expedient definition of the *lower semi–continuous hull* of a $\hat{\mathcal{C}}^\star$–valued function. An analogous assertion for the upper limit is not true, see the Example 3.2.15 below.

Proposition 3.2.9 *Let $f : X \to \hat{\mathcal{C}}^\star$ and $\bar{x} \in X$. Then it holds*

$$\liminf_{x\to\bar{x}} f(x) = \liminf_{x\to\bar{x}} \Big(\liminf_{w\to x} f(w)\Big).$$

Proof. Clearly, we have $f(x) \geq \liminf_{w\to x} f(w)$ for all $x \in X$ and hence

$$\liminf_{x\to\bar{x}} f(x) \geq \liminf_{x\to\bar{x}} \Big(\liminf_{w\to x} f(w)\Big).$$

In order to show the opposite inequality let $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \to \bar{x}$ be arbitrarily given. For all $y^* \in \operatorname{ri}\left(0^+ \liminf_{n\to\infty} f(x_n)\right)^\circ$ it holds

$$\begin{aligned}
-\delta^*\left(y^* \middle| \liminf_{n\to\infty}\big(\liminf_{w\to x_n} f(w)\big)\right) &\overset{\text{Pr. 1.3.7}}{\geq} \liminf_{n\to\infty}\left(\liminf_{w\to x_n} -\delta^*\big(y^*|f(w)\big)\right) \\
&= \liminf_{n\to\infty} -\delta^*\big(y^*|f(x_n)\big) \\
&\overset{\text{Lem. 2.3.5}}{=} -\delta^*\left(y^*| \liminf_{n\to\infty} f(x_n)\right).
\end{aligned}$$

Lemma 1.3.1 (iii) yields that $\liminf_{n\to\infty}\big(\liminf_{w\to x_n} f(w)\big) \subseteq \liminf_{n\to\infty} f(x_n)$. Hence $\liminf_{n\to\infty}\big(\liminf_{w\to x_n} f(w)\big) \geq \inf_{x_n\to\bar{x}} \liminf_{n\to\infty} f(x_n) = \liminf_{x\to\bar{x}} f(x)$. □

We next introduce the *lower semi–continuous hull* of a $\hat{\mathcal{C}}^\star$–valued function.

Definition 3.2.10 *Let $f : X \to \hat{\mathcal{C}}^\star$. The* lower semi–continuous hull *of f is the function $(\operatorname{lsc} f) : X \to \hat{\mathcal{C}}^\star$, defined by*

$$(\operatorname{lsc} f)(x) := \liminf_{x'\to x} f(x').$$

In order to show that the lower semi–continuous hull of a function $f : X \to \hat{\mathcal{C}}^\star$ has similar (but not completely the same) properties as the outer semi–continuous hull we define the epigraph of f as well as its closedness.

Definition 3.2.11 *The* epigraph *of a function $f : X \to \hat{\mathcal{C}}^\star$ is defined to be the set*

$$\operatorname{epi} f := \left\{(x, A) \in X \times \hat{\mathcal{C}}^\star \mid A \geq f(x)\right\}.$$

Definition 3.2.12 *A subset $\mathcal{A} \subseteq X \times \hat{\mathcal{C}}^\star$ is closed if for every sequence $\{(x_n, A_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ with $x_n \to \bar{x} \in X$ and $A_n \to \bar{A} \in \hat{\mathcal{C}}^\star$ (with respect to $\mathcal{C}$–convergence) it is true that $(\bar{x}, \bar{A}) \in \mathcal{A}$. The* closure *of a set $\mathcal{A} \subseteq X \times \hat{\mathcal{C}}^\star$ is defined to be the set of all $(\bar{x}, \bar{A}) \in X \times \hat{\mathcal{C}}^\star$ such that there is a sequence $\{(x_n, A_n)\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ with $x_n \to \bar{x}$ and $A_n \to \bar{A}$. The closure of $\mathcal{A}$ is denoted by $\operatorname{cl}\mathcal{A}$.*

Let us collect some properties of the lower semi–continuous hull of a $\hat{\mathcal{C}}^\star$–valued function.

Proposition 3.2.13 *For $f : X \to \hat{\mathcal{C}}^\star$ the following statements hold true:*

(i) $\operatorname{gr}(\operatorname{lsc} f) \supseteq \operatorname{cl}(\operatorname{gr} f)$,

(ii) $\operatorname{epi}(\operatorname{lsc} f) \supseteq \operatorname{cl}(\operatorname{epi} f)$,

(iii) $(\operatorname{lsc} f)$ *is lsc,*

(iv) $\forall x \in X : \quad (\operatorname{lsc} f)(x) \leq f(x)$,

(v) f *is lsc at* $\bar{x} \in X \quad\Leftrightarrow\quad (\operatorname{lsc} f)(\bar{x}) = f(\bar{x})$,

(vi) $\operatorname{gr}(\operatorname{lsc} f)$ *is closed,*

(vii) $\operatorname{epi}(\operatorname{lsc} f)$ *is closed.*

Proof. (i) Let $(\bar{x}, \bar{y}) \in \operatorname{cl}(\operatorname{gr} f)$. Then there exists a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq \operatorname{gr} f$ converging to $(\bar{x}, \bar{y})$. Let $\{y_n\}, \{\bar{y}\} \in \mathcal{C}^\star$. For all $n \in \mathbb{N}$, we have $\{y_n\} \geq f(x_n)$. Hence

$$\begin{aligned} \{\bar{y}\} &= \lim_{n\to\infty} \{y_n\} = \liminf_{n\to\infty} \{y_n\} \overset{\text{Pr. 2.2.7}}{=} \liminf_{n\to\infty} f(x_n) \\ &\geq \inf_{x'_n\to\bar{x}} \liminf_{n\to\infty} f(x'_n) = \liminf_{x\to\bar{x}} f(x) = (\operatorname{lsc} f)(\bar{x}), \end{aligned} \tag{3.3}$$

i.e., $(\bar{x}, \bar{y}) \in \operatorname{gr}(\operatorname{lsc} f)$.

(ii) Let $(\bar{x}, \bar{A}) \in \operatorname{cl}(\operatorname{epi} f)$. Then there exist $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ and $\{A_n\}_{n\in\mathbb{N}} \subseteq \hat{\mathcal{C}}^\star$ such that $\bar{x} = \lim_{n\to\infty} x_n$, $\bar{A} = \lim_{n\to\infty} A_n$ and $A_n \geq f(x_n)$ for all $n \in \mathbb{N}$. Similarly to (3.3), we obtain $(\bar{x}, \bar{A}) \in \operatorname{epi}(\operatorname{lsc} f)$.

(iii) Let $\bar{x} \in X$ be arbitrarily given. Then, by Proposition 3.2.9, we have

$$(\operatorname{lsc} f)(\bar{x}) = \liminf_{x\to\bar{x}} f(x) = \liminf_{x\to\bar{x}}\big(\liminf_{w\to x} f(w)\big) = \liminf_{x\to\bar{x}}(\operatorname{lsc} f)(x).$$

(iv) Choosing the special sequence $x_n \equiv x$ we obtain

$$(\operatorname{lsc} f)(x) = \liminf_{x'\to x} f(x') \leq \liminf_{n\to\infty} f(x_n) = \liminf_{n\to\infty} f(x) = f(x).$$

(v) By definition, f is lsc at $\bar{x}$ if and only if $f(\bar{x}) \leq (\operatorname{lsc} f)(\bar{x})$. By (iv) this equivalent to $f(\bar{x}) = (\operatorname{lsc} f)(\bar{x})$.

(vi) Let $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq \operatorname{gr}(\operatorname{lsc} f)$ with $(x_n, y_n) \to (\bar{x}, \bar{y}) \in X \times \mathbb{R}^p$ be given. Proceeding as in (i), but replacing f by $(\operatorname{lsc} f)$, we obtain $\{\bar{y}\} \geq (\operatorname{lsc}(\operatorname{lsc} f))(\bar{x})$. From (iii) we conclude that $(\operatorname{lsc}(\operatorname{lsc} f))(\bar{x}) = (\operatorname{lsc} f)(\bar{x})$. Hence $(\bar{x}, \bar{y}) \in \operatorname{gr}(\operatorname{lsc} f)$.

(vii) Let $\{(x_n, A_n)\}_{n\in\mathbb{N}} \subseteq \operatorname{epi}(\operatorname{lsc} f)$ with $\bar{x} = \lim_{n\in\mathbb{N}} x_n$, $\bar{A} = \lim_{n\in\mathbb{N}} A_n$. Proceeding as in (ii), but replacing f by $(\operatorname{lsc} f)$, we obtain $(\bar{x}, \bar{A}) \in \operatorname{epi}(\operatorname{lsc}(\operatorname{lsc} f))$. From (iii) we conclude that $(x, \bar{A}) \in \operatorname{epi}(\operatorname{lsc} f)$. □

The next example shows that neither the closedness of epi f nor the closedness of gr f implies that f is lsc.

Example 3.2.14 Let $f : \mathbb{R} \to \hat{\mathcal{C}}^\star(\mathbb{R})$ be defined by

$$f(x) = \begin{cases} \{1/x\} & \text{if } \; x \neq 0 \\ \emptyset & \text{if } \; x = 0. \end{cases}$$

It can be easily seen that $\operatorname{gr} f \subseteq \mathbb{R}\times\mathbb{R}$ is closed. Moreover, we observe that $\operatorname{epi} f \subseteq \mathbb{R}\times\hat{\mathcal{C}}^\star(\mathbb{R})$ is closed. Indeed, let $\{(x_n, A_n)\}_{n\in\mathbb{N}} \subseteq \operatorname{epi} f$ be a sequence converging to $(0, \bar{A})$. (the case $\bar{x} \neq 0$ is obvious). We can assume that there is a subsequence $\{x_n\}_{n\in\hat{N}}$ $(\hat{N} \in \mathcal{N}_\infty^\#)$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n > 0$ for all $n \in \hat{N}$ or $x_n < 0$ for all $n \in \hat{N}$ (otherwise we obviously have $\bar{A} = \emptyset$). Then, by Proposition 2.2.6, it follows $\bar{A} = \lim_{n\to\infty} A_n = \limsup_{n\to\infty} A_n \geq \limsup_{n\in\hat{N}} A_n \geq \liminf_{n\in\hat{N}} A_n$. Now it is easy to see that $\liminf_{n\in\hat{N}} A_n = \emptyset$ and hence $\bar{A} = \emptyset$. Of course, it holds $(0, \emptyset) \in \operatorname{epi} f$. However, f is not lsc, because $(\operatorname{lsc} f)$ can be easily calculated as

$$(\operatorname{lsc} f)(x) = \begin{cases} \{1/x\} & \text{if } \; x \neq 0 \\ \mathbb{R} & \text{if } \; x = 0. \end{cases}$$

In Remark 3.1.10 (due to [68]) we noticed that an inner semi–continuous hull of a $\hat{\mathcal{F}}$–valued function that is analogously defined to the outer semi–continuous hull is not necessarily inner semi–continuous. There are analogous problems with the upper semi–continuous hull of $\hat{\mathcal{C}}^\star$–valued functions. This is due to the fact that there is no analogous assertion to Proposition 3.2.9 for upper limits, as the following example shows.

Example 3.2.15 For functions $f : X \to \hat{\mathcal{C}}^\star$, in general, we have

$$\limsup_{x\to\bar{x}} f(x) \neq \limsup_{x\to\bar{x}}\left(\limsup_{w\to x} f(w)\right).$$

Indeed, consider the function $f : \mathbb{R}^2 \to \hat{\mathcal{C}}^\star(\mathbb{R})$, defined by

$$f(x) := \begin{cases} \{\|x\|\} & \text{if} \quad x_1 \geq 0 \\ \{-\|x\|\} & \text{if} \quad x_1 < 0. \end{cases}$$

Then it holds

$$\limsup_{w\to x} f(w) := \begin{cases} \{\|x\|\} & \text{if} \quad x_1 > 0 \text{ or } x_2 = 0 \\ \{-\|x\|\} & \text{if} \quad x_1 < 0 \\ \emptyset & \text{if} \quad x_1 = 0 \text{ and } x_2 \neq 0. \end{cases}$$

Hence we obtain

$$\{0\} = \limsup_{x\to 0} f(x) \neq \limsup_{x\to 0}\left(\limsup_{w\to x} f(w)\right) = \emptyset.$$

3.3 Locally bounded functions

The concept of local boundedness of a set–valued map plays an important role in Variational Analysis, see [68]. As an easy consequence of the definition ([68, Definition 5.14]), local boundedness of a map $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ at $\bar{x}$ implies that $f(\bar{x})$ is a bounded subset of $\mathbb{R}^p$. This means, local boundedness is (at least locally) adapted to set–valued maps with bounded values. Therefore we introduce a slightly generalized concept, adapted to the framework of $\hat{\mathcal{C}}^\star$–valued functions. It turns out that this concept provides a sufficient condition for the coincidence of lower (upper) semi–continuity with outer (inner) semi–continuity. In this section, we set $X = \mathbb{R}^n$.

Definition 3.3.1 *A function $f : X \to \hat{\mathcal{C}}^\star$ is said to be* locally bounded *at $\bar{x} \in \operatorname{dom} f$ if there exists a neighborhood $V \in \mathcal{N}(\bar{x})$ such that the following conditions are satisfied:*

(i) $0^+ \inf_{x\in V} f(x) \subseteq 0^+ f(\bar{x})$,

(ii) $\forall x \in V \cap \operatorname{dom} f : \quad 0^+ f(x) \supseteq 0^+ f(\bar{x})$.

Remark 3.3.2 If $f : X \to \hat{\mathcal{C}}^\star$ locally bounded at $\bar{x} \in \operatorname{dom} f$, (i) and (ii) of the previous definition are always satisfied with equality. (Indeed, the opposite inclusion in (i) follows from $\inf_{x\in V} f(x) \leq f(\bar{x})$ and Proposition A.9 and the opposite inclusion in (ii) follows from (i).) In view of the embedding theorem in the form of Corollary 1.3.4 this means, locally, the values of f can be embedded into a partially ordered linear space (extended by a largest element) and the embedding yields a set in the linear space which is bounded below.

We next clarify the relations between our local boundedness concept and the classical one.

Proposition 3.3.3 *Let $f : X \to \hat{\mathcal{C}}^\star$ such that $f(\bar{x})$ is bounded for some $\bar{x} \in \operatorname{dom} f$. Then f is locally bounded in the sense of the above definition if and only if f is locally bounded in the sense of [68] (where f is understood to be a set–valued map $f : X \rightrightarrows \mathbb{R}^p$ in the latter case).*

Proof. In the present situation, condition (i) of Definition 3.3.1 has the meaning of $0^+\operatorname{cl}\operatorname{conv}\bigcup_{x\in V} f(x) = \{0\}$ and condition (ii) is always satisfied. This is equivalent to the boundedness of $\bigcup_{x\in V} f(x)$. □

Theorem 3.3.4 *Let $f : X \to \hat{\mathcal{C}}^\star$ be locally bounded at $\bar{x} \in \operatorname{dom} f$. Then,*

$$\liminf_{x\to\bar{x}} f(x) = \operatorname{cl}\operatorname{conv} \operatorname*{LIM\,SUP}_{x\to\bar{x}} f(x).$$

Proof. Clearly, we have $\liminf_{x\to\bar{x}} f(x) \supseteq \operatorname{cl}\operatorname{conv} \operatorname{LIM\,SUP}_{x\to\bar{x}} f(x)$. To show the opposite inclusion let $y \in \liminf_{x\to\bar{x}} f(x)$ be given. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \to \bar{x}$ such that $y \in \liminf_{n\to\infty} f(x_n)$. Assuming that there exists some $n_0 \in \mathbb{N}$ such that $f(x_n) = \emptyset$ for all $n \geq n_0$, we obtain $\liminf_{n\to\infty} f(x_n) = \emptyset$, which contradicts $y \in \liminf_{x\to\bar{x}} f(x)$. Hence, by $\{x_{n_k}\}_{k\in\mathbb{N}} := \{x_n\}_{n\in\mathbb{N}} \cap \operatorname{dom} f$, we obtain a subsequence of $\{x_n\}_{n\in\mathbb{N}}$. By the definition of the lower limit, we have $\liminf_{n\to\infty} f(x_n) = \liminf_{k\to\infty} f(x_{n_k})$. By the local boundedness, we find $k_0 \in \mathbb{N}$ such that, setting $K := 0^+f(\bar{x})$, $f(x_{n_k})$ and $\inf_{k\geq k_0} f(x_{n_k})$ belong to $\hat{\mathcal{C}}^\star_K$ for all $k \geq k_0$. Theorem 2.2.18 yields $y \in \operatorname{cl}\operatorname{conv} \operatorname{LIM\,SUP}_{k\to\infty} f(x_{n_k})$, hence we obtain $y \in \operatorname{cl}\operatorname{conv} \bigcup_{x_n\to\bar{x}} \operatorname{LIM\,SUP}_{n\to\infty} f(x_n) = \operatorname{LIM\,SUP}_{x\to\bar{x}} f(x)$. □

In the next example we show that the assertion of the preceeding theorem can fail if one of the conditions in the definition of the local boundedness concept is not satisfied.

Example 3.3.5 Let $f : \mathbb{R} \to \mathcal{C}^\star(\mathbb{R})$ be defined by

$$f(x) = \begin{cases} \{\frac{1}{x}\} & \text{if} \quad x \neq 0 \\ \{0\} & \text{if} \quad x = 0, \end{cases}$$

i.e., (ii) is satisfied, but (i) is not. Then we have

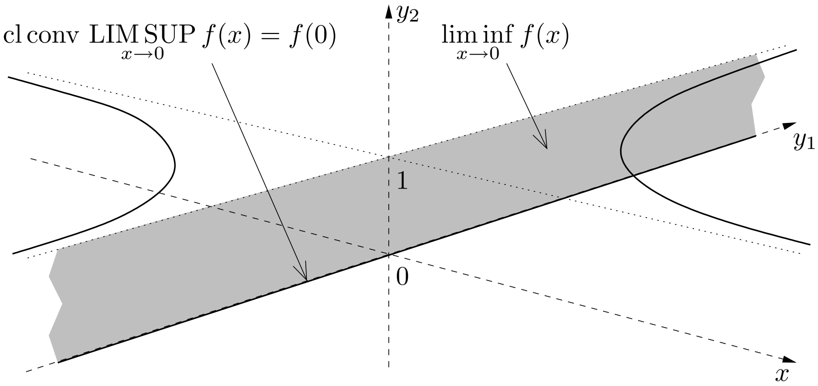
$$\mathbb{R} = \liminf_{x\to 0} f(x) \neq \operatorname{cl}\operatorname{conv} \operatorname*{LIM\,SUP}_{x\to 0} f(x) = \{0\}\,.$$

Example 3.3.6 Let $f : \mathbb{R} \to \mathcal{C}^\star(\mathbb{R}^2)$ be defined by

$$f(x) = \begin{cases} \left\{y \in \mathbb{R}^2 |\; y_2 = 1, y_1 = 1/x\right\} & \text{if} \quad x \neq 0 \\ \left\{y \in \mathbb{R}^2 |\; y_2 = 0\right\} & \text{if} \quad x = 0, \end{cases}$$

i.e., (i) is satisfied, but (ii) is not. An easy calculation shows that

$$\left\{y \in \mathbb{R}^2 |\; 0 \leq y_2 \leq 1\right\} = \liminf_{x\to 0} f(x) \neq \operatorname{cl}\operatorname{conv} \operatorname*{LIM\,SUP}_{x\to 0} f(x) = \left\{y \in \mathbb{R}^2 |\; y_2 = 0\right\}.$$



The local boundedness of a function $f : X \to \hat{\mathcal{C}}^\star$ at a point $\bar{x} \in \operatorname{dom} f$ also implies that $\limsup_{x\to\bar{x}} f(x) = \operatorname{LIM\,INF}_{x\to\bar{x}} f(x)$ (see Corollary 3.3.9 below). However, as shown in the next theorem, a weaker assumption is already sufficient.

Theorem 3.3.7 *Let $f : X \to \hat{\mathcal{C}}^\star$ be a function and let $\bar{x} \in \operatorname{dom} f$ such that for all sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ with $x_n \to \bar{x}$ there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ and a nonempty closed convex cone $K \subseteq \mathbb{R}^p$ with*

$$\forall k \in \mathbb{N}: \; f(x_{n_k}) \in \mathcal{C}_K^\star \qquad \textit{and} \qquad \inf_{k\in\mathbb{N}} f(x_{n_k}) \in \mathcal{C}_K^\star.$$

Then it holds

$$\limsup_{x\to\bar{x}} f(x) = \operatorname*{LIM\,INF}_{x\to\bar{x}} f(x).$$

Proof. Of course, $\limsup_{x\to\bar{x}} f(x) \supseteq \operatorname{LIM\,INF}_{x\to\bar{x}} f(x)$. In order to show the opposite inclusion let $y \in \mathbb{R}^p \setminus \operatorname{LIM\,INF}_{x\to\bar{x}} f(x)$ be given (the case $\operatorname{LIM\,INF}_{x\to\bar{x}} f(x) = \mathbb{R}^p$ is obvious). Hence there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ with $x_n \to \bar{x}$ such that $y \notin \operatorname{LIM\,INF}_{n\to\infty} f(x_n)$. Every subsequence of $\{x_n\}_{n\in\mathbb{N}}$ is again a sequence converging to $\bar{x}$, hence our assumption ensures that Theorem 2.2.19 is applicable. It follows that $y \notin \limsup_{n\to\infty} f(x_n)$. □

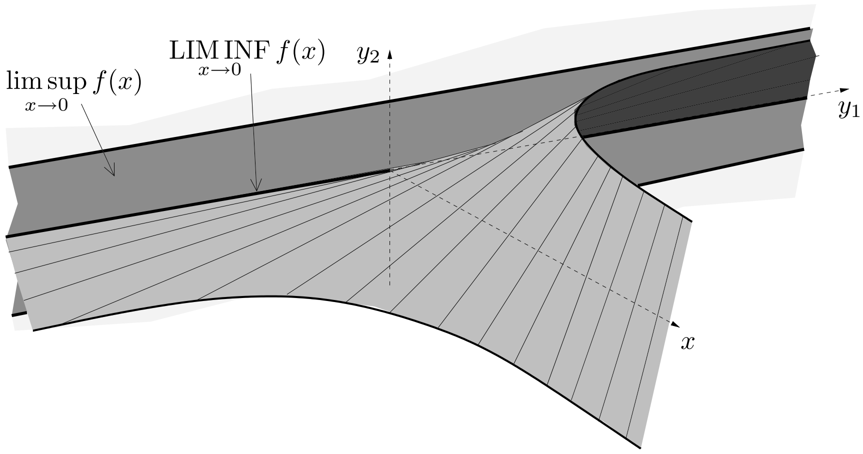
The next example shows that the assertion of the previous theorem can fail if the assumption is not satisfied.

Example 3.3.8 Let $f : \mathbb{R} \to \mathcal{C}^\star(\mathbb{R}^2)$ be defined by

$$f(x) = \begin{cases} \mathbb{R}^2 & \text{if} \quad x \le 0 \\ \operatorname{conv}\left\{\left(-1, -\frac{1}{x}\right), \left(1, \frac{1}{x}\right)\right\} & \text{if} \quad x > 0, \end{cases}$$

i.e., the condition in the previous theorem is not satisfied. Then we have

$$\left\{y \in \mathbb{R}^2 \,|\, -1 \le y_2 \le 1\right\} = \limsup_{x\to 0} f(x) \ne \operatorname*{LIM\,INF}_{x\to 0} f(x) = \left\{y \in \mathbb{R}^2 \,|\, y_2 = 0\right\}.$$



Corollary 3.3.9 *Let $f : X \to \hat{\mathcal{C}}^\star$ be locally bounded at $\bar{x} \in \operatorname{dom} f$. Then,*

$$\limsup_{x\to\bar{x}} f(x) = \operatorname{LIM\,INF}_{x\to\bar{x}} f(x).$$

Proof. By the local boundedness of f at $\bar{x}$, for every sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \to \bar{x}$ there exists some $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 : \; f(x_n) \in \mathcal{C}_K^\star \qquad \text{and} \qquad \inf_{n\geq n_0} f(x_n) \in \mathcal{C}_K^\star,$$

where $K := 0^+ f(\bar{x})$. Hence, Theorem 3.3.7 yields the desired assertion. □

Corollary 3.3.10 *Let $f : X \to \hat{\mathcal{C}}^\star$ be locally bounded at every $\bar{x} \in \operatorname{dom} f$. Then the following statements are equivalent:*

(i) $\operatorname{epi} f \subseteq X \times \hat{\mathcal{C}}^\star$ *is closed,*

(ii) f *is lsc,*

(iii) $\operatorname{gr} f \subseteq X \times \mathbb{R}^p$ *is closed.*

Proof. (i) ⇒ (iii). Elementary (using Proposition 2.2.13 (i)).

(iii) ⇒ (ii). Proposition 3.1.5 yields that f is osc. By Theorem 3.3.4, f is lsc.

(ii) ⇒ (i). Follows from Proposition 3.2.13 (v), (vii). □

3.4 Convex functions

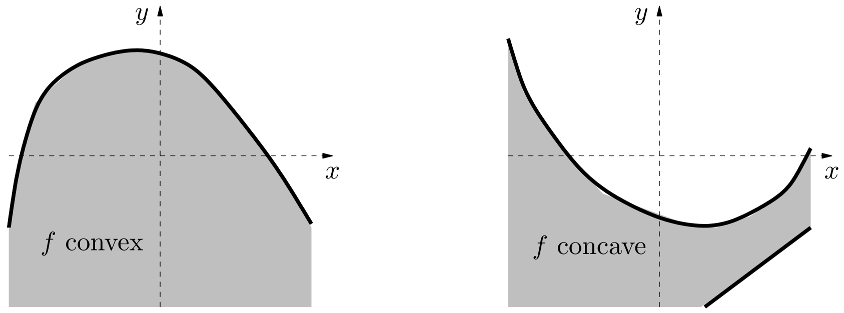
In this section, we investigate convex and concave $\hat{\mathcal{C}}$–valued functions. As a main result, we show that semi–continuity in the sense of Section 3.2 coincides with the classical concepts of outer and inner semi–continuity. We next introduce convex and concave $\hat{\mathcal{C}}$–valued functions. We proceed analogously to the scalar case. We set $X = \mathbb{R}^n$ even though many assertions are also valid in more general context.

Definition 3.4.1 *A function $f : X \to \hat{\mathcal{C}}$ is said to be* convex *if*

$$\forall \lambda \in [0,1],\, \forall x_1, x_2 \in X : \quad f(\lambda \cdot x_1 + (1-\lambda)\cdot x_2) \leq \lambda\, f(x_1) \oplus (1-\lambda)\, f(x_2).$$

A concave *function is analogously defined (replacing $\leq$ by $\geq$).*

The following figure shows (the graph of) a convex and a concave function $f : \mathbb{R} \to \hat{\mathcal{C}}^\star(\mathbb{R})$.



Proposition 3.4.2 *A function $f : X \to \hat{\mathcal{C}}$ is convex (concave) if and only if* $\operatorname{epi} f \subseteq X \times \hat{\mathcal{C}}$ *(*$\operatorname{hyp} f \subseteq X \times \hat{\mathcal{C}}$*) is convex.*

Proof. The proof is immediate. □

Usually, convexity of a set–valued map is defined by convexity of its graph. The relationship is as follows.

Proposition 3.4.3 *A function $f : X \to \hat{\mathcal{C}}^\star$ is convex if and only if* $\operatorname{gr} f$ *is likewise. A function $f : X \to \hat{\mathcal{C}}^\diamond$ is concave if and only if* $\operatorname{gr} f$ *is convex.*

Proof. Follows from the definition. □

It can be easily seen that a concave function $f : X \to \hat{\mathcal{C}}^\star$ and a convex function $f : X \to \hat{\mathcal{C}}^\diamond$ do not have convex graphs, in general. Functions of this (or similar) type are called "fans" by Ioffe, compare [38]. In the most cases these functions have worse properties than its counterparts (with convex graph). In Chapter 4, for instance, they occur as objective functions of dual problems.

The following proposition shows that the values of a convex $\hat{\mathcal{C}}^\star$–valued function essentially have the same recession cone. An analogous result for concave $\hat{\mathcal{C}}^\star$–valued functions is not true. A systematic study of assertions of this type can be found in [54].

Proposition 3.4.4 *Let $f : X \to \hat{\mathcal{C}}^\star$ be convex. If $\bar{x} \in \operatorname{ri} \operatorname{dom} f$, then $0^+ f(x) \subseteq 0^+ f(\bar{x})$ for all $x \in \operatorname{dom} f$ and $0^+ f(x) = 0^+ f(\bar{x})$ for all $x \in \operatorname{ri}(\operatorname{dom} f)$.*

Proof. Note that $\operatorname{dom} f$ is convex. Let $x \in \operatorname{dom} f$ be arbitrarily given and, by hypothesis, $\bar{x} \in \operatorname{ri} \operatorname{dom} f$. By [1, Theorem 6.4], there exists $\mu > 1$ such that $\hat{x} := \mu\bar{x} + (1-\mu)x \in \operatorname{dom} f$.

Set $\lambda := 1/\mu \in (0,1)$. The convexity of f yields $f(\bar{x}) \supseteq \lambda f(\hat{x}) \oplus (1-\lambda) f(x)$. Since $\hat{x} \in \operatorname{dom} f$ we can choose some $\hat{y} \in f(\hat{x})$, hence $f(\bar{x}) \supseteq \lambda \{\hat{y}\} + (1-\lambda) f(x) := C_x$. Of course, $f(x) \subseteq \mathbb{R}^p$ is a nonempty closed convex set for each $x \in \operatorname{dom} f$. Therefore, Proposition A.9 yields $0^+ C_x \subseteq 0^+ f(\bar{x})$. With the aid of [1, Theorem 8.1] we conclude that $0^+ C_x = 0^+ f(x)$, hence $0^+ f(x) \subseteq 0^+ f(\bar{x})$.

Assume there is some $\tilde{x} \in \operatorname{ri} \operatorname{dom} f$ with $0^+ f(\tilde{x}) \subsetneq 0^+ f(\bar{x})$, then the first part yields $0^+ f(x) \subseteq 0^+ f(\tilde{x})$ for all $x \in \operatorname{dom} f$, whence the contradiction $0^+ f(\bar{x}) \subsetneq 0^+ f(\bar{x})$. □

Theorem 3.4.5 *Let $f : X \to \hat{\mathcal{C}}^\star$ be convex. Then, for all $\bar{x} \in X$ it holds*

$$\liminf_{x \to \bar{x}} f(x) = \operatorname*{LIM\,SUP}_{x \to \bar{x}} f(x).$$

Proof. Of course, we always have $\liminf_{x \to \bar{x}} f(x) \supseteq \operatorname{LIM\,SUP}_{x \to \bar{x}} f(x)$. To show the opposite inclusion let $y \in \liminf_{x \to \bar{x}} f(x)$ be given. Hence there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ with $x_n \to \bar{x}$ such that $y \in \liminf_{n \to \infty} f(x_n)$. By Proposition 2.2.15 this can be written as

$$\exists \left\{ (\lambda_0^{(m)}, ..., \lambda_p^{(m)}), (k_0^{(m)}, ..., k_p^{(m)}), (z_0^{(m)}, ..., z_p^{(m)}) \right\}_{m \in \mathbb{N}} \subseteq [0,1]^{p+1} \times \mathbb{N}^{p+1} \times (\mathbb{R}^p)^{p+1} :$$

$$y = \lim_{m \to \infty} \sum_{j=0}^{p} \lambda_j^{(m)} z_j^{(m)},$$

$$\forall j \in \{0,1,...,p\}, \forall m \in \mathbb{N} : \quad z_j^{(m)} \in f(x_{k_j^{(m)}}),$$

$$\forall j \in \{0,1,...,p\}, \forall m \in \mathbb{N} : \quad k_j^{(m)} \geq m,$$

$$\forall m \in \mathbb{N} : \quad \sum_{j=0}^{p} \lambda_j^{(m)} = 1.$$

We define two sequences $\{y_m\}_{m \in \mathbb{N}} \subseteq \mathbb{R}^p$ and $\{\tilde{x}_m\}_{m \in \mathbb{N}} \subseteq X$ by

$$y_m := \sum_{j=0}^{p} \lambda_j^{(m)} z_j^{(m)} \qquad \tilde{x}_m := \sum_{j=0}^{p} \lambda_j^{(m)} x_{k_j^{(m)}}.$$

Then we have $y_m \to y$, $\tilde{x}_m \to \bar{x}$ and the convexity of f yields that

$$y_m = \sum_{j=0}^{p} \lambda_j^{(m)} z_j^{(m)} \in \sum_{j=0}^{p} \lambda_j^{(m)} f\left(x_{k_j^{(m)}}\right) \subseteq f\left(\sum_{j=0}^{p} \lambda_j^{(m)} x_{k_j^{(m)}}\right) = f(\tilde{x}_m)$$

for all $m \in \mathbb{N}$. By [68, 5(1)], this means $y \in \operatorname{LIM\,SUP}_{x \to \bar{x}} f(x)$. □

Corollary 3.4.6 *Let $f : X \to \hat{\mathcal{C}}^\star$ be convex. Then the following statements hold true:*

(i) $(\operatorname{lsc} f) = (\operatorname{osc} f)$,

(ii) $(\operatorname{lsc} f)$ *is convex,*

(iii) $(\operatorname{lsc} f) : X \to \hat{\mathcal{C}}^\star_K$ *for some nonempty closed convex cone* $K \subseteq \mathbb{R}^p$*.*

Proof. (i) Follows from Theorem 3.4.5.

(ii) f convex $\Leftrightarrow$ $\operatorname{gr} f$ convex $\Rightarrow$ $\operatorname{cl}(\operatorname{gr} f) = \operatorname{gr}(\operatorname{osc} f)$ convex $\Leftrightarrow$ $\operatorname{osc} f = \operatorname{lsc} f$ convex.

(iii) Since $\operatorname{lsc} f$ is osc and convex, its graph is closed and convex. If $\operatorname{dom}(\operatorname{lsc} f) = \emptyset$ there is nothing to prove, otherwise, there exists some $\bar{x} \in \operatorname{ri}\operatorname{dom}(\operatorname{lsc} f)$. From Proposition 3.4.4 we deduce that $0^+(\operatorname{lsc} f)(x) \subseteq 0^+(\operatorname{lsc} f)(\bar{x}) =: K$ for all $x \in \operatorname{dom}(\operatorname{lsc} f)$. It remains to prove the opposite inclusion for all $x \in \operatorname{dom}(\operatorname{lsc} f)$. Indeed, let $\hat{y} \in 0^+(\operatorname{lsc} f)(\bar{x})$ and $\bar{y} \in (\operatorname{lsc} f)(\bar{x})$ be arbitrarily chosen. By [1, Theorem 8.3] we have $\bar{y} + \lambda\hat{y} \in (\operatorname{lsc} f)(\bar{x})$ for all $\lambda \geq 0$ and equivalently $(0, \hat{y}) \in 0^+\operatorname{gr}(\operatorname{lsc} f)$. Given some $x \in \operatorname{dom}(\operatorname{lsc} f)$ we can choose $y \in (\operatorname{lsc} f)(x)$. Since $(0, \hat{y}) \in 0^+\operatorname{gr}(\operatorname{lsc} f)$, [1, Theorem 8.3] yields that $y + \lambda\hat{y} \in (\operatorname{lsc} f)(x)$ for all $\lambda \geq 0$ and equivalently $\hat{y} \in 0^+(\operatorname{lsc} f)(x)$. □

Corollary 3.4.7 *For some nonempty closed convex cone* $K \subseteq \mathbb{R}^p$ *and a function* $f : X \to \hat{\mathcal{C}}^\star_K$*, the following statements are equivalent:*

(i) f *is convex and lsc,*

(ii) $\operatorname{gr} f \subseteq X \times \mathbb{R}^p$ *is convex and closed,*

(iii) $\operatorname{epi} f \subseteq X \times \hat{\mathcal{C}}^\star$ *is convex and closed,*

(iv) *For all* $y^* \in \operatorname{ri} K^\circ$ *the function*

$$\bar{f}_{y^*} : X \to \mathbb{R} \cup \{-\infty, +\infty\}, \quad \bar{f}_{y^*}(x) := -\delta^*\big(y^*|f(x)\big)$$

is convex and closed.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). The equivalence of the convexity assertions follows from Proposition 3.4.2 and Proposition 3.4.3. The equivalence of the lower semi–continuity and closedness assertions follows similarly to the proof of Corollary 3.3.10 (using Corollary 3.4.6 (i) instead of Theorem 3.3.4).

(i) $\Leftrightarrow$ (iv). From Corollary 1.3.4 (convexity) and Theorem 3.2.8 taking into account Corollary 3.4.6 (iii). □

Theorem 3.4.8 *Let* $f : X \to \hat{\mathcal{C}}^\star$ *be convex. Then the following assertions hold true:*

(i) f *is lsc at every* $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$*,*

(ii) f *is continuous at every* $\bar{x} \in \operatorname{int}(\operatorname{dom} f)$*.*

Proof. (i) Let $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$ be given and let $K := 0^+ f(\bar{x})$. By Theorem 3.2.8, it remains to show that, for all $y^* \in \operatorname{ri} K^\circ$, $\bar{f}_{y^*}$ is lsc at $\bar{x}$. From Proposition 3.4.4 we deduce that $0^+ f(x) = K$ for all $x \in \operatorname{ri}(\operatorname{dom} f)$. Hence, for all $y^* \in \operatorname{ri} K^\circ$ it is true that $\bar{x} \in \operatorname{ri} \operatorname{dom} \bar{f}_{y^*}$, whence, by [1, Theorem 7.4], $\bar{f}_{y^*}$ is lsc at $\bar{x}$.

(ii) By [68, Theorem 5.9 (b)], f is isc at $\bar{x} \in \operatorname{int}(\operatorname{dom} f)$. Hence, by Proposition 3.2.7, f is usc at $\bar{x}$. Now the assertion follows from (i). □

We close this section with some assertions with respect to local boundedness of convex functions. These statements are not used in the following, however, they illuminate some additional connections between the previous assertions and known results for set–valued maps and could be of independent interest.

Theorem 3.4.9 *Let $f : X \to \hat{\mathcal{C}}^\star$ be convex and lsc. Then, f is locally bounded at every $\bar{x} \in \operatorname{dom} f$.*

Proof. Let $\bar{x} \in \operatorname{dom} f$, $V := \{\bar{x}\} + \mathbb{B}$ and $K := 0^+ f(\bar{x})$. By Proposition 3.2.13 (v) and Corollary 3.4.6 (iii) we have $0^+ f(x) = K$ for all $x \in \operatorname{dom} f$. Hence, condition (ii) in the definition of the local boundedness is satisfied. It remains to show $0^+ \inf_{x \in V} f(x) \subseteq K$. First we show that

$$\inf_{x \in V} f(x) = \bigcup_{x \in V} f(x). \tag{3.4}$$

We have to show that $\bigcup_{x \in V} f(x)$ is convex and closed. Since V and f are convex, the set $\bigcup_{x \in V} f(x)$ is convex. Since V is compact and $\operatorname{gr} f$ is closed, we deduce that $\bigcup_{x \in V} f(x)$ is closed. From (3.4) it follows that

$$0^+ \inf_{x \in V} f(x) = 0^+ \bigcup_{x \in V} f(x).$$

Let $k \in 0^+ \bigcup_{x \in V} f(x)$ be given. By [1, Theorem 8.2], k is the limit of a sequence $\{\lambda_n y_n\}$ where $\lambda_n \downarrow 0$ and $y_n \in \bigcup_{x \in V} f(x)$. Clearly, for all $n \in \mathbb{N}$ there exists $x_n \in V$ such that $y_n \in f(x_n)$. Since V is bounded, we have $(\lambda_n x_n, \lambda_n y_n) \to (0, k)$. Applying [1, Theorem 8.2] to the closed convex set $\operatorname{gr} f \subseteq X \times \mathbb{R}^p$, we obtain $(0, k) \in 0^+ \operatorname{gr} f$. With the aid of [1, Theorem 8.3] we deduce that $\bar{y} + \lambda k \in f(\bar{x} + \lambda \cdot 0) = f(\bar{x})$ for all $\lambda \geq 0$ and arbitrary $\bar{y} \in f(\bar{x})$, which is equivalent to $k \in 0^+ f(\bar{x}) = K$. □

Corollary 3.4.10 *If $f : X \to \hat{\mathcal{C}}^\star$ is convex, then f locally bounded at every $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$.*

Proof. Theorem 3.4.9 yields that $\operatorname{lsc} f$ is locally bounded at every $x \in \operatorname{dom}(\operatorname{lsc} f)$. By Theorem 3.4.8 (i), we know that $f(\bar{x}) = (\operatorname{lsc} f)(\bar{x})$ for all $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$. □

Corollary 3.4.11 *Let $f : X \to \hat{\mathcal{C}}^\star$ be convex. Then, for all $\bar{x} \in X$ it holds*

$$\limsup_{x \to \bar{x}} f(x) = \operatorname*{LIM\,INF}_{x \to \bar{x}} f(x).$$

Proof. If $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$, this follows from Corollary 3.4.10 and Corollary 3.3.9. Otherwise, we have $\limsup_{x\to\bar{x}} f(x) = \operatorname{LIM\,INF}_{x\to\bar{x}} f(x) = \emptyset$. □

3.5 Conjugates

In this section, we introduce conjugates of $\hat{\mathcal{C}}$–valued functions. We observe that a lot of properties being well–known for the classical conjugate notion remain valid. The lack of linearity with respect to the image space is compensated by the concept of oriented sets against the background of the considerations on embedding, see Section 1.2 and Section 1.3. We set $X = X^* = \mathbb{R}^n$ even though many assertions are also valid in more general spaces.

Definition 3.5.1 *Let $f : X \to \hat{\mathcal{C}}$ and $c \in \mathbb{R}^p$. The function $f_c^* : X^* \to \hat{\mathcal{C}}$, defined by*

$$f_c^*(x^*) := \sup_{x\in X} \left\{ \langle x^*, x\rangle \cdot \{c\} \boxminus f(x) \right\},$$

where the set $\{c\}$ is considered to have the opposite orientation of f, is said to be the conjugate *of f with respect to c. The function $f_c^{**} : X \to \hat{\mathcal{C}}$, defined by*

$$f_c^{**}(x) := (f_c^*)_c^*,$$

is said to be the biconjugate *of f with respect to c.*

Clearly, f and f_c^* are contrarily oriented, but f and f_c^{**} have the same orientation. For instance, if $f : X \to \hat{\mathcal{C}}^\star$ and $\{c\} \in \mathcal{C}^\star$, we can express the biconjugate by using supremum oriented sets only:

$$f_c^{**}(x) = \sup_{x^*\in X^*} \left\{ \langle x, x^*\rangle \{c\} + \inf_{x'\in X} \left\{ - \left\langle x^*, x' \right\rangle \{c\} + f(x') \right\} \right\}. \tag{3.5}$$

As an easy consequence of the definition we obtain the Fenchel–Young inequality

$$\forall x \in X,\, x^* \in X^*,\, c \in \mathbb{R}^p : \quad f_c^*(x^*) \geq \langle x^*, x\rangle \cdot \{c\} \boxminus f(x). \tag{3.6}$$

The following proposition collects some further properties of conjugates.

Proposition 3.5.2 *Let $f, f_1, f_2 : X \to \hat{\mathcal{C}}$ (with the same orientation) be given. Then, for all $c \in \mathbb{R}^p$ the following statements hold true:*

(i) *f_c^* is convex and lsc (even if f is not),*

(ii) *f_c^{**} is convex and lsc,*

(iii) $f_1 \leq f_2 \Rightarrow (f_1)_c^* \geq (f_2)_c^*$,

(iv) $f_1 \leq f_2 \Rightarrow (f_1)_c^{**} \leq (f_2)_c^{**}$,

(v) $f_c^{**} \leq f$,

(vi) $f_c^{**} = f_{-c}^{**}$.

Proof. (i) and (ii). *Convexity.* For all $\lambda \in [0,1]$ and all $x_1^*, x_2^* \in X^*$ it holds

$$\begin{aligned}
&f_c^*\big(\lambda x_1^* + (1-\lambda)x_2^*\big) \\
&\quad= \sup_{x\in X}\Big\{\langle \lambda x_1^* + (1-\lambda)x_2^*, x\rangle \{c\} \boxminus f(x)\Big\} \\
&\quad= \sup_{x\in X}\Big\{\lambda\big(\langle x_1^*, x\rangle \{c\} \boxminus f(x)\big) \oplus (1-\lambda)\big(\langle x_2^*, x\rangle \{c\} \boxminus f(x)\big)\Big\} \\
&\quad\leq \sup_{x_1,x_2\in X}\Big\{\lambda\big(\langle x_1^*, x_1\rangle \{c\} \boxminus f(x_1)\big) \oplus (1-\lambda)\big(\langle x_2^*, x_2\rangle \{c\} \boxminus f(x_2)\big)\Big\} \\
&\quad\overset{(1.1)}{\leq} \sup_{x_1\in X}\Big\{\lambda\big(\langle x_1^*, x_1\rangle \{c\} \boxminus f(x_1)\big)\Big\} \oplus \sup_{x_2\in X}\Big\{(1-\lambda)\big(\langle x_2^*, x_2\rangle \{c\} \boxminus f(x_2)\big)\Big\} \\
&\quad= \lambda f_c^*\big(x_1^*\big) \oplus (1-\lambda) f_c^*\big(x_2^*\big).
\end{aligned}$$

Lower semi–continuity. For all $\bar{x}^* \in X^*$ it holds

$$\begin{aligned}
\liminf_{x^*\to\bar{x}^*} f_c^*(x^*) &= \inf_{x_n^*\to\bar{x}^*}\liminf_{n\to\infty}\sup_{x\in X}\big\{\langle x_n^*, x\rangle \{c\} \boxminus f(x)\big\} \\
&\overset{\text{Pr. 2.2.8 (iii)}}{\geq} \inf_{x_n^*\to\bar{x}^*}\sup_{x\in X}\liminf_{n\to\infty}\big\{\langle x_n^*, x\rangle \{c\} \boxminus f(x)\big\} \\
&\overset{\text{Pr. 2.2.14}}{=} \sup_{x\in X}\big\{\langle \bar{x}^*, x\rangle \{c\} \boxminus f(x)\big\} = f_c^*(\bar{x}^*).
\end{aligned}$$

(iii) and (iv). Since $f_1 \leq f_2$ implies $\boxminus f_1 \geq \boxminus f_2$, this is obvious.

(v) From (3.5) we deduce

$$\begin{aligned}
f_c^{**}(x) &= \sup_{x^*\in X^*}\Big\{\langle x, x^*\rangle \{c\} + \inf_{x'\in X}\big\{-\langle x^*, x'\rangle \{c\} + f(x')\big\}\Big\} \\
&\leq \sup_{x^*\in X^*}\Big\{\langle x, x^*\rangle \{c\} - \langle x^*, x\rangle \{c\} + f(x)\Big\} = f(x).
\end{aligned}$$

(vi) Replace $\langle x, x^*\rangle \{c\}$ and $\langle x^*, x'\rangle \{c\}$ in formula (3.5), respectively, by $\langle x, -x^*\rangle \{-c\}$ and $\langle -x^*, x'\rangle \{-c\}$. □

Example 3.5.3 (set–valued support function) The following set–valued map is often used in the literature. Given a subset $S \subseteq X$, the *set–valued indicator function* is defined by

$$\Delta(x \,|\, S) := \begin{cases} \{0\} & \text{if} \quad x \in S \\ \emptyset & \text{else.} \end{cases}$$

If $\Delta(\cdot \,|\, S)$ is considered to be a function with values in $\hat{\mathcal{C}}^\star$, $\Delta(\cdot \,|\, S)$ is convex if and only if the set S is likewise. The conjugate of $\Delta(\cdot \,|\, S) : X \to \hat{\mathcal{C}}^\star$ with respect to $c \in \mathbb{R}^p$ is the function

$\Delta_c^*(\,\cdot\,|S) : X^* \to \mathcal{C}^\diamond$,

$$\Delta_c^*(x^*|\,S) = \sup_{x\in S}\big\{\langle x^*, x\rangle\,\{c\}\big\} = \big[-\delta^*(x^*,-S), \delta^*(x^*,S)\big]\cdot\{c\}\,,$$

where $\{c\} \in \mathcal{C}^\diamond$ and, by convention,

$$\forall \alpha \in \mathbb{R}: \quad [-\infty, \alpha]\,\{c\} := (-\infty, \alpha]\,\{c\} = \bigcup_{\lambda\le\alpha}\big(\lambda\,\{c\}\big),$$

$$\forall \alpha \in \mathbb{R}: \quad [\alpha, +\infty]\,\{c\} := [\alpha, +\infty)\,\{c\} = \bigcup_{\lambda\ge\alpha}\big(\lambda\,\{c\}\big),$$

$$[-\infty,+\infty]\,\{c\} := (-\infty,+\infty)\,\{c\} = \mathbb{R}\cdot\{c\} = \bigcup_{\lambda\in\mathbb{R}}\big(\lambda\,\{c\}\big),$$

$$[+\infty,-\infty]\,\{c\} := \emptyset \quad \text{(this case occurs if } S=\emptyset\text{)}.$$

In order to emphasize the analogy to the scalar case, $\Delta_c^*(\,\cdot\,|S)$ is called the *(set–valued) support function (with respect to $c \in \mathbb{R}^p$) of f*. If S is closed and convex, one can easily verify (using [1, Theorem 13.1]) that the biconjugate $\Delta_c^{**}(\,\cdot\,|S)$ is equal to $\Delta(\,\cdot\,|S)$. Of course, this is also a consequence of the set–valued biconjugation theory to be developed in the next section.

Remark 3.5.4 An axiomatic approach to duality and conjugation was given by Martínez–Legaz and Singer [62]. Recently, this theory was extended up to the framework of complete lattices, see [26], which is general enough in order to apply these results to optimization with set relations. The spaces $\hat{\mathcal{C}}^\star$ and $\hat{\mathcal{C}}^\diamond$ can be considered as two complete lattices. The change of orientation corresponds to the bijection s in [26]. In axiomatic duality a so–called "condition inf–d" plays a crucial role. In our framework this is condition is equivalent to the requirement that (1.1) is satisfied with equality. Another observation is that in [26, Theorem 4.1] coupling functions with a special form are considered, which corresponds to the form of our coupling functions, namely $\langle\,\cdot\,,\,\cdot\,\rangle\,\{c\}$.

3.6 Biconjugation theorem

In this section, we prove a biconjugation theorem for functions $f : X \to \hat{\mathcal{C}}^\star$. In the next chapter, this result is used to prove strong duality assertions for convex optimization problems based on set relations. As in the previous section, we set $X = X^* = \mathbb{R}^n$. We begin with an auxiliary assertion.

Lemma 3.6.1 *Let $K \subseteq \mathbb{R}^p$ be a nonempty closed convex cone which is not a linear subspace of $\mathbb{R}^p$. Then it holds*

$$\big(y \in \operatorname{ri} K \wedge y^* \in \operatorname{ri} K^\circ\big) \;\Rightarrow\; \langle y, y^*\rangle < 0.$$

Proof. Assume the contrary. By the definition of the polar cone, this means that there exists $\bar{y} \in \operatorname{ri} K$ and $\bar{y}^* \in \operatorname{ri} K^\circ$ such that $\langle \bar{y}, \bar{y}^*\rangle = 0$. We show that

$$\forall y^* \in K^\circ: \;\; \langle \bar{y}, y^*\rangle = 0. \tag{3.7}$$

Assuming that (3.7) is not true, we find some $\tilde{y}^* \in K^\circ$ such that $\langle \bar{y}, \tilde{y}^* \rangle < 0$. Since $\bar{y}^* \in \operatorname{ri} K^\circ$, there exists some $\mu > 1$ such that $\hat{y}^* := \mu \bar{y}^* + (1-\mu)\tilde{y}^* \in K^\circ$. Hence $\langle \bar{y}, \hat{y}^* \rangle > 0$, which contradicts $\hat{y}^* \in K^\circ$.

With the aid of (3.7) and the bipolar theorem [1, Theorem 14.1] we obtain $-\bar{y} \in K^{\circ\circ} = K$. Since K is a convex cone and $\bar{y} \in K \cap (-K)$, we have $K = K + \{-\bar{y}\}$. From $\bar{y} \in \operatorname{ri} K$ we conclude $0 \in \operatorname{ri} K + \{-\bar{y}\} = \operatorname{ri}(K + \{-\bar{y}\}) = \operatorname{ri} K$. This implies $\operatorname{lin} K = \operatorname{aff} K = K$, i.e. K is a linear subspace of $\mathbb{R}^p$, a contradiction. □

In order to formulate the biconjugation theorem, the vector $c \in \mathbb{R}^p$, which is involved in the definition of the biconjugate, has to be chosen appropriately. Therefore we define the set C_f, which depends on the function $f : X \to \hat{\mathcal{C}}^\star$, as follows:

$$K_f := \inf_{x \in \operatorname{dom} f} 0^+ f(x),$$

$$C_f := \begin{cases} \operatorname{ri} K_f \cup -\operatorname{ri} K_f & \text{if} \quad K_f \subsetneq \mathbb{R}^p \text{ is a not linear subspace of } \mathbb{R}^p \text{ or } K_f = \mathbb{R}^p \\ \mathbb{R}^p \setminus K_f & \text{if} \quad K_f \subsetneq \mathbb{R}^p \text{ is a linear subspace of } \mathbb{R}^p \text{ or } K_f = \emptyset. \end{cases}$$

Let us enumerate some basic properties of the sets K_f and C_f. These statements easily follow from the definition and some results of Section 3.4.

(i) C_f is nonempty,

(ii) If $f(\cdot) \equiv \emptyset$, then $C_f = \mathbb{R}^p$,

(iii) If $f(x) = \mathbb{R}^p$ for some $x \in X$, then $C_f = \mathbb{R}^p$ (this case does not occur in Theorem 3.6.2, but in Corollary 3.6.3),

(iv) If $f : X \to \hat{\mathcal{C}}^\star_K$, then $K = K_f$,

(v) If f is convex, then $K_f = 0^+ f(x)$ for all $x \in \operatorname{ri} \operatorname{dom} f$,

(vi) If f is convex and lsc, then $K_f = 0^+ f(x)$ for all $x \in \operatorname{dom} f$.

It follows the main result of this section and one of the main results of this work.

Theorem 3.6.2 (Biconjugation theorem) *For all functions $f : X \to \hat{\mathcal{C}}^\star \setminus \{\mathbb{R}^p\}$ the following statements are equivalent:*

(i) *f is convex and lsc,*

(ii) *For all $c \in C_f$ it holds $f = f_c^{**}$,*

(iii) *There exists $c \in \mathbb{R}^p$ such that $f = f_c^{**}$.*

Proof. If f is identically $\emptyset$ we directly conclude that $f = f_c^{**}$ for all $c \in C_f = \mathbb{R}^p$ and, of course, f is convex and lsc. Therefore, $\operatorname{dom} f$ is assumed to be nonempty, and hence $K_f \neq \emptyset$.

(i) ⇒ (ii): From Corollary 3.4.6 (iii) and Proposition 3.2.13 (v) we deduce that $x \longmapsto 0^+ f(x)$ is constant on $\operatorname{dom} f$, i.e., setting $K := K_f$, we have $f : X \to \hat{\mathcal{C}}^\star_K$. From Corollary 3.4.7 we deduce that for all $y^* \in \operatorname{ri} K^\circ$ the function $\bar{f}_{y^*} : X \to \mathbb{R} \cup \{-\infty, +\infty\}$, $\bar{f}_{y^*}(x) := -\delta^*\big(y^*|f(x)\big)$ is convex and closed. By a classical biconjugation theorem, e.g. [1, Theorem 12.2], it follows that

$$\forall y^* \in \operatorname{ri} K^\circ : \qquad \bar{f}_{y^*} = \bar{f}^{**}_{y^*}. \tag{3.8}$$

Let $x \in X$ and $c \in C_f$ be arbitrarily chosen. In dependence on $K \neq \mathbb{R}^p$, we distinguish between two cases.

(A) Let $K \subsetneq \mathbb{R}^p$ be a not a linear subspace of $\mathbb{R}^p$. In view of Proposition 3.5.2 (vi) we can assume that $c \in \operatorname{ri} K$. For arbitrarily given $y^* \in \operatorname{ri} K^\circ$, Lemma 3.6.1 yields that $\langle y^*, c\rangle < 0$. Hence there exists $\alpha_{y^*} > 0$ such that $\langle \alpha_{y^*} y^*, c\rangle = -1$. This can be rewritten as

$$\forall t \in \mathbb{R} : \ \ -\delta^*\big(\alpha_{y^*} y^*|\,\{t \cdot c\}\big) = -\left\langle \alpha_{y^*} y^*, t\cdot c\right\rangle = t. \tag{3.9}$$

For $\alpha := \alpha_{y^*} > 0$ we have

$$\begin{aligned}
& \alpha \cdot \big(-\delta^*\left(y^*|f(x)\right)\big) = -\delta^*\left(\alpha y^*|f(x)\right) \quad = \quad \bar{f}_{\alpha\, y^*}(x) \quad \overset{(3.8)}{=} \quad \bar{f}^{**}_{\alpha\, y^*}(x) \\
= \quad & \sup_{x^*\in X^*} \left\{ \langle x, x^*\rangle + \inf_{x'\in X} \left\{ -\left\langle x^*, x'\right\rangle + \bar{f}_{\alpha\, y^*}(x') \right\} \right\} \\
\overset{(3.9)}{=} \quad & \sup_{x^*\in X^*} \left\{ -\delta^*\big(\alpha y^*|\, \langle x, x^*\rangle \{c\}\big) + \inf_{x'\in X} \left\{ -\delta^*\big(\alpha y^*|-\left\langle x^*, x'\right\rangle\{c\}\big) - \delta^*\big(\alpha y^*|f(x')\big) \right\} \right\} \\
= \quad & \sup_{x^*\in X^*} \left\{ -\delta^*\big(\alpha y^*|\, \langle x, x^*\rangle \{c\}\big) + \inf_{x'\in X} \left\{ -\delta^*\big(\alpha y^*|-\left\langle x^*, x'\right\rangle\{c\} + f(x')\big) \right\} \right\} \\
\overset{\text{Pr. 1.3.7}}{=} \quad & \sup_{x^*\in X^*} \left\{ -\delta^*\big(\alpha y^*|\, \langle x, x^*\rangle \{c\}\big) - \delta^*\left(\alpha y^* \middle| \inf_{x'\in X} \left\{ -\left\langle x^*, x'\right\rangle\{c\} + f(x')\right\} \right) \right\} \\
= \quad & \sup_{x^*\in X^*} \left\{ -\delta^*\left(\alpha y^* \middle| \langle x, x^*\rangle \{c\} + \inf_{x'\in X} \left\{ -\left\langle x^*, x'\right\rangle\{c\} + f(x')\right\} \right) \right\} \\
\overset{\text{Pr. 1.3.7}}{\leq} \quad & -\delta^*\left(\alpha y^* \middle| \sup_{x^*\in X^*} \left\{ \langle x, x^*\rangle \{c\} + \inf_{x'\in X} \left\{ -\left\langle x^*, x'\right\rangle\{c\} + f(x')\right\} \right\} \right) \\
\overset{(3.5)}{=} \quad & -\delta^*\big(\alpha y^*|f^{**}_c(x)\big) \quad = \quad \alpha \cdot \big(-\delta^*\left(y^*|f^{**}_c(x)\right)\big).
\end{aligned}$$

It follows that $\delta^*(y^*|f^{**}_c(x)) \leq \delta^*(y^*|f(x))$ for all $y^* \in \operatorname{ri} K^\circ$. If $f(x) = \emptyset$ we conclude that $\delta^*(y^*|f^{**}_c(x)) = -\infty$ for some y^* and hence $f^{**}_c(x) = \emptyset$. If $x \in \operatorname{dom} f$, we have $0^+ f(x) = K$. Hence, by Lemma 1.3.1 (iii) we deduce that $f(x) \leq f^{**}_c(x)$. Finally, 3.5.2 (v) yields $f(x) = f^{**}_c(x)$.

(B) Let $K \subsetneq \mathbb{R}^p$ be a linear subspace of $\mathbb{R}^p$ and let $c \in \mathbb{R}^p \setminus K$. Letting $\{c\} \in \mathcal{C}^\star$, we define the set $B := \mathbb{R}_+ \{c\} \in \mathcal{C}^\star$. Consider the function $\tilde{f} : X \to \hat{\mathcal{C}}^\star$, $\tilde{f}(x) := f(x) + B$ (by [1, Corollary 9.1.2] this sum is closed). Of course, $\tilde{f}$ is convex and lsc and $\tilde{f}(x) \neq \mathbb{R}^p$. For $x \in \operatorname{dom} f$, [1, Corollary 9.1.2] yields $\tilde{K} := 0^+\tilde{f}(x) = 0^+ f(x) + B = K + B$. In particular, $\tilde{K}$ is not a linear space and $c \in \operatorname{ri} \tilde{K}$. Therefore, $\tilde{f}$ satisfies the assumptions of case (A). For arbitrary $x \in X$ it follows

$$\begin{aligned} f(x)+B &= \tilde{f}(x) \overset{\text{part (A)}}{=} \tilde{f}^{**}_c(x) \\ &= \sup_{x^*\in X^*}\left\{\langle x,x^*\rangle\{c\} + \inf_{x'\in X}\left\{-\langle x^*,x'\rangle\{c\}+f(x')+B\right\}\right\} \\ &\overset{\text{Pr. 1.2.4, (1.1)}}{\leq} f^{**}_c(x)\oplus B \leq f^{**}_c(x). \end{aligned}$$

By the same arguments (replace c by $-c$) we obtain $f(x)-B \leq f^{**}_{-c}(x) = f^{**}_c(x)$. Together we have $(f(x)-B)\cap(f(x)+B) \leq f^{**}_c(x)$. It remains to show that $f(x) \leq (f(x)-B)\cap(f(x)+B)$. Let $y \in (f(x)+B)\cap(f(x)-B)$ be given (the case $f(x)=\emptyset$ is immediate). This means $y = y_1 + r_1 c = y_2 - r_2 c$ for $y_1, y_2 \in f(x)$ and real numbers $r_1, r_2 \geq 0$. If $r_1 + r_2 = 0$ there is nothing to prove. For $r_1 + r_2 > 0$ it follows

$$y = \frac{r_2}{r_1+r_2}(y_1 + r_1 c) + \frac{r_1}{r_1+r_2}(y_2 - r_2 c) = \frac{r_2}{r_1+r_2}y_1 + \frac{r_1}{r_1+r_2}y_2 \in f(x).$$

Hence $f(x) \leq f^{**}_c(x)$ and Proposition 3.5.2 (v) yields equality.

(ii) $\Rightarrow$ (iii): Since C_f is always nonempty, this is obvious.

(iii) $\Rightarrow$ (i): Follows from Proposition 3.5.2 (ii). □

It follows a local variant of the biconjugation theorem. In this assertion, the assumption $f(x) \neq \mathbb{R}^p$ can be omitted.

Corollary 3.6.3 *Let $f : X \to \hat{\mathcal{C}}^\star$ be convex and let $\bar{x} \in \operatorname{dom} f$. Then, the following statements are equivalent:*

(i) *f is lsc at $\bar{x} \in X$,*

(ii) *For all $c \in C_f$ it holds $f(\bar{x}) = f^{**}_c(\bar{x})$,*

(iii) *There exists $c \in \mathbb{R}^p$ such that $f(\bar{x}) = f^{**}_c(\bar{x})$.*

Proof. (i) $\Rightarrow$ (ii). If $f(\bar{x}) = \mathbb{R}^p$, Proposition 3.5.2 (v) yields that $f(\bar{x}) = f^{**}_c(\bar{x}) = \mathbb{R}^p$ for all $c \in C_f$ $(= \mathbb{R}^p)$. Therefore let $f(\bar{x}) \neq \mathbb{R}^p$. By Corollary 3.4.6 and Proposition 3.2.13 (iii), $(\operatorname{lsc} f) : X \to \hat{\mathcal{C}}^\star$ is convex, lsc and $\hat{\mathcal{C}}^\star_K$–valued for some nonempty closed convex cone $K \subseteq \mathbb{R}^p$. Since f is lsc at $\bar{x} \in \operatorname{dom} f$ we obtain $K = 0^+ f(\bar{x}) \neq \mathbb{R}^p$. Hence $(\operatorname{lsc} f)(x) \neq \mathbb{R}^p$ for all $x \in X$. Theorem 3.6.2 yields $(\operatorname{lsc} f) = (\operatorname{lsc} f)^{**}_c$ for all $c \in C_{(\operatorname{lsc} f)}$. Using Proposition 3.2.13 (iv) and Proposition 3.5.2 (iv) we conclude that $f(\bar{x}) = (\operatorname{lsc} f)(\bar{x}) = (\operatorname{lsc} f)^{**}_c(\bar{x}) \leq f^{**}_c(\bar{x})$. Proposition 3.5.2 (v) yields that $f(\bar{x}) = f^{**}_c(\bar{x})$ for all $c \in C_{(\operatorname{lsc} f)}$. It remains to show that $C_{(\operatorname{lsc} f)} = C_f$. Therefore, we show that $K = K_f$ $\big(= \inf_{x\in\operatorname{dom} f} 0^+ f(x)\big)$. Since $K = 0^+ f(\bar{x})$ we have $K \geq K_f$. From $(\operatorname{lsc} f) \leq f$ we conclude that $K = 0^+(\operatorname{lsc} f)(x) \leq 0^+ f(x)$ for all $x \in \operatorname{dom} f$. Hence $K \leq K_f$.

(ii) $\Rightarrow$ (iii). Since C_f is always nonempty, this is obvious.

(iii) $\Rightarrow$ (i). By Proposition 3.5.2 (ii), f^{**}_c is lsc. From $f^{**}_c \leq f$ we obtain $\liminf_{x\to\bar{x}} f^{**}_c(x) \leq \liminf_{x\to\bar{x}} f(x)$. Hence $f(\bar{x}) = f^{**}_c(\bar{x}) \leq \liminf_{x\to\bar{x}} f^{**}_c(x) \leq \liminf_{x\to\bar{x}} f(x)$. □

Finally, we express the biconjugation theorem by conventional notations. The set C_f is defined as above, but in the definition of K_f, $0^+ f(x)$ is replaced by $0^+(\operatorname{cl}\operatorname{conv} f(x))$ (in order to have a correct definition of C_f for a set–valued map $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, which has not closed convex values a priori).

Corollary 3.6.4 *Let $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set–valued map with nonempty graph and $f(x) \neq \mathbb{R}^p$ for all $x \in X$. Then, the following statements are equivalent:*

(i) $\operatorname{gr} f$ *is closed and convex,*

(ii) *For all $c \in C_f$ it holds*

$$\forall x \in \mathbb{R}^n : \quad f(x) = \bigcap_{x^* \in \mathbb{R}^n} \left\{ \langle x, x^* \rangle \{c\} - \operatorname{cl}\operatorname{conv} \bigcup_{x' \in \mathbb{R}^n} \{ \langle x^*, x' \rangle \{c\} - f(x') \} \right\},$$

(iii) *There exists $c \in \mathbb{R}^p$ such that f can be expressed as in (ii).*

Proof. Follows from Theorem 3.6.2 and the results of Section 3.4. □

Chapter 4

Duality

In this chapter, we investigate optimization problems with $\hat{\mathcal{C}}$–valued objective function. Given an objective function $f : X \to \hat{\mathcal{C}}$ and a set $S \subseteq X$ of feasible points, we are interested in the sets $\inf_{x \in S} f(x)$ and $\sup_{x \in S} f(x)$.

We already know that these problems are not of the same type, because one of them involves the union and the other one the intersection. In this chapter, we prove weak as well as strong duality assertions. In order to obtain strong duality it is essential to start either with an infimum problem for $\hat{\mathcal{C}}^{\star}$–valued functions or with a supremum problem for $\hat{\mathcal{C}}^{\diamond}$–valued functions. Otherwise, it seems to be not possible to obtain comparable duality assertions, see Remark 4.1.4 below. However, the other type of problem occurs as the dual problem. Exemplary, we consider primal problems with $\hat{\mathcal{C}}^{\star}$–valued functions.

The main tool for proving strong duality assertions is the biconjugation theorem developed in Section 3.6. We consider a Fenchel duality approach as well as a Lagrange duality approach. The problem formulation as well as the proofs are very analogous to the well–known scalar case. In the last section of this chapter, we calculate some special cases.

4.1 Fenchel duality

In this section, we prove weak and strong duality assertions for optimization problems with $\hat{\mathcal{C}}^{\star}$–valued objective function. We speak about Fenchel duality because our duality theorem has the same structure as the classical Fenchel duality theorem for extended real–valued functions, for instance, see [1, Theorem 31.1] or [10, Theorem 3.3.5]. This means, for instance, that the dual problem of a given problem of the form $\inf_{x \in X} \big\{ f(x) - g(x) \big\}$ (or similar) is expressed by the conjugates of f and g. In contrast to that, this structure gets lost in many other generalizations of the classical Fenchel duality, for instance, see [78]. In this section, we set $X = X^* = \mathbb{R}^n$ and $U = U^* = \mathbb{R}^m$, because our main attention is drawn to the image space. Nevertheless, an extension to more general spaces should be possible. As in the scalar theory, the constraint qualification has to be strengthened in case of infinite dimensional X and U.

For given functions $f: X \to \hat{\mathcal{C}}^\star$ and $g: U \to \hat{\mathcal{C}}^\star$, a linear map $A: X \to U$ and a vector $c \in \mathbb{R}^p$, let

$$p : X \to \hat{\mathcal{C}}^\star \quad \text{and} \quad d_c : U^* \to \hat{\mathcal{C}}^\star$$

be defined, respectively, by

$$p(x) = f(x) \oplus g(Ax) \quad \text{and} \quad d_c(u^*) = \boxminus\big(f_c^*(A^*u^*) \oplus g_c^*(-u^*)\big).$$

We consider the following optimization problems, the primal problem

$$\text{(P)} \qquad P := \inf_{x\in X} p\,(x),$$

and the dual problem associated to (P)

$$(\mathrm{D_c}) \qquad D_c := \sup_{u^*\in U^*} d_c(u^*).$$

Analogous to the scalar optimization theory, we introduce the *value function* by

$$v : U \to \hat{\mathcal{C}}^\star, \qquad v(u) := \inf_{x\in X}\big\{f(x) \oplus g(Ax - u)\big\}.$$

The following proposition collects some properties of the value function.

Proposition 4.1.1 *The value function $v : U \to \hat{\mathcal{C}}^\star$ has the following properties:*

(i) *If f and g are convex, then v is convex,*

(ii) $v(0) = P$,

(iii) $\forall c \in \mathbb{R}^p,\ \forall u^* \in U^*: \quad v_c^*(u^*) = \boxminus d_c(u^*)$,

(iv) $\forall c \in \mathbb{R}^p: \quad v_c^{**}(0) = D_c$.

Proof. (i) For arbitrary $u_1, u_2 \in U$ and $\lambda \in (0,1)$ it holds

$$\begin{aligned}
&\lambda v(u_1) \oplus (1-\lambda)v(u_2)\\
&\quad = \lambda \inf_{x_1\in X}\big\{f(x_1) \oplus g(Ax_1 - u_1)\big\} \oplus (1-\lambda)\inf_{x_2\in X}\big\{f(x_2) \oplus g(Ax_2 - u_2)\big\}\\
&\quad \overset{\text{Pr. 1.2.4}}{=} \inf_{x_1,x_2\in X}\big\{\lambda f(x_1) \oplus \lambda g(Ax_1 - u_1) f(x_1) \oplus (1-\lambda) f(x_2) \oplus (1-\lambda) g(Ax_2 - u_2)\big\}\\
&\quad \overset{f,\,g\text{ convex}}{\geq} \inf_{x_1,x_2\in X}\Big\{f\big(\lambda x_1 + (1-\lambda)x_2\big) \oplus g\big(A\,(\lambda x_1 + (1-\lambda)x_2) - (\lambda u_1 + (1-\lambda)u_2)\big)\Big\}\\
&\quad = \inf_{x\in X}\big\{f\,(x) \oplus g\big(Ax - (\lambda u_1 + (1-\lambda)u_2)\big)\big\} \quad = \quad v(\lambda u_1 + (1-\lambda)u_2).
\end{aligned}$$

(ii) Obvious.

(iii) Let $c \in \mathbb{R}^p$ ($\{c\} \in \mathcal{C}^\star$) and $u^* \in U^*$ be given. Then we have

$$
\begin{aligned}
v_c^*(u^*) &= \sup_{u\in U}\left\{\boxplus \langle u^*, u\rangle \{c\} \boxminus v(u)\right\} \\
&\overset{(1.2)}{=} \boxminus \inf_{u\in U}\left\{\langle -u^*, u\rangle \{c\} + v(u)\right\} \\
&= \boxminus \inf_{u\in U}\left\{\langle -u^*, u\rangle \{c\} + \inf_{x\in X}\left\{f(x)\oplus g(Ax-u)\right\}\right\} \\
&\overset{\text{Pr. 1.2.4}}{=} \boxminus \inf_{u\in U}\inf_{x\in X}\left\{\langle -u^*, u\rangle \{c\} + f(x)\oplus g(Ax-u)\right\} \\
&\overset{\text{Pr. A.4 (v)}}{=} \boxminus \inf_{x\in X}\inf_{u\in U}\left\{\left(f(x) - \langle A^*u^*, x\rangle \{c\}\right) \oplus \left(g(Ax-u) - \langle -u^*, Ax-u\rangle \{c\}\right)\right\} \\
&\overset{\text{Pr. 1.2.4}}{=} \boxminus \inf_{x\in X}\left\{f(x) - \langle A^*u^*, x\rangle \{c\}\right\} \oplus \boxminus \inf_{u'\in U}\left\{g(u') - \left\langle -u^*, u'\right\rangle \{c\}\right\} \\
&= f_c^*(A^*u^*) \oplus g_c^*(-u^*) \quad = \quad \boxminus d_c(u^*).
\end{aligned}
$$

(iv) It holds $v_c^{**}(0) = \sup_{u^*\in U^*} \boxminus v_c^*(u^*) \overset{\text{(iii)}}{=} \sup_{u^*\in U^*} d_c(u^*) = D_c$. □

It follows the main result of this section. This assertion was announced in the form of Corollary 4.1.3 in [55] and is to be published in the present form in [56]. We present two different proofs, the first one makes use of the biconjugation theorem developed in Section 3.6 and the second one is a "direct proof" using a scalar duality theorem.

Theorem 4.1.2 (Duality theorem) *The problems* (P) *and* ($\mathrm{D_c}$) *(with arbitrary $c \in \mathbb{R}^p$) satisfy the weak duality inequality, i.e., $D_c \leq P$.*

Furthermore, let f and g be convex, let

$$0 \in \operatorname{ri}(\operatorname{dom} g - A \operatorname{dom} f) \tag{4.1}$$

and, in dependence on $K := 0^+P$, let the element $c \in \mathbb{R}^p$ be chosen as follows:

(i) $c \in \operatorname{ri} K \cup -\operatorname{ri} K$, *if $K \subsetneq \mathbb{R}^p$ is not a linear subspace of $\mathbb{R}^p$ or $K = \mathbb{R}^p$,*

(ii) $c \in \mathbb{R}^p \setminus K$, *if $K \subsetneq \mathbb{R}^p$ is a linear subspace of $\mathbb{R}^p$.*

Then, we have strong duality, i.e., $D_c = P$.

Proof. Consider the value function $v : U \to \hat{\mathcal{C}}^\star$ of problem (P). By Proposition 4.1.1 (ii), (iv), for all $c \in \mathbb{R}^p$ we have $P = v(0)$ and $v_c^{**}(0) = D_c$. Since $v_c^{**} \leq v$, we obtain the weak duality inequality. From Proposition 4.1.1 (i) we conclude that v is convex. It is easy to check that $\operatorname{dom} v = \operatorname{dom} g - A\operatorname{dom} f$. By virtue of (4.1), this implies $0 \in \operatorname{ri}\operatorname{dom} v$. Theorem 3.4.8 (i) yields that v is lsc at 0. By Proposition 3.4.4, we conclude that $K = 0^+P = 0^+v(0) = \inf_{u\in\operatorname{dom} v} 0^+v(u)$, i.e., the choice of the vector c by (i) and (ii) ensures that c belongs to the set C_v, defined in Section 3.6. The local variant of the biconjugation theorem (Corollary 3.6.3) yields that $v(0) = v_c^{**}(0)$, i.e., we have strong duality. □

We continue with a second proof of Theorem 4.1.2 wherein the strong duality assertion is obtained from a corresponding scalar strong duality result using the embedding procedure as in the proof of the biconjugation theorem. This proof is independent of the considerations about semi–continuity in Chapter 3 and it was already published in the author's paper [56].

Second proof of Theorem 4.1.2. The weak duality can be proven as above or by the Fenchel–Young inequality (3.6). The proof of the strong duality assertion is organized as follows. We start with case (i). Then we show that case (ii) is a consequence of case (i).

(i) In case of $K = \mathbb{R}^p$ there is nothing to prove because the strong duality immediately follows from the weak duality assertion. Therefore, let $K \subsetneq \mathbb{R}^p$ be not a linear subspace of $\mathbb{R}^p$. It is easy to verify that $D_c = D_{-c}$. Hence it suffices to consider the case $c \in \operatorname{ri} K$.

With the aid of Proposition 1.3.7 it follows that

$$\forall y^* \in \mathbb{R}^p : \; -\delta^*(y^*|P) = -\delta^*\big(y^*|\inf_{x\in X} p(x)\big) = \inf_{x\in X}\big\{-\delta^*\big(y^*|\,p(x)\big)\big\}.$$

By the extended real–valued functions $\bar f_{y^*} : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ and $\bar g_{y^*} : U \to \mathbb{R} \cup \{-\infty, +\infty\}$ being defined, respectively, by $\bar f_{y^*}(x) := -\delta^*\big(y^*|\,f(x)\big)$ and $\bar g_{y^*}(u) := -\delta^*\big(y^*|\,g(u)\big)$ this can be rewritten as a collection of scalar optimization problems

$$\forall y^* \in \mathbb{R}^p : \; -\delta^*(y^*|P) = \inf_{x\in X}\big\{\bar f_{y^*}(x) + \bar g_{y^*}(Ax)\big\}. \tag{4.2}$$

The convexity of f and g implies the convexity of $\bar f_{y^*}$ and $\bar g_{y^*}$, respectively. Clearly, we have $\operatorname{dom} f = \operatorname{dom} \bar f_{y^*}$ and $\operatorname{dom} g = \operatorname{dom} \bar g_{y^*}$. Hence, (4.1) implies that $0 \in \operatorname{ri}(\operatorname{dom} \bar g_{y^*} - A \operatorname{dom} \bar f_{y^*})$. A scalar duality theorem, for instance [10, Theorem 3.3.5], yields that

$$\forall y^* \in \mathbb{R}^p : \; -\delta^*(y^*|P) = \sup_{u^*\in U^*}\left\{-\bar f^*_{y^*}(A^*u^*) - \bar g^*_{y^*}(-u^*)\right\}. \tag{4.3}$$

Let $y^* \in \operatorname{ri} K^\circ$ be arbitrarily given. Since $c \in \operatorname{ri} K$, Lemma 3.6.1 yields that $\langle y^*, c\rangle < 0$. Hence, there exists $\alpha_{y^*} > 0$ such that $\langle \alpha_{y^*} y^*, c\rangle = -1$. This can be rewritten as

$$\forall t \in \mathbb{R} : \;\; -\delta^*\big(\alpha_{y^*} y^*|\,\{t\cdot c\}\big) = -\left\langle \alpha_{y^*} y^*, t\cdot c\right\rangle = t. \tag{4.4}$$

For $\alpha := \alpha_{y^*} > 0$ and letting $\{c\} \in \mathcal{C}^\star$ we have

$$\begin{aligned}
\alpha \cdot \big(-\delta^*(y^*|P)\big) &= -\delta^*(\alpha y^*|P) \overset{(4.3)}{=} \sup_{u^*\in U^*}\left\{-\bar f^*_{\alpha y^*}(A^*u^*) - \bar g^*_{\alpha y^*}(-u^*)\right\}\\
&= \sup_{u^*\in U^*}\left\{\inf_{x\in X}\left\{-\langle A^*u^*, x\rangle + \bar f_{\alpha y^*}(x)\right\} + \inf_{u\in U}\left\{\langle u^*, u\rangle + \bar g_{\alpha y^*}(u)\right\}\right\}\\
&\overset{(4.4)}{=} \sup_{u^*\in U^*}\left\{\inf_{x\in X}\left\{-\delta^*\big(\alpha y^*\big|-\langle A^*u^*, x\rangle\cdot\{c\}\big) - \delta^*\big(\alpha y^*\big|f(x)\big)\right\}\right.\\
&\qquad\left. + \inf_{u\in U}\left\{-\delta^*\big(\alpha y^*\big|\langle u^*, u\rangle\cdot\{c\}\big) - \delta^*\big(\alpha y^*\big|g(u)\big)\right\}\right\}\\
&= \sup_{u^*\in U^*}\left\{\inf_{x\in X}\left\{-\delta^*\big(\alpha y^*\big|-\langle A^*u^*, x\rangle\{c\}+f(x)\big)\right\} + \inf_{u\in U}\left\{-\delta^*\big(\alpha y^*\big|\langle u^*, u\rangle\{c\}+g(u)\big)\right\}\right\}\\
&\overset{\text{Pr. 1.3.7}}{=} \sup_{u^*\in U^*}\left\{-\delta^*\big(\alpha y^*\big|\inf_{x\in X}\{-\langle A^*u^*, x\rangle\{c\}+f(x)\}\big) - \delta^*\big(\alpha y^*\big|\inf_{u\in U}\{\langle u^*, u\rangle\{c\}+g(u)\}\big)\right\}\\
&= \sup_{u^*\in U^*}\left\{-\delta^*\left(\alpha y^*\left|\inf_{x\in X}\{-\langle A^*u^*, x\rangle\{c\}+f(x)\}\oplus\inf_{u\in U}\{\langle u^*, u\rangle\{c\}+g(u)\}\right.\right)\right\}\\
&\overset{\text{Pr. 1.3.7}}{\le} -\delta^*\left(\alpha y^*\left|\sup_{u^*\in U^*}\left\{\inf_{x\in X}\{-\langle A^*u^*, x\rangle\{c\}+f(x)\}\oplus\inf_{u\in U}\{\langle u^*, u\rangle\{c\}+g(u)\}\right\}\right.\right)\\
&= -\delta^*\left(\alpha y^*\left|\sup_{u^*\in U^*}\left\{\boxminus f_c^*(A^*u^*)\oplus\boxminus g_c^*(-u^*)\right\}\right.\right)\\
&= -\delta^*(\alpha y^*|D_c) = \alpha\cdot\big(-\delta^*(y^*|D_c)\big).
\end{aligned}$$

It follows that $\delta^*(y^*|D_c) \le \delta^*(y^*|P)$ for all $y^* \in \operatorname{ri} K^\circ$. Lemma 1.3.1 (iii) yields $P \supseteq D_c$. By the weak duality inequality we obtain $P = D_c$.

(ii) Let $K \subsetneq \mathbb{R}^p$ be a linear subspace of $\mathbb{R}^p$ and let $c \in \mathbb{R}^p\setminus K$. Consider the set $B := \mathbb{R}_+\{c\} \subseteq \mathcal{C}^\star$. We define a new objective function by $\tilde p : X \to \hat{\mathcal{C}}^\star$, $\tilde p(x) := p(x)\oplus B = f(x)\oplus(g(Ax)\oplus B)$. By Proposition 1.2.4, we have $\tilde P := \inf_{x\in X}\tilde p(x) = \big(\inf_{x\in X} p(x)\big)\oplus B = P\oplus B$. With the aid of [1, Corollary 9.1.2] we conclude that $\tilde P = P\oplus B = P+B$ and $\tilde K := 0^+\tilde P = 0^+P + B = K+B$. Clearly, $\tilde K$ is not a linear space and $c \in \operatorname{ri}\tilde K$. It is an easy task to show that $\tilde g : U \to \hat{\mathcal{C}}^\star$, $\tilde g(\cdot) := g(\cdot)\oplus B$ is convex and (4.1) remains true for the new problem, hence, we have strong duality by part (i) of this theorem. For the conjugate $\tilde g_c^* : U^* \to \hat{\mathcal{C}}^\diamond$ of $\tilde g$ it holds

$$\tilde g_c^*(u^*) = \boxminus\inf_{u\in U}\{-\langle u, u^*\rangle\{c\}\oplus g(u)\oplus B\} \overset{\text{Pr. 1.2.4}}{=} g_c^*(u^*)\oplus\boxminus B.$$

Hence, the dual objective function $\tilde d : U^* \to \hat{\mathcal{C}}^\star$ for the problem $\inf_{x\in X}\tilde p(x)$ is given by

$$\tilde d_c(u^*) = \boxminus f_c^*(A^*u^*)\oplus\boxminus g_c^*(-u^*)\oplus B = d_c(u^*)\oplus B.$$

Since $0\in B$ we deduce that $\tilde d_c \le d_c$, hence $\tilde D_c := \sup_{u^*\in U^*}\tilde d_c(u^*) \le D_c$. The strong duality assertion for the problem $\inf_{x\in X}\tilde p(x)$ yields $P + B = \tilde P \le \tilde D_c \le D_c$. Likewise (replace c by $-c$), it follows $P - B \le D_{-c} = D_c$. Hence, $(P+B)\cap(P-B) \le D_c$. As in the last part of the proof of Theorem 3.6.2 we can show that $P \le (P+B)\cap(P-B)$, hence we have $P \le D_c$. By the weak duality assertion it follows $P = D_c$. □

We next express the preceeding theorem by conventional notations. Although the analogy to the scalar theory is more difficult to see, this form could be more convenient for possible applications. Moreover, we try to avoid the convex hull and closure operations if they are superfluous. Let $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set–valued map. We say f has closed (convex) values if $f(x) \subseteq \mathbb{R}^p$ is closed (convex) for all $x \in \mathbb{R}^n$. Clearly, if f has a closed (convex) graph, then f has closed (convex) values. The opposite implication is not true, in general. The map f has closed values and a convex graph if and only if f can be interpreted as a convex function $f : \mathbb{R}^n \to \hat{\mathcal{C}}^\star$.

Corollary 4.1.3 *For given set–valued maps $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and $g : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$, a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ and a vector $c \in \mathbb{R}^p$, we have*

$$\bigcup_{x\in\mathbb{R}^n} \big(f(x) + g(Ax)\big) \subseteq \bigcap_{u^*\in\mathbb{R}^m} \Bigg(\bigcup_{x\in\mathbb{R}^n} \big(f(x) - \langle A^*u^*, x\rangle \{c\}\big) + \bigcup_{u\in\mathbb{R}^m} \big(g(u) + \langle u^*, u\rangle \{c\}\big) \Bigg).$$

If f and g additionally have convex graphs and closed values and satisfy the condition $0 \in \operatorname{ri}(\operatorname{dom} g - A \operatorname{dom} f)$ and, in dependence on $K := 0^+(\operatorname{cl}\operatorname{conv} \bigcup_{x\in X} \big(f(x) + g(Ax)\big)$, the vector $c \in \mathbb{R}^p$ is chosen as in Theorem 4.1.2, we have strong duality, i.e.,

$$\operatorname{cl} \bigcup_{x\in\mathbb{R}^n} \big(f(x) + g(Ax)\big) = \bigcap_{u^*\in\mathbb{R}^m} \operatorname{cl} \Bigg(\bigcup_{x\in\mathbb{R}^n} \big(f(x) - \langle A^*u^*, x\rangle \{c\}\big) + \bigcup_{u\in\mathbb{R}^m} \big(g(u) + \langle u^*, u\rangle \{c\}\big) \Bigg).$$

Proof. For all $u^* \in \mathbb{R}^m$, we have

$$\begin{aligned} \bigcup_{x\in\mathbb{R}^n} \big(f(x) + g(Ax)\big) &= \bigcup_{x\in\mathbb{R}^n} \big(f(x) - \langle A^*u^*, x\rangle \{c\} + g(Ax) + \langle u^*, Ax\rangle \{c\}\big) \\ &\subseteq \bigcup_{x\in\mathbb{R}^n} \big(f(x) - \langle A^*u^*, x\rangle \{c\}\big) + \bigcup_{u\in\mathbb{R}^m} \big(g(u) + \langle u^*, u\rangle \{c\}\big). \end{aligned}$$

Taking the intersection over all $u^* \in \mathbb{R}^m$ we obtain the weak duality inclusion.

Let f and g have convex graphs and closed values. This means f and g can be interpreted as convex functions $f : \mathbb{R}^n \to \hat{\mathcal{C}}^\star$ and $g : \mathbb{R}^m \to \hat{\mathcal{C}}^\star$. In the present case, we can show that

$$\inf_{x\in X} \{f(x) \oplus g(Ax)\} = \operatorname{cl} \bigcup_{x\in X} \big(f(x) + g(Ax)\big). \tag{4.5}$$

Indeed, by elementary arguments we deduce that $P := \bigcup_{x\in X} \operatorname{cl}\big(f(x) + g(Ax)\big)$ is convex. Furthermore, from Proposition A.2 (ii) we conclude that $\operatorname{cl} \bigcup_{x\in X} \operatorname{cl} p(x) = \operatorname{cl} \bigcup_{x\in X} p(x)$, where $p(x) = f(x) + g(Ax)$. Together we obtain (4.5). By analogous arguments the right–hand side of the strong duality equality equals D_c in Theorem 4.1.2. □

Remark 4.1.4 It seems to be not possible to obtain comparable duality assertions if we start with an infimum problem with a $\hat{\mathcal{C}}^\diamond$–valued objective function. The reason for that seems to be that (1.1) is not satisfied with equality in this case. Equality in (1.1) is equivalent to the condition "inf-d", which plays an important role in the axiomatic duality theory by Getán, Martinez–Legaz and Singer [26].

4.2 Lagrange duality

This section is devoted to Lagrange duality of optimization problems with $\hat{\mathcal{C}}^\star$–valued objective function and set–valued constraints. As before we set $X = X^* = \mathbb{R}^n$ and $U = U^* = \mathbb{R}^m$.

Let $f : X \to \hat{\mathcal{C}}^\star$, let $G : X \rightrightarrows U$ be a set–valued map and let $C_U \subseteq U$ be a nonempty closed convex cone. We consider the following optimization problem:

$$\text{(P)} \qquad P := \inf_{x \in S} f(x), \qquad S := \{x \in X |\; G(x) \cap -C_U \neq \emptyset\}.$$

Constraints of this type have been considered by many authors such as [7], [20], [57], [32], [41], [22]. We next recall the notion of C–convexity. In the following, this concept is used to describe convexity with respect to the constraints.

Definition 4.2.1 (e.g. [41]) *Let X, Y be real linear spaces and let $C \subseteq Y$ be a convex cone. A set–valued map $H : X \rightrightarrows Y$ is said to be C–*convex *if*

$$\forall x_1, x_2 \in X,\ \forall \lambda \in [0,1] : \qquad H\big(\lambda x_1 + (1-\lambda)x_2\big) + C \supseteq \lambda H(x_1) + (1-\lambda) H(x_2).$$

The Lagrangian of the problem (P) (with respect to $c \in \mathbb{R}^p$) is defined by

$$L_c : X \times U^* \to \hat{\mathcal{C}}^\star, \qquad L_c(x,u^*) := f(x) \oplus \boxminus \Delta_c^*\big(u^* \big| - C_U - G(x)\big),$$

where $\Delta_c^*\big(\cdot \big| - C_U - G(x)\big) : U^* \to \hat{\mathcal{C}}^\diamond$ is the set–valued support function (see Section 3.5) of the set $-C_U - G(x)$. Hence, the Lagrangian can be expressed (for $\{c\} \in \mathcal{C}^\star$) by

$$L_c(x,u^*) = f(x) \oplus \inf_{u \in G(x) + C_U} \Big\{ \langle u^*, u\rangle \,\{c\} \Big\}.$$

Perhaps, this definition looks a little unusual. If f is an extended real–valued function and G a vector–valued function, our constraint reduces to $G(x) \in -C_U$ (or $G(x) \leq_{C_U} 0$) and the Lagrangian is usually defined by $L(x,u^*) := f(x) + \langle u^*, G(x)\rangle$, but only those $u^* \in U^*$ are involved into the theory that belong to $-C_U^\circ$. If we define the Lagrangian analogously to above by $L(x,u^*) := f(x) - \delta^*\big(u^* \big| - C_U - G(x)\big)$ (using the convention $-\infty + \infty = +\infty$), we obtain

$$L(x,u^*) = f(x) + \langle u^*, G(x)\rangle - \delta^*\big(u^* \big| - C_U\big),$$

where the support function of $-C_U$ has the simple form

$$\delta^*\big(u^* \big| - C_U\big) = \begin{cases} 0 & \text{if} \quad u^* \in -C_U^\circ \\ +\infty & \text{else.} \end{cases}$$

The set–valued support function, however, has a more complicated form, namely, letting L be the lineality space of C_U° and $\{c\} \in \mathcal{C}^\diamond$, it can be expressed by

$$\Delta_c^*\big(u^* \big| - C_U\big) = \begin{cases} \mathbb{R}_+ \cdot \{c\} & \text{if} \quad u^* \in C_U^\circ \setminus L \\ \mathbb{R}_- \cdot \{c\} & \text{if} \quad u^* \in -C_U^\circ \setminus L \\ 0 & \text{if} \quad u^* \in L \\ \mathbb{R} \cdot \{c\} & \text{else.} \end{cases}$$

This means, we do not obtain a meaningful analogue to the Lagrangian in the classical form.

By Proposition 1.2.4, the Lagrangian can be written as

$$L_c(x,u^*) = f(x) \oplus \inf_{u\in G(x)} \Big\{ \langle u^*,u\rangle \{c\} \Big\} \oplus \inf_{u \in C_U} \Big\{ \langle u^*,u\rangle \{c\} \Big\},$$

where $\{c\} \in \mathcal{C}^\star$. An easy calculation shows that

$$L_c(x,u^*) = f(x) \oplus \left[\inf_{u\in G(x)} \langle u^*,u\rangle\, ,\, \sup_{u\in G(x)} \langle u^*,u\rangle \right] \{c\} \oplus \begin{cases} \mathbb{R}_+ \cdot \{c\} & \text{if} \quad u^* \in -C_U^\circ \setminus L \\ \mathbb{R}_- \cdot \{c\} & \text{if} \quad u^* \in C_U^\circ \setminus L \\ 0 & \text{if} \quad u^* \in L \\ \mathbb{R} \cdot \{c\} & \text{else,} \end{cases}$$

where the interval $[\alpha,\beta]$ for $\alpha,\beta \in \mathbb{R}\cup\{-\infty,+\infty\}$ is defined as in Section 3.5.

In the scalar theory we re–obtain the original problem from the Lagrangian by taking the supremum over all $u^* \in U^*$ (or all $u^* \in -C_U^\circ$). A corresponding result is given in the following two propositions.

Proposition 4.2.2 *For all $x \in S$ and all $c \in \mathbb{R}^p$ it holds*

$$\sup_{u^*\in U^*} L_c(x,u^*) = f(x).$$

Proof. From $x \in S$ we deduce that $0 \in -C_U - G(x)$. For arbitrary $c \in \mathbb{R}^p$ it holds

$$\begin{aligned} \sup_{u^*\in U^*} L_c(x,u^*) &\overset{(1.1)}{\leq} f(x) \oplus \sup_{u^*\in U^*} \boxminus \Delta_c^*\big(u^* \big| - C_U - G(x)\big) \\ &= f(x) \oplus \Delta_c^{**}\big(\,0\,\big| - C_U - G(x)\big) \\ &\overset{\text{Pr. 3.5.2 (v)}}{\leq} f(x) \oplus \Delta_c\big(\,0\,\big| - C_U - G(x)\big) = f(x). \end{aligned}$$

Since $L(x,0) = f(x)$, it follows that $\sup_{u^*\in U^*} L_c(x,u^*) = f(x)$. □

The preceeding proposition clarifies the relation between the problem (P) and the Lagrangian only for feasible points $x \in S$. It remains the question what happens if x is not feasible. By the analogy to scalar optimization, we expect that $\sup_{u^*\in U^*} L_c(x,u^*) = \emptyset$ in this case. This can be shown under additional assumptions to the constraints and to the choice of $c \in \mathbb{R}^p$. Note that the following proposition is not used in the proof of the duality theorem. In particular, the additional assumption $0^+G(x) \cap -C_U = \{0\}$ in the following proposition is not necessary for duality assertions.

Proposition 4.2.3 *If for some $x \in \operatorname{dom} f \cap \operatorname{dom} G$ the condition $0^+G(x) \cap -C_U = \{0\}$ is satisfied, then for all $c \in \mathbb{R}^p \setminus -0^+f(x)$ it holds*

$$\sup_{u^*\in U^*} L_c(x,u^*) = \begin{cases} f(x) & \textit{if} \quad x \in S \\ \emptyset & \textit{else.} \end{cases}$$

Proof. The case $x \in S$ follows from Proposition 4.2.2. Let $x \neq S$, i.e., $G(x) \cap -C_U = \emptyset$. Since $0^+G(x) \cap -C_U = \{0\}$, the separation theorem [1, Corollary 11.4.1] yields the existence of some $\bar{u}^* \in U^*$ such that

$$\alpha(x, \bar{u}^*) = \inf_{u \in G(x)} \langle \bar{u}^*, u \rangle > \sup_{u \in -C_U} \langle \bar{u}^*, u \rangle \geq 0. \tag{4.6}$$

Assuming that there exists some $u \in -C_U$ such that $\langle \bar{u}^*, u \rangle > 0$, we get $\sup_{u \in -C_U} \langle \bar{u}^*, u \rangle = +\infty$. This contradicts (4.6) because we supposed $G(x) \neq \emptyset$. Hence $\bar{u}^* \in -C_U^\circ$. Of course, it is true that $\sup_{u^* \in U^*} L_c(x, u^*) \geq \sup_{\lambda > 0} L_c(x, \lambda \bar{u}^*)$. Therefore, it remains to show that $\sup_{\lambda > 0} L_c(x, \lambda \bar{u}^*) = \emptyset$. Since $\bar{u}^* \in -C_U^\circ$, an easy calculation shows that $L_c(x, \lambda \bar{u}^*) \subseteq f(x) \oplus [\lambda \alpha(x, \bar{u}^*), +\infty] \{c\}$. Hence it remains to show that

$$\bigcap_{\lambda > 0} \Big(f(x) \oplus [\lambda \alpha(x, \bar{u}^*), +\infty] \{c\} \Big) = \emptyset.$$

Assume the contrary, i.e., there exists some $y \in \mathbb{R}^p$ such that

$$\forall \lambda > 0 : \quad y \in f(x) \oplus [\lambda \alpha(x, \bar{u}^*), +\infty] \{c\}.$$

Hence, for all $n \in \mathbb{N}$ we have $y \in f(x) + [n\alpha(x, \bar{u}^*), +\infty] \{c\} + \frac{1}{n}\mathbb{B}$. Consequently,

$$\forall n \in \mathbb{N},\ \exists\, y_n \in \{y\} + \tfrac{1}{n}\mathbb{B},\ \exists \mu_n \geq n \cdot \alpha(x, \bar{u}^*) > 0 : \quad y_n - \mu_n c \in f(x).$$

It follows

$$\frac{1}{\mu_n}\big(y_n - \mu_n \cdot c\big) = \frac{y_n}{\mu_n} - c \overset{n \to \infty}{\longrightarrow} -c.$$

By [1, Theorem 8.2] this implies $-c \in 0^+ f(x)$, a contradiction. □

The next example shows that the assumption $0^+G(x) \cap -C_U = \{0\}$ cannot be omitted in the preceeding proposition. However, this assumption is fulfilled in many important special cases, such as for vector–valued or compact–valued functions G.

Example 4.2.4 Let f, K and c as in Proposition 4.2.3 and let $0 \in \operatorname{dom} f$. We set $A := \Big\{ x \in \mathbb{R}^2_+ |\ x_2 \geq \frac{1}{x_1} \Big\}$, $G(x) := \{x\} + A$ and $C_U := \big\{ u \in \mathbb{R}^2 |\ u_1 \geq 0 \big\}$. Then we have $C_U^\circ \setminus \{0\} = \big\{ u^* \in \mathbb{R}^2 |\ u_1^* < 0, u_2^* = 0 \big\}$. An easy computation shows

$$L_c(0, u^*) := f(0) + \begin{cases} \mathbb{R}_+ \cdot \{c\} & \text{if} \quad u^* \in -C_U^\circ \setminus \{0\} \\ \mathbb{R}_- \cdot \{c\} & \text{if} \quad u^* \in C_U^\circ \setminus \{0\} \\ 0 & \text{if} \quad u^* = 0 \\ \mathbb{R} \cdot \{c\} & \text{else.} \end{cases} .$$

It follows that $\sup_{u^* \in U^*} L_c(0, u^*) = f(0) \neq \emptyset$, but $G(0) \cap -C_U = \emptyset$, i.e., $0 \notin S$.

We next define the dual problem. The dual objective function is defined by

$$\phi_c : U^* \to \hat{\mathcal{C}}^\star, \qquad \phi_c(u^*) := \inf_{x \in X} L_c(x, u^*).$$

The dual problem associated to (P) is defined by

$$(\mathrm{D_c}) \qquad D_c := \sup_{u^*\in U^*} \phi_c(u^*).$$

As in the scalar optimization theory, we introduce the *value function* by

$$v : U \to \hat{\mathcal{C}}^\star, \qquad \begin{aligned} v(u) &:= \inf\big\{f(x)|\ x\in X:\ (G(x)-\{u\})\cap -C_U \neq \emptyset\big\} \\ &= \inf\big\{f(x)|\ x\in X:\ u\in G(x)+C_U\big\}. \end{aligned}$$

The following proposition collects some properties of the value function.

Proposition 4.2.5 *The value function $v : U \to \hat{\mathcal{C}}^\star$ has the following properties:*

(i) *If f is convex and G is C_U–convex, then v is convex,*

(ii) $v(0) = P$,

(iii) $\forall c\in\mathbb{R}^p,\ \forall u^*\in U^*:\quad v_c^*(u^*) = \boxminus\phi_c(-u^*)$,

(iv) $\forall c\in\mathbb{R}^p:\quad v_c^{**}(0) = D_c$.

Proof. (i) For arbitrary $u_1, u_2 \in U$ and $\lambda\in(0,1)$ it holds

$$\begin{aligned}
&\lambda v(u_1)\oplus(1-\lambda)v(u_2)\\
&\quad= \quad\lambda \quad \inf\big\{f(x_1)\,\big|\,x_1\in X:\ u_1\in G(x_1)+C_U\big\}\\
&\quad\oplus\quad (1-\lambda)\inf\big\{f(x_2)\,\big|\,x_2\in X:\ u_2\in G(x_2)+C_U\big\}\\
&\quad\overset{\text{Pr. 1.2.4}}{=} \inf\big\{\lambda f(x_1)\oplus(1-\lambda)f(x_2)\,\big|\,x_1,x_2\in X:\ u_1\in G(x_1)+C_U, u_2\in G(x_2)+C_U\big\}\\
&\quad\overset{\text{Convexity}}{\geq} \inf\big\{f(\lambda x_1+(1-\lambda)x_2)\big|\,x_1,x_2\in X:\lambda u_1+(1-\lambda)u_2\in G(\lambda x_1+(1-\lambda)x_2)+C_U\big\}\\
&\quad= \inf\big\{f(x)\,\big|\,x\in X:\ \lambda u_1+(1-\lambda)u_2\in G(x)+C_U\big\}\\
&\quad= v(\lambda u_1+(1-\lambda)u_2).
\end{aligned}$$

(ii) Obvious.

(iii) Let $c\in\mathbb{R}^p$ (where $\{c\}\in\mathcal{C}^\star$) and $u^*\in U^*$ be given. Then,

$$\begin{aligned}
v_c^*(u^*) &= \boxminus\inf_{u\in U}\big\{\langle -u^*,u\rangle\,\{c\}+v(u)\big\}\\
&= \boxminus\inf_{u\in U}\Big\{\langle -u^*,u\rangle\,\{c\}+\inf\big\{f(x)\,\big|\,x\in X:\ u\in G(x)+C_U\big\}\Big\}\\
&\overset{\text{Pr. 1.2.4}}{=} \boxminus\inf\big\{\langle -u^*,u\rangle\,\{c\}+f(x)\,\big|\,x\in X,\ u\in U:\ u\in G(x)+C_U\big\}\\
&= \boxminus\inf_{x\in X}\Big\{\inf_{u\in G(x)+C_U}\big\{\langle -u^*,u\rangle\,\{c\}+f(x)\big\}\Big\}\\
&\overset{\text{Pr. 1.2.4}}{=} \boxminus\inf_{x\in X}\Big\{f(x)\oplus\inf_{u\in G(x)+C_U}\big\{\langle -u^*,u\rangle\,\{c\}\big\}\Big\} \quad=\quad \boxminus\phi_c(-u^*).
\end{aligned}$$

(iv) It holds $v_c^{**}(0) = \sup_{u^*\in U^*}\boxminus v_c^*(u^*) \overset{\text{(iii)}}{=} \sup_{u^*\in U^*}\phi_c^*(-u^*) = \sup_{u^*\in U^*}\phi_c^*(u^*) = D_c$. □

Since $\hat{\mathcal{C}}^\star$ is order complete, we have

$$\forall c \in \mathbb{R}^p : \qquad \sup_{u^* \in U^*} \inf_{x \in X} L_c(x,u^*) \leq \inf_{x \in X} \sup_{u^* \in U^*} L_c(x,u^*), \tag{4.7}$$

even if L_c is replaced by an arbitrary function from $X \times U^*$ into $\hat{\mathcal{C}}^\star$. By Proposition 4.2.2 we know that $\inf_{x \in X} \sup_{u^* \in U^*} L_c(x,u^*) \leq P$. Hence, (4.7) yields weak duality between (P) and ($\mathrm{D_c}$), i.e., $D_c \leq P$. Furthermore, it is easy to see that strong duality (i.e., $P = D_c$) implies that (4.7) is satisfied with equality. It follows the main result of this section, a strong duality theorem.

Theorem 4.2.6 *Let f be convex and let G be C_U–convex, let*

$$G(\operatorname{dom} f) \cap -\operatorname{int} C_U \neq \emptyset, \tag{4.8}$$

and, in dependence on $K := 0^+ P$, let the vector $c \in \mathbb{R}^p$ be chosen as follows:

(i) $c \in \operatorname{ri} K \cup -\operatorname{ri} K$, *if $K \subsetneq \mathbb{R}^p$ is not a linear subspace of $\mathbb{R}^p$ or $K = \mathbb{R}^p$,*

(ii) $c \in \mathbb{R}^p \setminus K$, *if $K \subsetneq \mathbb{R}^p$ is a linear subspace of $\mathbb{R}^p$.*

Then, we have strong duality, i.e., $D_c = P$.

Proof. The proof is exactly the same as the (first) proof of Theorem 4.1.2, but using Proposition 4.2.5 instead of Proposition 4.1.1. □

4.3 Some special cases

In this section, we indicate some special optimization problems and calculate the corresponding dual problem. We consider the problem of minimizing a convex function $f : \mathbb{R}^n \to \hat{\mathcal{C}}^\star$ with respect to a nonempty closed convex set $S \subseteq \mathbb{R}^n$. This problem can be formulated as

$$\inf_{x \in \mathbb{R}^n} \big\{ f(x) + \Delta(x \,|\, S) \big\},$$

where $\Delta(\,\cdot\,|\,S) : \mathbb{R}^n \to \hat{\mathcal{C}}^\star$ is the set–valued indicator function, defined in Section 3.5. Recall that its conjugate $\Delta^*_c(\,\cdot\,|\,S) : \mathbb{R}^n \to \hat{\mathcal{C}}^\diamond$ was already considered in Section 3.5 and in Section 4.2.

As a special case for the function f, let us consider $f(x) = \{M \cdot x\}$, where M is a real $p \times n$ matrix. An easy computation shows that

$$f^*_c(x^*) = \bigcup_{x \in \mathbb{R}^n} \big\{ \big(M - c \cdot (x^*)^T\big) \cdot x \big\} =: \big(M - c \cdot (x^*)^T\big) \cdot \mathbb{R}^n.$$

In the special case of $n = p$ (recall that $\hat{\mathcal{C}}^\star = \hat{\mathcal{C}}^\star(\mathbb{R}^p)$) and $M := I$ being the $n \times n$ unit matrix, Corollary 4.1.3 yields the following dual description of a nonempty closed convex set $S \subseteq \mathbb{R}^n$.

In contrast to the usual dual description $S = \bigcap_{x^* \in \mathbb{R}^n} \{x \in \mathbb{R}^n |\ \langle x^*, x\rangle \le \delta^*(x^*|S)\}$, we have a different parameterization in the following formula, i.e., the same x^* may generate different sets. This parameterization depends on the choice of c. If 0^+S is not a linear subspace of $\mathbb{R}^n$, for all $c \in \operatorname{ri}(0^+S)$ it holds

$$S = \bigcap_{x^* \in \mathbb{R}^n} \Big(\big(I - c \cdot (x^*)^T\big) \cdot \mathbb{R}^n + \big[-\delta^*(x^*, -S), \delta^*(x^*, S)\big] \cdot \{c\} \Big). \tag{4.9}$$

Moreover, if 0^+S is a linear subspace of $\mathbb{R}^n$, (4.9) is valid for all $c \in \mathbb{R}^n \setminus 0^+S$. Note that in (4.9) the constraint qualification (4.1) is superfluous, see Remark 4.3.1 below.

We next turn to the case of linear inequality constraints. Let A be a real $m \times n$ matrix and $b \in \mathbb{R}^m$ a given vector. We write $u \le v$ if $v - u \in \mathbb{R}^m_+$. Consider the problem

$$\inf_{x \in S} \{Mx\}, \quad S = \{x \in \mathbb{R}^n |\ Ax \ge b\}. \tag{4.10}$$

In problem (P) in Section 4.1, we set $g(\cdot) = \Delta(\cdot\,|\,\bar{S})$, where $\bar{S} := \{u \in \mathbb{R}^m |\ u \ge b\}$. The (set–valued) support function of $\bar{S}$ (where $\{c\} \in \mathcal{C}^\diamond$) can be expressed by

$$\begin{aligned} \Delta^*_c(u^*|\bar{S}) &= \bigcup_{u \ge b} \big(\langle u^*, u\rangle \{c\}\big) = \bigcup_{u \ge 0} \big(\langle u^*, u + b\rangle \{c\}\big) \\ &= \big\{c \cdot (u^*)^T \cdot b\big\} + \bigcup_{u \ge 0} \big(\langle u^*, u\rangle \{c\}\big) = \big\{c \cdot (u^*)^T \cdot b\big\} + \Delta^*_c(u^*|\mathbb{R}^m_+), \end{aligned}$$

where

$$\Delta^*_c(u^*|\mathbb{R}^m_+) = \begin{cases} \mathbb{R}_+ \cdot \{c\} & \text{if} \quad u^* \in \mathbb{R}^m_+ \setminus \{0\} \\ \mathbb{R}_- \cdot \{c\} & \text{if} \quad u^* \in \mathbb{R}^m_- \setminus \{0\} \\ \{0\} & \text{if} \quad u^* = 0 \\ \mathbb{R} \cdot \{c\} & \text{else.} \end{cases}$$

Note that $\Delta^*_c(u^*|\,\mathbb{R}^m_+) = -\Delta^*_c(-u^*|\,\mathbb{R}^m_+)$, hence the dual objective function $d_c : \mathbb{R}^m \to \hat{\mathcal{C}}^\star$ (see Section 4.1) is given by

$$d_c(u^*) = \boxplus\big(M - c \cdot (A^T \cdot u^*)^T\big) \cdot \mathbb{R}^n \boxplus \big\{c \cdot (u^*)^T \cdot b\big\} \boxplus \Delta^*_c(u^*|\,\mathbb{R}^m_+).$$

By Theorem 4.1.2 (Fenchel duality theorem) we obtain the following strong duality assertion. Since S is polyhedral, we have $\inf_{x \in S} \{Mx\} = \bigcup_{x \in S} \{Mx\} =: M \cdot S$. Let $K := 0^+(M \cdot S)$. If there exists some $x \in \mathbb{R}^n$ such that $Ax \ge b$, then, for all $c \in \operatorname{ri} K$ if K is not a linear subspace of $\mathbb{R}^p$ and for all $c \in \mathbb{R}^p \setminus K$ if K is a linear subspace of $\mathbb{R}^p$ it is true that (we omit the orientation)

$$M \cdot S = \bigcap_{u^* \in \mathbb{R}^m} \Big(\big(M - c \cdot (A^T \cdot u^*)^T\big) \cdot \mathbb{R}^n + \big\{c \cdot (u^*)^T \cdot b\big\} + \Delta^*_c(u^*|\,\mathbb{R}^m_+) \Big). \tag{4.11}$$

Remark 4.3.1 In Theorem 4.1.2 (Fenchel duality theorem) we suppose the constraint qualification (4.1). In the second proof of this theorem we use this condition in order to obtain the corresponding condition for the scalar problems (4.2). If all these problems are polyhedral, (4.1) can be replaced by $\operatorname{dom} g \cap A \operatorname{dom} f \neq \emptyset$, compare e.g. [10, Corollary 5.1.9]. Hence, in (4.9) and (4.11) the constraint qualification reduces to $S \neq \emptyset$.

Chapter 5

Relationship to vector optimization

In this chapter, we discuss the relationship between the duality theory developed in the previous chapter and duality theory in vector optimization and set optimization with point relations. Duality assertions for vector optimization problems have been investigated by many authors such as Gale, Kuhn and Tucker [24], van Slyke and Wets [72], Breckner [12], Zowe [86], Brumelle [14], Gerstewitz (Tammer) [25], Jahn [39], [40], [41], Sawaragi, Nakayama and Tanino [71], Göpfert and Gerth (Tammer) [28], Göpfert and Nehse [30], Tammer [75], Boţ and Wanka [11] and many others. References with respect to the extention to set–valued objective maps are already enumerated in the introduction.

Duality for vector optimization problems (and the extension to set optimization problems with point relations) have been developed in a very general setting, so the image space is often a general linear topological space. In contrast to this, up to now, our duality results have been developed for finite dimensional "image spaces" only (more precisely, the image space is based on a finite dimensional space). So our comparison is made in a finite dimensional context.

This chapter is organized as follows. We start with recalling some notions such as that of a supremal and an infimal set and we collect some related auxiliary assertion. The second section is devoted to a comparison of duality assertions of both types of set optimization problems, i.e., a comparison between point relation approach and set relation approach. In the last section, we continue the discussion about the structure of vector optimization problems, started in the introduction. We use the duality results of the previous chapter in order to give an alternative representation of a vector optimization problem.

5.1 Basic concepts

In this section, we recall some basic concepts of vector optimization and prove some auxiliary assertions that will be used in the following two sections. Throughout this section, let $C \subseteq \mathbb{R}^p$ be a closed convex pointed cone with nonempty interior. The cone C has the meaning of the ordering cone in the following sense:

$$y_1 \leq y_2 :\Leftrightarrow y_2 - y_1 \in C \qquad \text{and} \qquad y_1 < y_2 :\Leftrightarrow y_2 - y_1 \in \operatorname{int} C.$$

The duality assertions considered in Section 5.2 below are mainly based on the concepts of supremal and infimal sets, which seem to be due to Nieuwenhuis [63] and was extended by Tanino [77]. The notion of an infimal point is closely related to the concept of a *weakly minimal point* (or weakly efficient point), for instance, see [63],[40],[77],[57]. Given a set $A \subseteq \mathbb{R}^p$, a point $y \in \mathbb{R}^p$ is said to be a *weakly minimal point* of A if

$$y \in A \qquad \text{and} \qquad (\{y\} - \operatorname{int} C) \cap A = \emptyset.$$

The set of all weakly minimal points of A is denoted by $\operatorname{wMin}[A, C]$ or simply by $\operatorname{wMin} A$. A point $y \in \mathbb{R}^p$ is said to be an *infimal point* of A if

$$y \notin A + \operatorname{int} C \qquad \text{and} \qquad \{y\} + \operatorname{int} C \subseteq A + \operatorname{int} C.$$

The set of all infimal points of A is denoted by $\operatorname{Inf}[A, C]$ or by $\operatorname{Inf} A$. Likewise, by replacing C by $-C$, we define *weakly maximal points* and *supremal points* of A as well as the sets $\operatorname{wMax}[A, C]$ and $\operatorname{Sup}[A, C]$.

In contrast to [77], we avoid the extension of the space $\mathbb{R}^p$ by two imaginary elements $+\infty$ and $-\infty$, because this makes the proofs a little bit easier. As a consequence, it may happen that the infimal set is empty. In particular, we have $\operatorname{Inf}[\emptyset, C] = \emptyset$ (instead of $\operatorname{Inf}[\emptyset, C] = +\infty$) and $\operatorname{Inf}[A, C] = \emptyset$ (instead of $\operatorname{Inf}[A, C] = -\infty$) if $A + C = \mathbb{R}^p$. Otherwise, the infimal set is nonempty, see [63, Theorem I-18].

We next summarize some properties of supremal and infimal sets.

Proposition 5.1.1 ([63],[77]) *Let $A \subseteq \mathbb{R}^p$. Then it holds*

(i) $\operatorname{Inf}[A, C] = \operatorname{Inf}[\operatorname{cl} A, C]$,

(ii) $\operatorname{Inf} A = \operatorname{wMin} \operatorname{cl}(A + \operatorname{int} C)$,

(iii) *If $A + C \neq Y$, then $A + \operatorname{int} C = \operatorname{Inf} A + \operatorname{int} C$.*

Proof. (i) See [63, Theorem I-15] or apply Proposition A.8.
(ii) See [77, Proposition 4.3] or apply Proposition A.8.
(iii) See [77, Proposition 4.4]. □

Similar to the relation $\preccurlyeq_K$, discussed in the introduction, we define a relation in $\mathcal{P}(\mathbb{R}^p)$ by

$$A \preccurlyeq_C B \qquad :\Leftrightarrow \qquad B \subseteq \operatorname{cl}(A + C).$$

Of course, this relation is reflexive and transitive, but not antisymmetric. In the following two corollaries we present some simple properties concerning infimal sets.

Corollary 5.1.2 *Let $A \subseteq B \subseteq \mathbb{R}^p$ such that $B + C \neq \mathbb{R}^p$. Then,* $\operatorname{Inf} B \preccurlyeq_C \operatorname{Inf} A$.

Proof. Let $y \in \operatorname{Inf} A$. By Proposition 5.1.1 (iii) we conclude that $\{y\} + \operatorname{int} C \subseteq A + \operatorname{int} C \subseteq B + \operatorname{int} C = \operatorname{Inf} B + \operatorname{int} C$. Letting $c \in \operatorname{int} C$, for all $\lambda > 0$ we have $y + \lambda c \in \operatorname{Inf} B + \operatorname{int} C$, hence $y \in \operatorname{cl}(\operatorname{Inf} B + \operatorname{int} C) \subseteq \operatorname{cl}(\operatorname{Inf} B + C)$. □

Corollary 5.1.3 *Let $A, B \subseteq \mathbb{R}^p$. Then,*

$$\big(\operatorname{Inf} B \preccurlyeq_C \operatorname{Inf} A,\ \operatorname{Inf} A \preccurlyeq_C \operatorname{Inf} B\big) \qquad \Rightarrow \qquad \operatorname{Inf} A = \operatorname{Inf} B.$$

Proof. We can assume that $\operatorname{Inf} A$ and $\operatorname{Inf} B$ are nonempty, hence $A + C \neq Y$ and $B + C \neq Y$. Using Proposition A.8, we obtain $\operatorname{Inf} A + \operatorname{int} C = \operatorname{Inf} B + \operatorname{int} C$. Proposition 5.1.1 (iii) yields $A + \operatorname{int} C = B + \operatorname{int} C$. Hence, the result follows from the definition of the infimal set. □

We close this section with two results on the existence of weakly minimal elements of closed convex subsets of $\mathbb{R}^p$.

Proposition 5.1.4 *Let $A \subseteq \mathbb{R}^p$ be a nonempty closed convex set with $\mathbb{R}^p \neq 0^+A \supseteq C$. Then,*

$$\forall c \in \operatorname{int} C : \qquad A = \operatorname{wMin}[A, C] + \mathbb{R}_+ \cdot \{c\}\,.$$

Proof. "$\supseteq$". $\operatorname{wMin}[A, C] + \mathbb{R}_+ \cdot \{c\} \subseteq A + C \subseteq A + 0^+A = A$.

"$\subseteq$". For given $y \in A$, define $\alpha := \inf\{\gamma \in \mathbb{R}|\ y + \gamma c \in A\}$. If $\alpha = -\infty$, we conclude that $-c \in 0^+A$. Hence, $0 = c - c \in \operatorname{int} C + 0^+A \subseteq \operatorname{int} 0^+A$. This contradicts $0^+A \neq \mathbb{R}^p$. Hence we have $-\infty < \alpha \leq 0$. The closedness of A implies that $y' := y + \alpha c \in A$. We next show that y' even belongs to $\operatorname{wMin}[A, C]$. Assuming the contrary yields some $y'' \in (\{y'\} - \operatorname{int} C) \cap A$. We have $y'' = y' - c'$ for some $c' \in \operatorname{int} C$. Furthermore, there exists some $\delta > 0$ such that $c'' := c' - \delta c \in \operatorname{int} C$. Hence we obtain $y''' := y'' + c'' \in A + C \subseteq A + 0^+A = A$. On the other hand we have $y''' = y' - \delta c = y + (\alpha - \delta)c \in A$ where $(\alpha - \delta) < \alpha$. This contradicts the minimality of α. Consequently, we obtain $y' \in \operatorname{wMin}[A, C]$ and hence $y \in \operatorname{wMin}[A, C] + \mathbb{R}_+ \cdot \{c\}$. □

Proposition 5.1.5 *Let $A \subseteq \mathbb{R}^p$ be a nonempty closed convex set with $0^+A \supseteq C$. Then, for all $y \in \mathbb{R}^p \setminus A$ and all $c \in \operatorname{int} C$ there exists some $\alpha > 0$ such that $y + \alpha c \in \operatorname{wMin}[A, C]$.*

Proof. Let $\bar{y} \in \mathbb{R}^p \setminus A$ and $\bar{c} \in \operatorname{int} C$ be given. Define the set $B := \{\bar{y}\} + \mathbb{R}_+ \{\bar{c}\}$. Assume that $A \cap B = \emptyset$. Then, the separation theorem [1, Theorem 11.3] implies the existence of some $y^* \in \mathbb{R}^p \setminus \{0\}$ such that

$$-\infty < \sup_{a \in A} \langle y^*, a\rangle \leq \inf_{b \in B} \langle y^*, b\rangle < \infty. \tag{5.1}$$

By Lemma 1.3.1 (i) we know that $y^* \in (0^+A)^\circ$. Since $0^+A \supseteq C$, we obtain $(0^+A)^\circ \subseteq C^\circ$ and hence $y^* \in C^\circ \setminus \{0\}$. Proposition 2.3.2 yields that $\langle y^*, \bar{c}\rangle < 0$. It follows that $\inf_{b \in B} \langle y^*, b\rangle = -\infty$, which contradicts (5.1). Hence, we have $A \cap B \neq \emptyset$. Consequently, there exists some $\bar{\alpha} > 0$ such that $\bar{y} + \bar{\alpha}\bar{c} \in A$. From Proposition 5.1.4 we deduce that there exists $\hat{y} \in \operatorname{wMin}[A, C]$ and $\hat{\alpha} \geq 0$ such that $\bar{y} + \bar{\alpha}\bar{c} = \hat{y} + \hat{\alpha}\bar{c}$. It remains to show that $\alpha := \bar{\alpha} - \hat{\alpha} > 0$. Assuming the contrary, we obtain $\bar{y} \in \operatorname{wMin}[A, C] + C \subseteq A + 0^+A = A$, a contradiction. □

5.2 Point relation vs. set relation approach

The aim of this section is to show that a strong duality assertion with respect to set relation approach follows from a strong duality assertion with respect to point relation approach and vice versa. This is expressed by the "equivalence theorem", which states that, without the typical constraint qualification, both types of strong duality assertions are equivalent. We begin with the definition of the dual pair of set optimization problems with point relations. Of course, as a special case we can consider vector optimization problems instead.

For given set–valued maps $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, a closed convex pointed cone $C \subseteq \mathbb{R}^p$ with nonempty interior and a nonempty closed convex cone $C_U \subseteq \mathbb{R}^m$, we define a primal set optimization problem by (the superscript "PR" stands for "point relation")

$$(\mathrm{P}^{\mathrm{PR}}) \qquad \bar{P} := \operatorname{Inf}\Big[\bigcup_{x\in S} F(x), C\Big], \qquad S := \{x \in \mathbb{R}^n \mid G(x) \cap -C_U \neq \emptyset\}.$$

For given $c \in \mathbb{R}^p$, consider the Lagrangian $\bar{L}_c : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p$, defined similar to the Lagrangian $L_c : \mathbb{R}^n \times \mathbb{R}^m \to \hat{\mathcal{C}}^\star$ in Section 4.2, by

$$\bar{L}_c(x, u^*) := F(x) + C + \bigcup_{u\in G(x)+C_U} \{\langle u^*, u\rangle \{c\}\}.$$

The dual objective function $\Phi_c : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ is defined by

$$\Phi_c(u^*) := \operatorname{Inf} \bigcup_{x\in\mathbb{R}^n} \bar{L}_c(x, u^*),$$

and the dual problem associated to $(\mathrm{P}^{\mathrm{PR}})$ is as follows:

$$(\mathrm{D}_c^{\mathrm{PR}}) \qquad \bar{D}_c := \operatorname{Sup} \bigcup_{u^*\in\mathbb{R}^m} \Phi_c(u^*).$$

Let us first prove some auxiliary assertions.

Proposition 5.2.1 *Let X be an arbitrary set and $H : X \rightrightarrows \mathbb{R}^p$ be a set–valued map. Then it holds*

(i) $\displaystyle\Big(\bigcup_{x\in X} \operatorname{wMin} H(x) - \operatorname{int} C\Big) \cap \Big(\bigcap_{x\in X} H(x)\Big) = \emptyset,$

If H has nonempty closed convex values and $0^+H(x) \supseteq C$ for all $x \in X$, then it holds

(ii) $\displaystyle\Big(\bigcup_{x\in X} \operatorname{wMin} H(x) - \operatorname{int} C\Big) \cup \Big(\bigcap_{x\in X} H(x)\Big) = \mathbb{R}^p,$

(iii) $\displaystyle\operatorname{Sup} \bigcup_{x\in X} \operatorname{Inf} H(x) = \operatorname{wMin} \bigcap_{x\in X} H(x).$

Proof. (i) Assume there exists some $y \in \big(\bigcup_{x\in X} \operatorname{wMin} H(x) - \operatorname{int} C\big) \cap \big(\bigcap_{x\in X} H(x)\big)$. Hence there is some $\bar{x} \in X$ such that $y \in \big(\operatorname{wMin} H(\bar{x}) - \operatorname{int} C\big) \cap H(\bar{x})$. Consequently, there exists some $\bar{y} \in \operatorname{wMin} H(\bar{x})$ such that $y \in \{\bar{y}\} - \operatorname{int} C$. Since $\bar{y} \in \operatorname{wMin} H(\bar{x})$ we have $(\{\bar{y}\} - \operatorname{int} C) \cap H(\bar{x}) = \emptyset$ and hence $y \notin H(\bar{x})$, a contradiction.

(ii) Let $y \in \mathbb{R}^p \setminus \bigcap_{x\in X} H(x)$. Then there exists some $\bar{x} \in X$ such that $y \notin H(\bar{x})$. Let $c \in \operatorname{int} C$ be given. By Proposition 5.1.5 there exists $\alpha > 0$ such that $y + \alpha c \in \operatorname{wMin}[H(\bar{x}), C]$. Hence, $y \in \bigcup_{x\in X} \operatorname{wMin} H(x) - \operatorname{int} C$.

(iii) By Proposition 5.1.1 (ii) we have $\operatorname{Inf} H(x) = \operatorname{wMin} \operatorname{cl}\big(H(x) + \operatorname{int} C\big)$. By virtue of Proposition A.3 (iii) and A.10 and since $C \subseteq 0^+ H(x)$ it follows $\operatorname{Inf} H(x) = \operatorname{wMin} H(x)$. Let $y \in \operatorname{Sup} \bigcup_{x\in X} \operatorname{Inf} H(x) = \operatorname{Sup} \bigcup_{x\in X} \operatorname{wMin} H(x)$. By the definition of the supremal set this is equivalent to

$$y \notin \bigcup_{x\in X} \operatorname{wMin} H(x) - \operatorname{int} C\,, \qquad \{y\} - \operatorname{int} C \subseteq \bigcup_{x\in X} \operatorname{wMin} H(x) - \operatorname{int} C$$

By (i) and (ii), this is equivalent to

$$y \in \bigcap_{x\in X} H(x)\,, \qquad \big(\{y\} - \operatorname{int} C\big) \cap \left(\bigcap_{x\in X} H(x)\right) = \emptyset,$$

i.e., $y \in \operatorname{wMin} \bigcap_{x\in X} H(x)$. □

We next investigate convex problems. We suppose that the objective map F is C–convex (see Definition 4.2.1) and, as in Section 4.2, the map G is supposed to be C_U–convex. Of course, F is C–convex if and only if $f(\cdot) := \operatorname{cl}(F(\cdot) + C)$ is convex function $f : \mathbb{R}^n \to \hat{\mathcal{C}}^\star$ (see Corollary 3.4.7). Furthermore, if $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is C_U–convex, we deduce that for all $u^* \in \mathbb{R}^m$ the map

$$g_{u^*}(\,\cdot\,) := \operatorname{cl} \bigcup_{u\in G(\,\cdot\,)+C_U} \big(\langle u^*, u\rangle \{c\}\big)$$

can be interpreted as a convex function $g_{u^*} : \mathbb{R}^n \to \hat{\mathcal{C}}^\star$. Consequently, we have the following relationship between the objective function Φ_c of problem $(\mathrm{D}_\mathrm{c}^{\mathrm{PR}})$ and the objective function ϕ_c of problem (D_c), defined in Section 4.2.

Proposition 5.2.2 *If F is C–convex and G is C_U–convex, we have*

$$\forall u^* \in \mathbb{R}^m : \qquad \Phi_c(u^*) = \operatorname{Inf} \phi_c(u^*).$$

Proof. For arbitrary $u^* \in \mathbb{R}^m$ it holds

$$\begin{aligned}
\Phi_c(u^*) &= \operatorname{Inf} \bigcup_{x\in\mathbb{R}^n} \bar{L}(x, u^*) \overset{\text{Pr. 5.1.1 (i)}}{=} \operatorname{Inf} \operatorname{cl} \bigcup_{x\in\mathbb{R}^n} \bar{L}(x, u^*) \\
&\overset{\text{Pr. A.2 (iii), A.4 (vii)}}{=} \operatorname{Inf} \operatorname{cl} \bigcup_{x\in\mathbb{R}^n} \big(f(x) \oplus g_{u^*}(x)\big) \\
&\overset{f,\, g_{u^*}\text{ convex}}{=} \operatorname{Inf} \inf_{x\in\mathbb{R}^n} L(x, u^*) = \operatorname{Inf} \phi_c(u^*).
\end{aligned}$$

□

We now compare strong duality between the problems $(\mathrm{P}^{\mathrm{PR}})$ and $(\mathrm{D}_\mathrm{c}^{\mathrm{PR}})$ with strong duality between the problems (P) and (D_c), defined in Section 4.2.

Theorem 5.2.3 (Equivalence theorem) *Let F be C–convex and let G be C_U–convex. If $S \neq \emptyset$, then for all $c \in \mathbb{R}^p$ the following strong duality assertions are equivalent:*

(i) $\operatorname{Inf} \bigcup_{x \in S} F(x) = \operatorname{Sup} \bigcup_{u^* \in \mathbb{R}^m} \Phi_c(u^*)$,

(ii) $\inf_{x \in S} f(x) = \sup_{u^* \in \mathbb{R}^m} \phi_c(u^*)$.

Proof. The C–convexity of F and the C_U–convexity of G imply that the set $\bigcup_{x \in S} f(x)$ is convex, hence

$$\operatorname{Inf} \bigcup_{x \in S} F(x) \overset{\text{Pr. 5.1.1 (ii)}}{=} \operatorname{wMin} \operatorname{cl}\Big(\bigcup_{x \in S} F(x) + \operatorname{int} C\Big)$$

$$\overset{\text{Pr. A.1 (iii), A.2 (iii), A.4 (vii)}}{=} \operatorname{wMin} \operatorname{cl} \bigcup_{x \in S} f(x) = \operatorname{wMin} \inf_{x \in S} f(x)$$

Moreover, taking into account that $0^+ \phi_c(u^*) \supseteq C$ for all $u^* \in \mathbb{R}^m$, we have

$$\operatorname{Sup} \bigcup_{u^* \in \mathbb{R}^m} \Phi_c(u^*) \overset{\text{Pr. 5.2.2}}{=} \operatorname{Sup} \bigcup_{u^* \in \mathbb{R}^m} \operatorname{Inf} \phi_c(u^*) \overset{\text{Pr. 5.2.1 (iii)}}{=} \operatorname{wMin} \sup_{u^* \in \mathbb{R}^m} \phi_c(u^*).$$

We directly conclude that (ii) implies (i). In order to show the opposite inclusion, let (i) be satisfied. Note that $S \neq \emptyset$ implies that $0^+ \inf_{x \in S} f(x) \supseteq C$ and by the weak duality inequality it follows that $0^+ \sup_{u^* \in \mathbb{R}^m} \phi_c(u^*) \supseteq C$. If $\inf_{x \in S} f(x) \neq \mathbb{R}^p$, Proposition 5.1.4 yields ($c \in \operatorname{int} C$)

$$\inf_{x \in S} f(x) = \operatorname{wMin} \inf_{x \in S} f(x) + \mathbb{R}^+ \cdot \{c\} = \operatorname{wMin} \sup_{u^* \in \mathbb{R}^m} \phi_c(u^*) + \mathbb{R}^+ \cdot \{c\} = \sup_{u^* \in \mathbb{R}^m} \phi_c(u^*),$$

otherwise (ii) follows from the weak duality inequality of Section 4.2. □

Finally, we prove a duality theorem for the problems (P^{PR}) and ($\mathrm{D}_{\mathrm{c}}^{\mathrm{PR}}$) by applying the duality theorem of Section 4.2, which is based on set relation approach.

Theorem 5.2.4 (Duality Theorem) *For the problems* (P^{PR}) *and* ($\mathrm{D}_{\mathrm{c}}^{\mathrm{PR}}$), *for all $c \in \mathbb{R}^p$ it holds weak duality, i.e.,*

$$(\bar{D}_c - \operatorname{int} C) \cap \bar{P} = \emptyset.$$

If, additionally, F is C–convex and G is C_U–convex and the constraint qualification

$$G(\operatorname{dom} F) \cap -\operatorname{int} C_U \neq \emptyset \tag{5.2}$$

is satisfied, for all $c \in \operatorname{int} C$, we have strong duality between (P^{PR}) *and* ($\mathrm{D}_{\mathrm{c}}^{\mathrm{PR}}$), *i.e., $\bar{P} = \bar{D}_c$.*

Proof. *Weak duality.* It holds

$$\begin{aligned}
\bar{D}_c - \operatorname{int} C &= \operatorname{Sup} \bigcup_{u^* \in \mathbb{R}^m} \Phi_c(u^*) - \operatorname{int} C \quad \subseteq \quad \bigcup_{u^* \in \mathbb{R}^m} \Phi_c(u^*) - \operatorname{int} C \\
&= \bigcup_{u^* \in \mathbb{R}^m} \operatorname{Inf} \bigcup_{x \in \mathbb{R}^n} \bar{L}_c(x, u^*) - \operatorname{int} C \\
&\overset{\text{Pr. 5.1.1 (ii)}}{=} \bigcup_{u^* \in \mathbb{R}^m} \operatorname{wMin} \operatorname{cl} \left(\bigcup_{x \in \mathbb{R}^n} \bar{L}_c(x, u^*) + \operatorname{int} C \right) - \operatorname{int} C \\
&\overset{\text{Pr. A.1 (iii), A.2 (iii), A.4 (vii)}}{=} \bigcup_{u^* \in \mathbb{R}^m} \operatorname{wMin} \operatorname{cl} \bigcup_{x \in \mathbb{R}^n} \bar{L}_c(x, u^*) - \operatorname{int} C \quad =: \quad A_1.
\end{aligned}$$

On the other hand, we have

$$\bar{P} \quad = \quad \operatorname{Inf} \bigcup_{x \in S} F(x) \overset{\text{Pr. 5.1.1 (ii)}}{=} \operatorname{wMin} \operatorname{cl} \left(\bigcup_{x \in S} F(x) + \operatorname{int} C \right) \subseteq \operatorname{cl} \bigcup_{x \in S} \big(F(x) + C \big).$$

For $x \in S$ and arbitrary $u^* \in \mathbb{R}^m$, it is easy to check that $F(x) + C \subseteq \bar{L}(x, u^*)$ and hence $F(x) + C \subseteq \bigcap_{u^* \in \mathbb{R}^m} \bar{L}(x, u^*)$. Using a (very easy to prove) weak duality assertion analogous to that in Section 4.2, but for $\hat{\mathcal{F}}$–valued functions, we obtain

$$\bar{P} \subseteq \operatorname{cl} \bigcup_{x \in S} \bigcap_{u^* \in \mathbb{R}^m} \bar{L}_c(x, u^*) \subseteq \operatorname{cl} \bigcup_{x \in \mathbb{R}^n} \bigcap_{u^* \in \mathbb{R}^m} \bar{L}_c(x, u^*) \subseteq \bigcap_{u^* \in \mathbb{R}^m} \operatorname{cl} \bigcup_{x \in \mathbb{R}^n} \bar{L}_c(x, u^*) =: A_2$$

Proposition 5.2.1 (i) yields that A_1 and A_2 are disjoint. Hence $(\bar{D}_c - \operatorname{int} C) \cap \bar{P} = \emptyset$.

Strong duality. It is easy to verify that the assumptions of Theorem 4.2.6 are satisfied. For instance, we have $K := 0^+ \inf_{x \in S} f(x) \supseteq C$. Since $c \in \operatorname{int} C$ we deduce that either $K = \mathbb{R}^p$ or $K \subsetneq \mathbb{R}^p$ is not a linear subspace of $\mathbb{R}^p$ and $c \in \operatorname{ri} K$. Theorem 4.2.6 yields strong duality between (P) and ($\mathrm{D_c}$). Hence the result follows from Theorem 5.2.3. □

5.3 On the structure of vector optimization problems

In this section, we discuss the structure of a vector optimization problems and set optimization problems with point relations. We already started this discussion in the introduction.

Let $C \subseteq \mathbb{R}^p$ be a closed convex pointed cone and $A \subseteq \mathbb{R}^p$. Recall that an element $y \in A$ is said to be *minimal* (or *efficient*) if $y \in A$ and $(\{y\} - C \setminus \{0\}) \cap A = \emptyset$. The set of all minimal elements of A is denoted by $\operatorname{Min}[A, C]$.

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ set–valued objective map and let $S \subseteq \mathbb{R}^n$ a set which describes the constraints. Then we are interested to know the set

$$\operatorname{Min}\left[\bigcup_{x \in S} F(x), C \right], \tag{5.3}$$

We observe that problem (5.3) consists of the following two components:

(i) $A := \bigcup_{x \in S} F(x)$ (an optimization problem based on set inclusion),

(ii) $\mathrm{Min}\,[A, C]$ (the determination of the set of minimal points).

Note that the union in (i) can be interpreted as the infimum in the ordered conlinear space $(\hat{\mathcal{P}}(\mathbb{R}^p), \supseteq)$ of all subsets of $\mathbb{R}^p$.

We next investigate convex problems. Recall that a set–valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is said to be *C–closed* (e.g. [57]) if $\mathrm{gr}\,(F(\cdot) + C)$ is closed. It is easy to verify that F is C–convex (see Definition 4.2.1) and C–closed if and only if $f(\cdot) := \mathrm{cl}\,(F(\cdot)+C)$ can be interpreted as a lower semi–continuous convex function $f : \mathbb{R}^n \to \hat{\mathcal{C}}^\star$ (see Corollary 3.4.7). Moreover, the following assertion holds true.

Proposition 5.3.1 *Let F be C–convex and C–closed and let $S \subseteq \mathbb{R}^n$ be convex and compact. Then,*

$$\mathrm{Min}\left[\bigcup_{x \in S} F(x), C\right] = \mathrm{Min}\left[\inf_{x \in S} f(x), C\right].$$

Proof. We show that the set $P := \bigcup_{x \in S} F(x) + C$ is convex and closed. Indeed, let $y_i \in P$, $(i = 1, 2)$. It follows $y_i \in F(x_i) + C$ for some $x_i \in S$, $(i = 1, 2)$. Hence $\lambda y_1 + (1 - \lambda) y_2 \in \lambda\big(F(x_1) + C\big) + (1 - \lambda)\big(F(x_2) + C\big) \subseteq F(\lambda x_1 + (1 - \lambda) x_2) + C \subseteq P$. To show the closedness, let $\{y_n\}_{n \in \mathbb{N}} \subseteq P$ with $y_n \to \bar{y}$ be given. Then there is a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq S$ such that $(x_n, y_n) \in \mathrm{gr}\,\big(F(\cdot) + C\big)$. Since S is compact we can choose a convergent subsequence $x_{n_k} \to \bar{x} \in S$. Since $\mathrm{gr}\,\big(F(\cdot)+C\big)$ is closed, we deduce that $(\bar{x}, \bar{y}) \in \mathrm{gr}\,\big(F(\cdot)+C\big)$, i.e., $\bar{y} \in F(\bar{x})+C \subseteq P$. Hence we have $P = \mathrm{cl\,conv}\, P = \mathrm{cl\,conv} \bigcup_{x \in S}(F(x) + C) = \mathrm{cl\,conv} \bigcup_{x \in S} \mathrm{cl}\,(F(x) + C) = \inf_{x \in S} f(x)$. As an easy consequence of the definition of minimal elements we always have $\mathrm{Min}[A, C] = \mathrm{Min}[A + C, C]$ (for this, C has to be pointed). It follows that $\mathrm{Min}[P, C] = \mathrm{Min}[\bigcup_{x \in S} F(x) + C, C] = \mathrm{Min}[\bigcup_{x \in S} F(x), C]$. □

As in Section 4.2, let $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set–valued map, $C_U \subseteq \mathbb{R}^m$ a nonempty closed convex cone and $S := \{x \in \mathbb{R}^n |\; G(x) \cap -C_U \neq \emptyset\}$. Using the strong duality assertion of Theorem 4.2.6 we obtain an equivalent characterization of the problem (5.3). For simplicity we suppose that C has a nonempty interior.

Corollary 5.3.2 *Let F be C–convex and C–closed, let G be C_U–convex, S compact, $c \in \mathrm{int}\, C$ and $G(\mathrm{dom}\, F) \cap -\mathrm{int}\, C_U \neq \emptyset$. Then,*

$$\mathrm{Min}\left[\bigcup_{x \in S} F(x), C\right] = \mathrm{Min}\left[\bigcap_{u^* \in \mathbb{R}^m} \phi_c(u^*), C\right].$$

where $\phi_c : \mathbb{R}^m \to \hat{\mathcal{C}}^\star$ is the dual objective function of problem (P) *for $f(\cdot) := \mathrm{cl}\,(F(\cdot) + C)$, defined in Section 4.2.*

Proof. By Proposition 5.3.1 we know that $\mathrm{Min}\left[\bigcup_{x \in S} F(x), C\right] = \mathrm{Min}\,[\inf_{x \in S} f(x), C]$, where the convexity of S is an easy consequence of the C_U–convexity of G. In order to get

$\inf_{x\in S} f(x) = \sup_{u^*\in\mathbb{R}^m} \phi_c(u^*)$ we apply Theorem 4.2.6. The assumptions of Theorem 4.2.6 are easy to verify, for instance, we have $K := 0^+ \inf_{x\in S} f(x) \supseteq C$. Since $c \in \operatorname{int} C$ we deduce that either $K = \mathbb{R}^p$ or $K \subsetneq \mathbb{R}^p$ is not a linear subspace of $\mathbb{R}^p$ and $c \in \operatorname{ri} K$. □

We close this work with an illustration how duality of optimization problems based on set inclusion can be used to sandwich the infimal set of a vector optimization problem. In scalar optimization we can use approximate primal and dual problems in order to get upper and lower bounds for the infimum of the original problem. For instance, we can take the infimum (supremum) over finitely many primal (dual) feasible points in order to obtain such bounds. Let us proceed analogously. Instead of the problems (P) and (D$_\mathrm{c}$), defined in Section 4.2, we consider the following approximate problems

$$\widetilde{P} := \inf_{x\in\widetilde{S}} f(x), \quad \widetilde{S} \subseteq S; \qquad\qquad \widetilde{D}_c := \sup_{u^*\in\widetilde{S}_D} \phi_c(u^*), \quad \widetilde{S}_D \subseteq \mathbb{R}^m.$$

By the weak duality we always have $\widetilde{D}_c \leq D_c \leq P \leq \widetilde{P}$. If $\widetilde{D}_c + C \neq \mathbb{R}^p$, Corollary 5.1.2 yields that

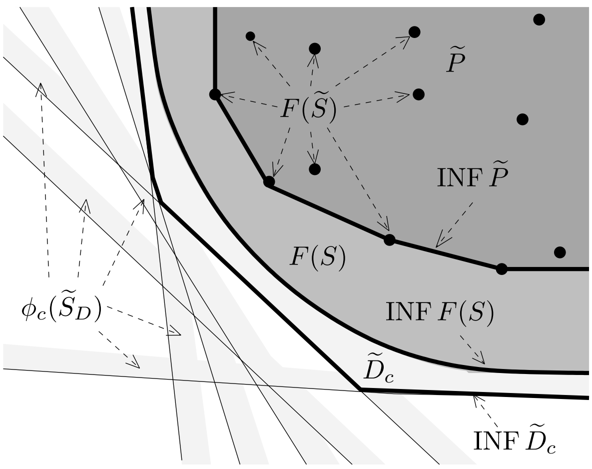
$$\operatorname{Inf} \widetilde{D}_c \preccurlyeq_C \operatorname{Inf} D_c \preccurlyeq_C \operatorname{Inf} P \preccurlyeq_C \operatorname{Inf} \widetilde{P}.$$

Of course, if F is C–convex we have

$$\operatorname{Inf} P = \operatorname{Inf} \inf_{x\in S} f(x) = \operatorname{Inf}\Big(\bigcup_{x\in S} F(x) + C\Big) = \operatorname{Inf} \bigcup_{x\in S} F(x) =: \operatorname{Inf} F(S).$$

Hence the set $\operatorname{Inf} F(S)$ is sandwiched in the following sense:

$$\operatorname{Inf} \widetilde{D}_c \preccurlyeq_C \operatorname{Inf} F(S) \preccurlyeq_C \operatorname{Inf} \widetilde{P}.$$



By Corollary 5.1.3, strong duality between (P) and (D$_\mathrm{c}$) implies that

$$\operatorname{Inf} D_c = \operatorname{Inf} F(S) = \operatorname{Inf} P.$$

Appendix

A Some calculus rules of sets

For the convenience of the reader we collect some simple calculus rules for sets. Of course, these rules are involved in many books (but unfortunately not all in one). Therefore, we present the most assertions with proof. Furthermore, in case of inclusions, we show that the opposite inclusion is not true, in general.

Proposition A.1 *Let X be a linear space, let $A_i, A, B \subseteq X$ and let I be an arbitrary index set. Then,*

(i) $\operatorname{conv} \bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} \operatorname{conv} A_i$,

(ii) $\operatorname{conv} \bigcup_{i \in I} A_i \supseteq \bigcup_{i \in I} \operatorname{conv} A_i$,

(iii) $\bigcup_{i \in I}(A_i + B) = \left(\bigcup_{i \in I} A_i\right) + B$,

(iv) $\bigcap_{i \in I}(A_i + B) \supseteq \left(\bigcap_{i \in I} A_i\right) + B$,

(v) $\operatorname{conv}(A + B) = \operatorname{conv} A + \operatorname{conv} B$.

Proof. The proof of (i) – (iv) is immediate, (v) can be found in [60]. □

In general, (i), (ii) and (iv) does not hold with equality as the following examples show:
(i) $A_1 = \{0, 2\}$, $A_2 = \{1\}$.
(ii) $A_1 = \{0, 1\}$, $A_2 = \{2, 3\}$.
(iv) $A_1 = \{0\}$, $A_2 = \{1\}$, $B = [-1, 1]$.

Proposition A.2 *Let X be a topological space, let $A_i, A, B \subseteq X$ and let I be an arbitrary index set. Then,*

(i) $\operatorname{cl} \bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} \operatorname{cl} A_i$,

(ii) $\operatorname{cl} \bigcup_{i \in I} A_i \supseteq \bigcup_{i \in I} \operatorname{cl} A_i$,

(iii) $\mathrm{cl} \bigcup_{i \in I} A_i = \mathrm{cl} \bigcup_{i \in I} \mathrm{cl}\, A_i$,

(iv) $\mathrm{cl}\,(A \cup B) = \mathrm{cl}\, A \cup \mathrm{cl}\, B$.

Proof. (i) $\mathrm{cl} \bigcap_{i \in I} A_i \subseteq \mathrm{cl} \bigcap_{i \in I} \mathrm{cl}\, A_i$. Since $\bigcap_{i \in I} \mathrm{cl}\, A_i$ is closed, $\mathrm{cl} \bigcap_{i \in I} \mathrm{cl}\, A_i = \bigcap_{i \in I} \mathrm{cl}\, A_i$.
(ii) For all $i \in I$ we have $\mathrm{cl} \bigcup_{i \in I} A_i \supseteq \mathrm{cl}\, A_i$. Hence $\mathrm{cl} \bigcup_{i \in I} A_i \supseteq \bigcup_{i \in I} \mathrm{cl}\, A_i$.
(iii) From (ii).
(iv) Since $\mathrm{cl}\, A \cup \mathrm{cl}\, B$ is closed, this follows from (iii). □

To see that, in general, (i) and (ii) does not hold with equality consider the following examples:
(i) $A_1 = (0,1)$, $A_2 = (1,2)$.
(ii) $I = \mathbb{N}$, $A_i = \{1/i\}$.

Proposition A.3 *Let X be a linear topological space and let $A \subseteq X$ be convex. Then,*

(i) $\mathrm{cl}\, A$ *is convex,*

(ii) $\mathrm{int}\, A$ *is convex,*

(iii) $\mathrm{int}\, A \neq \emptyset$ *implies that* $\mathrm{cl}\, A = \mathrm{cl}\,(\mathrm{int}\, A)$ *and* $\mathrm{int}\,(\mathrm{cl}\, A) = \mathrm{int}\, A$.

Proof. For instance, see [46]. □

Assertion (iii) of the previous proposition, in general, does not hold for nonconvex sets. For instance, consider $A = [0,1] \cup \{2\}$ and $A = \mathbb{R} \setminus \{0\} \subseteq \mathbb{R}$, respectively.

Proposition A.4 *Let X be a linear topological space, let $A_i, A, B \subseteq X$ and let I be an arbitrary index set. Then,*

(i) $\mathrm{cl}\,\mathrm{conv}\, A \supseteq \mathrm{conv}\,\mathrm{cl}\, A$,

(ii) $\mathrm{cl}\,\mathrm{conv}\, A = \mathrm{cl}\,\mathrm{conv}\,\mathrm{cl}\, A$,

(iii) $\mathrm{cl}\,\mathrm{conv} \bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} \mathrm{cl}\,\mathrm{conv}\, A_i$,

(iv) $\mathrm{cl}\,\mathrm{conv} \bigcup_{i \in I} A_i \supseteq \bigcup_{i \in I} \mathrm{cl}\,\mathrm{conv}\, A_i$,

(v) $\mathrm{cl}\,\mathrm{conv} \bigcup_{i \in I} A_i = \mathrm{cl}\,\mathrm{conv} \bigcup_{i \in I} \mathrm{cl}\,\mathrm{conv}\, A_i$,

(vi) $\mathrm{cl}\,(A + B) \supseteq \mathrm{cl}\, A + \mathrm{cl}\, B$,

(vii) $\mathrm{cl}\,(A + B) = \mathrm{cl}\,(\mathrm{cl}\, A + \mathrm{cl}\, B)$,

(viii) $\mathrm{cl}\,\mathrm{conv}\,(A + B) \supseteq \mathrm{cl}\,\mathrm{conv}\, A + \mathrm{cl}\,\mathrm{conv}\, B$,

(ix) $\mathrm{cl}\,\mathrm{conv}\,(A + B) = \mathrm{cl}\,(\mathrm{cl}\,\mathrm{conv}\, A + \mathrm{cl}\,\mathrm{conv}\, B)$.

Proof. (i) $\operatorname{cl}\operatorname{conv} A \supseteq \operatorname{cl} A$. Since $\operatorname{cl}\operatorname{conv} A$ is convex, we have $\operatorname{cl}\operatorname{conv} A \supseteq \operatorname{conv}\operatorname{cl} A$.
(ii) Follows from (i).
(iii) Since $\bigcap_{i\in I} A_i \subseteq \bigcap_{i\in I} \operatorname{cl}\operatorname{conv} A_i$ and $\bigcap_{i\in I} \operatorname{cl}\operatorname{conv} A_i$ is closed and convex.
(iv) For all $i \in I$ it holds $\operatorname{cl}\operatorname{conv} \bigcup_{i\in I} A_i \supseteq \operatorname{cl}\operatorname{conv} A_i$. Hence (iv) is true.
(v) From (iv).
(vi) Let $x \in \operatorname{cl} A + \operatorname{cl} B$, i.e., $x = \bar{a} + \bar{b}$ with $\bar{a} \in \operatorname{cl} A$ and $\bar{b} \in \operatorname{cl} B$. Let $\mathcal{B}$ be a neighborhood base of the origin of X, formed by balanced absorbing sets (for instance, see [31]). Then, for $A \subseteq X$ it holds $\operatorname{cl} A = \bigcap \{A + V |\ V \in \mathcal{B}\}$. Let $V \in \mathcal{B}$ be given. By the definition of a topological linear space there exists $W \in \mathcal{B}$ such that $W + W \subseteq V$. For such $W \in \mathcal{B}$ there exists $a \in A$ and $b \in B$ such that $\bar{a} - a \in W$ and $\bar{b} - b \in W$. Hence $(\bar{a} + \bar{b}) - (a + b) \in V$. Let us sum up these facts: For all $V \in \mathcal{B}$ there exists $y := a + b \in A + B$ such that $x - y \in V$, i.e., $x \in \operatorname{cl}(A + B)$. Consequently, we have $\operatorname{cl} A + \operatorname{cl} B \subseteq \operatorname{cl}(A + B)$.
(vii) From (vi).
(viii) From (vi) and Proposition A.1 (v).
(ix) From (viii). □

In general, (i), (iii), (iv), (vi) and (viii) do not hold with equality:
(i) $A = \{x \in \mathbb{R}^2 |\ x_2 \geq 1/x_1,\ x_1 > 0\} \cup \{0, 0\}$.
(iii) $A_1 = \{0, 2\}$, $A_2 = \{1, 3\}$.
(iv) $A_1 = \{0\}$, $A_2 = \{1\}$.
(vi) and (viii). $A = \{x \in \mathbb{R}^2 |\ x_2 \geq 1/x_1,\ x_1 > 0\}$, $B = \{x \in \mathbb{R}^2 |\ x_2 = 0, x_1 \leq 0\}$.

Proposition A.5 *Let X, Y be arbitrary sets, let $A_i \subseteq X$, let I be an arbitrary index set and $T : X \to Y$ a function. Then,*

(i) $T\big(\bigcup_{i\in I} A_i\big) = \bigcup_{i\in I} T(A_i)$,

(ii) $T\big(\bigcap_{i\in I} A_i\big) \subseteq \bigcap_{i\in I} T(A_i)$.

Proof. (i) $T\big(\bigcup_{i\in I} A_i\big) = \bigcup\{T(x) |\ x \in \bigcup_{i\in I} A_i\} = \bigcup_{i\in I} \bigcup_{x\in A_i} T(x) = \bigcup_{i\in I} T(A_i)$.
(ii) $y \in T\big(\bigcap_{i\in I} A_i\big) \ \Rightarrow\ \forall i \in I :\ y \in T\big(A_i\big) \ \Rightarrow\ \bigcap_{i\in I} T(A_i)$. □

To see that, in general, (ii) does not hold with equality, let $T : \mathbb{R} \to \mathbb{R}$, $T(x) = \|x\|$, $A_1 = \mathbb{R}_+$, $A_2 = \mathbb{R}_-$.

Proposition A.6 *Let X, Y be linear spaces, let $A \subseteq X$ and let $T : X \to Y$ be a linear operator. Then,* $\operatorname{conv} T(A) = T(\operatorname{conv} A)$.

Proof. Follows from the definition of the convex hull (by convex combinations). □

Proposition A.7 *Let X, Y be normed spaces, let $A \subseteq X$ and let $T : X \to Y$ be a linear continuous operator. Then,*

(i) $T(\operatorname{cl} A) \subseteq \operatorname{cl} T(A)$,

(ii) $T(\operatorname{cl} \operatorname{conv} A) \subseteq \operatorname{cl} \operatorname{conv} T(A)$,

(iii) $\operatorname{cl} T(\operatorname{cl} \operatorname{conv} A) = \operatorname{cl} \operatorname{conv} T(A)$.

Proof. (i) Let $y \in T(\operatorname{cl} A)$. Then there is some $x \in \operatorname{cl} A$ such that $y = T(x)$. There exists a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq A$ with $a_n \to x$. Hence $T(a_n) \to y$, i.e., $y \in \operatorname{cl} T(A)$.
(ii) Follows from (i) and Proposition A.6.
(iii) Follows from (ii) and Proposition A.6. □

In order to see that, in general, (i) and (ii) do not hold with equality consider the set $A = \left\{x \in \mathbb{R}^2 \mid x_2 \geq 1/x_1,\, x_1 > 0\right\}$ and the operator $T : \mathbb{R}^2 \to \mathbb{R}$, $T(x) = x_1$.

Proposition A.8 *Let X be a linear topological space, $A, B \subseteq X$ and $\operatorname{int} B \neq 0$. Then, $A + \operatorname{int} B = \operatorname{cl} A + \operatorname{int} B$.*

Proof. Of course, $A+\operatorname{int} B \subseteq \operatorname{cl} A+\operatorname{int} B$. To show the opposite inclusion let $y \in \operatorname{cl} A+\operatorname{int} B$. We have $y-a \in \operatorname{int} B$ for some $a \in \operatorname{cl} A$. For all neighborhoods N of 0 there exists some $\bar{a} \in A$ such that $-\bar{a} \in \{-a\}+N$. Since $\operatorname{int} B$ is nonempty and open, there exists some neighborhood N of 0 such that $y - \bar{a} \in \{y - a\} + N \subseteq \operatorname{int} B$. □

Proposition A.9 *Let $A, B \subseteq \mathbb{R}^p$ be nonempty closed convex sets. Then, $A \subseteq B$ implies $0^+A \subseteq 0^+B$.*

Proof. [1, Corollary 8.3.3] yields $0^+A = 0^+(A \cap B) = 0^+A \cap 0^+B \subseteq 0^+B$. □

The previous assertion is not true for nonclosed sets, in general. For instance, setting $A := \left\{x \in \mathbb{R}^2 \mid x_1 > 0, x_2 > 0\right\}$ and $B := A \cup \{0, 0\}$, we obtain $0^+A = \mathbb{R}^2_+$ and $0^+B = B$.

Proposition A.10 *Let $A, B \subseteq \mathbb{R}^p$ be convex and let $\operatorname{int} B \neq \emptyset$. Then it holds*

$$A + \operatorname{int} B = \operatorname{int}(A + B).$$

Proof. Of course, $A+\operatorname{int} B \subseteq A+B$. Since $A+\operatorname{int} B$ is open, it follows $A+\operatorname{int} B \subseteq \operatorname{int}(A+B)$. Without loss of generality we can assume $A \neq \emptyset$. Hence we have $\operatorname{int}(A + B) \neq \emptyset$. The definition of the relative interior of a convex set implies $\operatorname{int}(A+B) = \operatorname{ri}(A+B)$, $\operatorname{int} B = \operatorname{ri} B$ and $\operatorname{ri} A \subseteq A$. Finally, from [1, Corollary 6.6.2] it follows $\operatorname{int}(A+B) = \operatorname{ri}(A+B) = \operatorname{ri} A+\operatorname{ri} B = \operatorname{ri} A + \operatorname{int} B \subseteq A + \operatorname{int} B$. □

In the previous assertion the assumptions to A and B cannot be omitted as the following examples show.

(i) $A, B \subseteq \mathbb{R}^2$, $A = \mathbb{B}$, $B = \{0\}$ $(\operatorname{int} B = \emptyset)$. Then $\emptyset = A + \operatorname{int} B \neq \operatorname{int}(A + B) = \operatorname{int} \mathbb{B}$.

(ii) $A, B \subseteq \mathbb{R}^2$, $A = \text{conv}\left\{(0,1)^T, (0,-1)^T\right\}$, $B = \mathbb{B} \cup \text{conv}\left\{(1,0)^T, (2,0)^T\right\}$ (B is not convex). The point $(0, 3/2)^T$ belongs to $\text{int}\,(A+B)$, but it does not belong to $A + \text{int}\, B$.

(iii) $A, B \subseteq \mathbb{R}^2$, $A = \text{conv}\left\{(-1,1)^T, (-1,-1)^T\right\} \cup \text{conv}\left\{(1,1)^T, (1,-1)^T\right\}$, $B = \mathbb{B}$ (A is not convex). The point $(0,0)^T$ belongs to $\text{int}\,(A+B)$, but it does not belong to $A + \text{int}\, B$.

B Partially ordered sets

In this section, we recall some basic ideas with respect to partially ordered sets, which can be found, for instance, in [6] or [84].

Let $(Y, \leq)$ be a partially ordered set, i.e., Y is equipped with a reflexive, transitive and antisymmetric relation.

If V is a subset of Y and the point $y_0 \in Y$ satisfies $v \leq y_0$ for all $v \in V$, then y_0 is called an *upper bound* of V. The subset V is now said to be *bounded above*.

If $y_0 \in Y$ is an upper bound of V such that $y_0 \leq \bar{y}$ for any other upper bound $\bar{y} \in Y$ of V, then y_0 is called *least upper bound* or *supremum* of V and is denoted by $\sup V$. If V has a supremum, then it is uniquely defined. This is an easy consequence of the antisymmetry of the relation $\leq$.

The definitions of *bounded below* and *lower bound* are analogous. The *greatest lower bound* or *infimum* is defined analogously and is denoted by $\inf V$.

A partially ordered set Y is said to be *order complete* (or a *complete lattice*) if every subset of Y has a supremum and an infimum. If Y is order complete and $V = \emptyset$, then $\sup V = \inf Y$ and $\inf V = \sup Y$. The set Y is called *Dedekind complete* if every nonempty subset of Y which is bounded above (bounded below) has a supremum (infimum). Note that for Dedekind completeness an one–sided condition is already sufficient, this means Y is Dedekind complete if and only if every nonempty subset of Y which is bounded above has a supremum [84, Theorem 1.4]. An element $\bar{y} \in Y$ is called the *largest* element of $(Y, \leq)$ if $y \leq \bar{y}$ for all $y \in Y$. The *smallest* element is defined analogously. If $(Y, \leq)$ has a largest (smallest) element, then it is uniquely defined.

Let $(Y, \leq)$, $(Y^\star, \leq^\star)$ be Dedekind complete partially ordered sets with $Y^\star \subseteq Y$ and such that $\leq$ and $\leq^\star$ coincide on $Y^\star$. We denote the supremum (infimum) of a subset $V \subseteq Y$ with respect to $(Y, \leq)$ by $\sup V$ ($\inf V$) and the supremum (infimum) of a subset $V \subseteq Y^\star$ with respect to $(Y^\star, \leq^\star)$ by $\sup^\star V$ ($\inf^\star V$).

Proposition B.1 *If* $\emptyset \neq V \subseteq Y^\star$ *is bounded above, then* $\sup V \leq \sup^\star V$. *Under the additional assumption* $\sup V \in Y^\star$ *we even have equality.*

Proof. By hypothesis, the set $V \subseteq Y^\star$ is bounded above (with respect to $(Y^\star, \leq^\star)$). Hence $V \subseteq Y$ is bounded above (with respect to $(Y, \leq)$). Since V is nonempty and $(Y, \leq)$ and $(Y^\star, \leq^\star))$ are Dedekind complete, $\sup V$ and $\sup^\star V$ exist. Of course, $\sup^\star V \in Y^\star$ is an upper bound of V with respect to $(Y, \leq)$. By the definition of the supremum we get $\sup V \leq \sup^{\star} V$. The second assertion follows from the definition. $\square$

C Ordered conlinear spaces

The concept of an *ordered conlinear space* provides the theoretical background of this work and seems to be the natural framework for convexity rather than linear spaces. A systematic study of this concept can be found in Hamel [33]. Similar concepts like semi–linear spaces and almost linear spaces are also discussed there. Here we just give some definitions.

Definition C.1 ([33]) *A set Y equipped with an addition $+ : Y \times Y \to Y$ is said to be a* (real) conlinear space $(Y, +)$ *if the following axioms are satisfied:*

(C1) $(Y, +)$ *is a commutative monoid with neutral element θ, i.e.,*

(i) $\forall y_1, y_2, y_3 \in Y:\ y_1 + (y_2 + y_3) = (y_1 + y_2) + y_3,$

(ii) $\exists \theta \in Y, \forall y \in Y:\ y + \theta = \theta + y = y,$

(iii) $\forall y_1, y_2 \in Y:\ y_1 + y_2 = y_2 + y_1.$

(C2) *There is mapping from $\mathbb{R}_+ \times Y$ into Y, assigning $\alpha \geq 0$ and $y \in Y$ the product $\alpha y := \alpha \cdot y \in Y$ such that the following conditions are satisfied:*

(i) $\forall y \in Y, \forall \alpha, \beta \geq 0:\ \alpha \cdot (\beta \cdot y) = (\alpha\beta) \cdot y,$

(ii) $\forall y \in Y:\ 1 \cdot y = y,$

(iii) $\forall y \in Y:\ 0 \cdot y = \theta.$

(iv) $\forall \alpha \geq 0, \forall y_1, y_2 \in Y:\ \alpha \cdot (y_1 + y_2) = (\alpha \cdot y_1) + (\alpha \cdot y_2).$

Definition C.2 ([33]) *Let $(Y, +)$ be a conlinear space and $\leq$ a partial ordering on Y satisfying the following conditions:*

(O1) $\big(y_1, y_2, y \in Y,\ y_1 \leq y_2\big) \quad \Rightarrow \quad y_1 + y \leq y_2 + y,$

(O2) $\big(y_1, y_2 \in Y,\ y_1 \leq y_2,\ \alpha \geq 0\big) \quad \Rightarrow \quad \alpha y_1 \leq \alpha y_2.$

Then, $(Y, +, \leq)$ is called an ordered conlinear space.

Definition C.3 ([33]) *Let $(Y, +)$ be a conlinear space. A subset $V \subseteq Y$ is said to be* convex *if $v_1, v_2 \in V, \lambda \in [0, 1]$ implies that $\lambda v_1 + (1 - \lambda) v_2 \in V$.*

In [33], a cone in conlinear space Y is defined to be an element $y \in Y$ satisfying $\alpha y = y$ for all $\alpha > 0$. In contrast to this, we define a cone in Y as a subset of Y, because this definition is closer related to the usual definition of a cone in linear spaces.

Definition C.4 *Let* $(Y, +)$ *be a conlinear space. A subset* $V \subseteq Y$ *is said to be* cone *if* $v \in V, \alpha > 0$ *implies that* $\alpha v \in V$*.*

Index of notation

$\mathbb{N}$	positive integers
$\mathbb{R}^p$	p–dimensional Euclidean space
$\Vert\cdot\Vert$	Euclidean norm in $\mathbb{R}^p$
$\mathbb{R}^p_+$	non–negative orthant of $\mathbb{R}^p$
$\mathbb{R}_+$	equals $\mathbb{R}^1_+$
$[a, b]$	closed interval in $\mathbb{R}$
(a, b)	open interval in $\mathbb{R}$
$\operatorname{cl} A$	closure of the set A
$\operatorname{int} A$	(topological) interior of the set A
$\operatorname{ri} A$	relative interior of the set A
$\operatorname{rb} A$	relative boundary of the set A
$\operatorname{conv} A$	convex hull of the set A
$\operatorname{lin} A$ ($\operatorname{aff} A$)	linear (affine) hull of the set A
K°	polar cone of a nonempty cone $K \subseteq \mathbb{R}^p$
$L^\perp$	orthogonal space of a linear subspace $L \subseteq \mathbb{R}^p$ ($L^\perp = L^\circ$)
$0^+ A$	recession cone of a nonempty convex set $A \subseteq \mathbb{R}^p$
$\hat{\mathcal{F}}$ ($\mathcal{F}$)	set of (nonempty) closed subsets of $\mathbb{R}^p$
$\hat{\mathcal{C}}$ ($\mathcal{C}$)	set of (nonempty) closed convex subsets of $\mathbb{R}^p$
$\mathcal{C}_K$	set of all $A \in \mathcal{C}$ with $0^+ A = K$
$\hat{\mathcal{C}}_K$	$\mathcal{C}_K \cup \{\emptyset\}$
$\hat{\mathcal{C}}^\star, \mathcal{C}^\star, \hat{\mathcal{C}}^\star_K, \mathcal{C}^\star_K$	$\hat{\mathcal{C}}, \mathcal{C}, \hat{\mathcal{C}}_K, \mathcal{C}_K$, but with supremum oriented members
$\hat{\mathcal{C}}^\diamond, \mathcal{C}^\diamond, \hat{\mathcal{C}}^\diamond_K, \mathcal{C}^\diamond_K$	$\hat{\mathcal{C}}, \mathcal{C}, \hat{\mathcal{C}}_K, \mathcal{C}_K$, but with infimum oriented members
Γ_K	set of positively homogeneous real–valued functions defined on $\operatorname{ri} K^\circ$
$\Gamma^\star_K$	set of convave members of Γ_K
$\Gamma^\diamond_K$	set of convex members of Γ_K
$\hat{\Gamma}^\star_K$	$\Gamma^\star_K$ extended by a largest element $+\infty_K$
$\hat{\Gamma}^\diamond_K$	$\Gamma^\diamond_K$ extended by a smallest element $-\infty_K$
$\mathcal{N}(x)$	set of all neighborhoods of x
$\mathcal{N}^\#_\infty$	set of infinite subsets of $\mathbb{N}$
$\mathcal{N}_\infty$	set of cofinal subsets of $\mathbb{N}$
inf (INF)	(alternative concept of) infimum
sup (SUP)	(alternative concept of) supremum
lim inf (LIM INF)	(alternative concept of) lower limit
lim sup (LIM SUP)	(alternative concept of) upper limit

lim (LIM)	(alternative concept of) limit
$\mathrm{Inf}[A, C]$	infimal set of A (with respect to the ordering cone C)
$\mathrm{Sup}[A, C]$	supremal set of A (with respect to the ordering cone C)
$\mathrm{Min}[A, C]$	set of minimal (= minimal efficient) elements of A
$\mathrm{Max}[A, C]$	set of maximal (= maximal efficient) elements of A
$\mathrm{wMin}[A, C]$	set of weakly minimal (= minimal weakly efficient) elements of A
$\mathrm{wMax}[A, C]$	set of weakly maximal (= maximal weakly efficient) elements of A
$\mathrm{Inf}\, A$ ($\mathrm{Sup}\, A$)	$= \mathrm{Inf}[A, C]$ $(= \mathrm{Sup}[A, C])$
$\mathrm{Min}\, A$ ($\mathrm{Max}\, A$)	$= \mathrm{Min}[A, C]$ $(= \mathrm{Max}[A, C])$
$\mathrm{wMin}\, A$ ($\mathrm{wMax}\, A$)	$= \mathrm{wMin}[A, C]$ $(= \mathrm{wMax}[A, C])$
$\oplus$	closed Minkowski addition
$\boxplus$	(Minkowski addition with) change of orientation
$\boxminus$	equals "$\boxplus -$"
$\mathrm{epi}\, f$ ($\mathrm{hyp}\, f$)	epigraph (hypograph) of a function f
$\mathrm{gr}\, f$	graph of a map f
$\mathrm{dom}\, f$	(effective) domain of a map f
$\mathrm{rg}\, f$	range of a map f
$\mathrm{isc}\, f$	inner semi–continuous hull of f
$\mathrm{osc}\, f$	outer semi–continuous hull of f
$\mathrm{lsc}\, f$	lower semi–continuous hull of f
$\mathrm{usc}\, f$	upper semi–continuous hull of f
$\delta(\cdot \mid A)$	(convex) indicator function of $A \subseteq \mathbb{R}^p$
$\delta^*(\cdot \mid A)$	support function of $A \subseteq \mathbb{R}^p$
$\Delta(\cdot \mid A)$	set–valued indicator function of $A \subseteq \mathbb{R}^p$
$\Delta^*_c(\cdot \mid A)$	set–valued support function of $A \subseteq \mathbb{R}^p$ with respect to $c \in \mathbb{R}^p$
$f^*(\cdot)$	the conjugate of a real–valued function f
$f^*_c(\cdot)$	the conjugate of a $\hat{\mathcal{C}}$–valued function f

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