Justification of the Nonlinear Schrödinger Equation for Interface Wave Packets in Maxwell's Equations with 2D Localization

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Contents

1. Introduction

Knowledge is the treasure of a wise man.

William Penn

おれの財宝か？
欲しけりゃくれてやるぜ...
探してみろ
この世の全てをそこに置いてきた

ゴール・D・ロジャー

Maxwell's equations, discovered by James Clerk Maxwell in 1865 [53], are one of the most important equations in human history [79, Chapter 11] and describe the connection between the electric field and the magnetic field. The analysis and understanding of these equations are the cornerstone for all modern electronic devices from the radio up to the computer and even after more than 150 years they are still an active field of research [47]. One part of this research is devoted to the construction of nanodevices, i.e. electronic devices with the size of only a couple of nanometers, with the help of electromganetic surface waves, e.g. surface plasmon polaritons (SPPs). These electromagnetic waves propagate at the interface between two media, are strongly localized perpendicular to the interface and are closely linked to nonlinear optical effects [43].

This gives us the main motivation for this thesis. We want to study small localized solutions of Maxwell's equations with a cubic nonlinearity at the boundary surface of two media. Since it is in general not possible to solve such a problem explicitly and the numerical calculation of a solution with standard methods, like the finite element method, is rather difficult and results in very high computational costs, we search for an approximative solution. Our approach is based on the method of amplitude equations and the shape of the approximative wave packet will be determined by localized solutions of a corresponding linear Maxwell problem and a nonlinear Schrödinger equation. Such approximative solutions are commonly used in the physics literature, e.g. [7, 18, 2, 19], but they are often only derived at a formal level. It is well-known that the formal derivation of such approximative solutions can fail to give satisfying approximations over long time intervals [70, 71]. Therefore, it is the goal of this thesis to not only derive a formal approximative solution, but also to rigorously prove its approximation properties and thereby justify the formal approach.

The thesis is structured as follows. After this brief introduction we use Chapter 2 to establish the necessary background from physics, give a brief introduction to the method of amplitude equations and collect some standard mathematical results for the nonlinear Schrödinger equation. We will then study the linear Maxwell problem from an analytical and numerical point of view. In Chapter 4 we formally derive a suitable approximative solution and estimate the residual. The main part of this thesis is then devoted to the rigorous justification of the found approximation. We start in Chapter 5 by adapting a recent local existence result for quasilinear Maxwell's equations from [67] to our concrete problem. In Chapter 6 we then extend this local result to a long time interval by means of an involved bootstrapping argument. Finally, in Chapter 7 we discuss the construction of suitable initial values for our Maxwell problem as required by our technical argument. Throughout the thesis, we will supplement our analytic results with numerical methods and examples.

There is a wide range of publications dedicated to Maxwell's equations, for an introduction we refer for example to [40], [57] and [14] to cover physics, numerics and analysis, respectively. Since we are mainly interested in the analytic point of view, let us give a short overview of the current state of the art.
The well-posedness and local existence of solutions of quasilinear Maxwell's equations has been investigated in [75] and [67] based on the theory of general hyperbolic boundary problems in [29]. These results form the cornerstone of our analysis in Chapter 6. Maxwell's equations with memory but without interface have been considered in [62, 56] within the framework of evolution equations, as well as in [4], where solutions were constructed via analytic power series. An approach to Maxwell's equations with interface based on spectral theory can be found in [13, 11, 12], where Maxwell's equations were transformed to a Schrödinger equation and the long-time behavior of special solutions was studied. Another common approach to Maxwell's equations comes from the study of time-harmonic solutions. In this case, separate equations for the electric and the magnetic field can be derived and individually analyzed, see e.g. [46, 23].
We want to study Maxwell's equations with the method of amplitude equations. This method was already applied to a wide range of different problems, e.g. problems concerning pattern-formations, water waves, Bose–Einstein condensates and nonlinear optics, see [15, 68, 6, 59, 22].
In [72, 49] this method was applied to time dependent Maxwell's equations, but a reduction to a one-dimensional equation was deployed. In [72] a linearly polarized electric field was studied and an approximative wave packet solution was constructed via a complex Ginzburg-Landau equation. In [49] transverse electric modes in photonic crystal waveguides were analyzed and the amplitude equation was given by a nonlinear Schrödinger equation. For 2D photonic crystals the time-harmonic Maxwell's equations were studied in [21, 24] and approximative solutions were constructed via coupled mode equations.

The aim of this thesis is the study of a time dependent 2D Maxwell problem where no reduction to a 1D problem is possible. To be precise, we study transverse magnetic modes at the interface of two materials with instantaneous material response as described by the time dependent, vector valued Maxwell's equations.
Parts of Chapters 3 and 7 are published in [25] as joint work with Dr. Tomáš Dohnal and Dr. Giulio Romani, and parts of Chapters 3 – 6 are published in [27] as joint work with Dr. Tomáš Dohnal and Dr. Roland Schnaubelt.

2. Preliminaries

In this chapter we collect some insightful background information for the overarching topics of this thesis.

From the viewpoint of applications we want to understand electromagnetic surface waves. We therefore study Maxwell's equations with an interface and introduce the reader to this topic.
From a mathematical point of view the method of amplitude equations stands at the center of our analysis. We therefore want to present a short introduction to this method and refer to some problems where this method has been successfully employed and where it fails to give satisfying results. This gives support to the importance of coupling the formal analysis with suitable justification results. Finally, we take a look at the nonlinear Schrödinger equation since it will play a fundamental role in our approximative solution.
But first, we explain the notation used throughout the thesis.

2.1. Notation

For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ (or $\mathbb{C}^n$) the euclidean norm is denoted by $|\cdot|$ and the scalar product as $\boldsymbol{x} \cdot \boldsymbol{y}$. For the corresponding matrix norm in $\mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) we also write $|\cdot|$. For a multi-index $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ we define $|\boldsymbol{\alpha}| := \sum_{j=1}^n \alpha_j$. Note that vectors are written in bold. For function spaces we use $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ for the norm and the scalar product, respectively, and indicate the precise space as a subscript, e.g. $\|\cdot\|_{L^2(\mathbb{R})}$ for the usual norm in the Lebesgue space $L^2(\mathbb{R})$. A collection of all the function spaces used can be found at the end of this thesis. As always we denote the space of bounded continuous functions by C_b, the space of arbitrary smooth functions with compact support by C_c^∞, the Lebesgue spaces with L^p and Sobolev spaces with $W^{m,p}$ and set $H^m := W^{m,2}$.
Since we are interested in an interface problem, we collect some helpful notation related to half-spaces and interfaces. First, we define the half-spaces

$$\mathbb{R}^n_\pm := \left\{ \boldsymbol{x} = (x_1, \dots, x_n)^\top \in \mathbb{R}^n \,\middle|\, \pm x_1 > 0 \right\}.$$

Note that we are mostly interested in the cases $n \in \{1, 2, 3\}$.

For functions we use the superscripts $\pm$ to indicate that the function is defined on the half-spaces $\mathbb{R}^n_\pm$, i.e.

$$f(\boldsymbol{x}) := \begin{cases} f^-(\boldsymbol{x}), & x_1 < 0, \\ f^+(\boldsymbol{x}), & x_1 > 0. \end{cases}$$

Such a function f will often appear in combination with an interface condition at $x_1 = 0$, i.e. the jump at the interface has to satisfy certain conditions.

Definition 2.1.1 (Jump-brackets)
Let $\Gamma_n := \{\boldsymbol{x} \in \mathbb{R}^n \mid x_1 = 0\}$. The jump of a function $f : \mathbb{R}^n \backslash \Gamma_n \to \mathbb{R}$ across Γ_n at the point $\boldsymbol{x} \in \Gamma_n$ will be denoted as

$$[\![f]\!]_{\mathrm{nD}}(\boldsymbol{x}) := \lim_{h \searrow 0} f(\boldsymbol{x} + h\boldsymbol{\nu}(\boldsymbol{x})) - \lim_{h \nearrow 0} f(\boldsymbol{x} + h\boldsymbol{\nu}(\boldsymbol{x})),$$

where $\boldsymbol{\nu}(\boldsymbol{x})$ the unit normal on Γ_n in $\boldsymbol{x}$ into $\mathbb{R}^n_+$.

Furthermore, we define the following function spaces.

Definition 2.1.2 (Function Spaces over $\mathbb{R}^n \backslash \Gamma_n$)
For $1 \leqslant p \leqslant \infty$, $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$ let

$$\begin{aligned} \mathcal{L}^p(\mathbb{R}^n) &:= \left\{ u : \mathbb{R}^n \to \mathbb{R} \,\middle|\, u^+ \in L^p\left(\mathbb{R}^n_+\right),\, u^- \in L^p\left(\mathbb{R}^n_-\right) \right\}, \\ \|u\|_{\mathcal{L}^p(\mathbb{R}^n)} &:= \left\|u^+\right\|_{L^p\left(\mathbb{R}^n_+\right)} + \left\|u^-\right\|_{L^p\left(\mathbb{R}^n_-\right)}, \\ \mathcal{W}^{m,p}(\mathbb{R}^n) &:= \left\{ u \in \mathcal{L}^p(\mathbb{R}^n) \,\middle|\, u^+ \in W^{m,p}\left(\mathbb{R}^n_+\right),\, u^- \in W^{m,p}\left(\mathbb{R}^n_-\right) \right\}, \\ \|u\|_{\mathcal{W}^{m,p}(\mathbb{R}^n)} &:= \left\|u^+\right\|_{W^{m,p}\left(\mathbb{R}^n_+\right)} + \left\|u^-\right\|_{W^{m,p}\left(\mathbb{R}^n_-\right)}. \end{aligned}$$

We also extend the usual notation and use $\mathcal{H}^m := \mathcal{W}^{m,2}$.
For a time interval $J \subset \mathbb{R}$ we analogously define the function spaces for time dependent functions, e.g.

$$\begin{aligned} \mathcal{L}^p(\mathbb{R}^n \times J) &:= \left\{ u : \mathbb{R}^n \times J \to \mathbb{R} \,\middle|\, u^+ \in L^p(\mathbb{R}^n_+ \times J),\, u^- \in L^p\left(\mathbb{R}^n_- \times J\right) \right\}, \\ \|u\|_{\mathcal{L}^p(\mathbb{R}^n \times J)} &:= \left\|u^+\right\|_{L^p\left(\mathbb{R}^n_+ \times J\right)} + \left\|u^-\right\|_{L^p\left(\mathbb{R}^n_- \times J\right)}. \end{aligned}$$

Since both $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ satisfy the cone condition, the standard Sobolev embeddings hold true for $\mathcal{W}^{m,p}(\mathbb{R}^n)$. We will often use that for $mp > n$ and $1 \leqslant p \leqslant q \leqslant \infty$ the embeddings

$$\begin{aligned} W^{j+m,p}\left(\mathbb{R}^n_\pm\right) &\hookrightarrow C^j_{\mathrm{b}}\left(\mathbb{R}^n_\pm\right), \\ W^{m,p}\left(\mathbb{R}^n_\pm\right) &\hookrightarrow L^q\left(\mathbb{R}^n_\pm\right), \end{aligned}$$

hold and that $W^{m,p}\left(\mathbb{R}^n_\pm\right)$ is a Banach algebra, see [1, Theorem 5.4, Corollary 5.16] and [30, Chapter 5].

Remark 2.1.3

Note that $u \in \mathcal{H}^1(\mathbb{R}^n)$ does not imply $u \in H^1(\mathbb{R}^n)$, since the weak derivative $\partial_{x_1} u$ may not exist across Γ_n. We can nevertheless use an arbitrary extension of $\partial_{x_1} u$ via

$$\widetilde{u}(\boldsymbol{x}) = \begin{cases} \partial_{x_1} u(\boldsymbol{x}), & \boldsymbol{x} \notin \Gamma_n, \\ \text{arbitrary}, & \boldsymbol{x} \in \Gamma_n \end{cases}$$

to get at least an $L^2(\mathbb{R}^n)$-function.
Since Γ_n is a set of Lebesgue measure zero in $\mathbb{R}^n$, we can always replace $\mathcal{L}^p(\mathbb{R}^n)$ with $L^p(\mathbb{R}^n)$.

As usual, we use $\partial_x f$ for the partial derivatives, ∇f for the gradient, $\nabla \cdot \boldsymbol{f}$ for the divergence and $\nabla \times \boldsymbol{f}$ for the curl. The convolution of two functions will be denoted by

$$(f * g)(x) := \int_{-\infty}^{\infty} f(\xi) g(x - \xi) \, \mathrm{d}\xi$$

and the Fourier transformation and its inverse are given by

$$\mathcal{F}(f)(k) := \widehat{f}(k) := (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x,$$
$$\mathcal{F}^{-1}(f)(x) := \check{f}(x) := (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(k) \mathrm{e}^{\mathrm{i}xk} \, \mathrm{d}k.$$

The identity operator will be written as I and for an operator L we write $\mathrm{D}(L)$ for the domain, $\mathrm{N}(L)$ for the kernel and $\mathrm{R}(L)$ for the range of L.
Throughout our asymptotic analysis we will use the usual Landau notation $f(\varepsilon) \in \mathcal{O}(\varepsilon)$ for $\varepsilon \to 0$. We will also often use a generic positive constant C that can change its value in every step of a calculation.

2.2. Maxwell's Equations and Electromagnetic Surface Waves

In this section we shortly discuss electromagnetic surface waves since they are the main motivation for the following mathematical analysis. A good introduction to optics and electrodynamics can be found in [37] and [40]. For more comprehensive material on surface waves we refer to [64, 51, 60].

One example for electromagnetic surface waves are surface plasmon polaritons (SPPs). They have many potential applications, for example in biosensors and photonic circuits in nanodevices, see e.g. [51, 66, 43]. Commonly, SPPs exist at the interface between a metal and a dielectric material, but similar electromagnetic surface waves can also be observed for specially constructed materials and structures like photonic crystals, waveguides and photonic metamaterials, see e.g. [85, 49, 74]. Typically, the electromagnetic fields have very small magnitude and decay exponentially at both sides of the interface, this results in highly localized wave packets, see Figure 2.1. Since the electromagnetic fields are described by Maxwell's equations, our main motivation for the rest of this thesis is as follows:

FIND SMALL, LOCALIZED WAVE PACKET SOLUTIONS OF 2D MAXWELL'S EQUATIONS LOCALIZED AT AN INTERFACE BETWEEN TWO MEDIA.

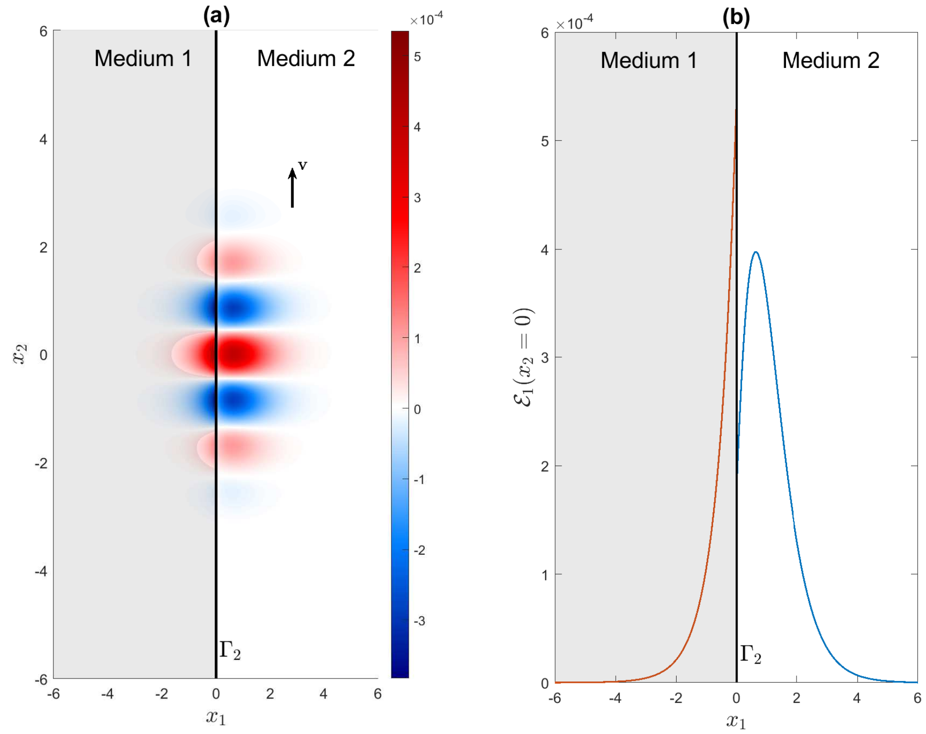

Figure 2.1.: (a) Schematic of pulse propagation in direction $\mathbf{v} = (0,1)^\top$ along an interface. (b) Profile of the $\mathcal{E}_1$ component of the same pulse as in (a) at $x_2 = 0$.

Maxwell's equations in $\mathbb{R}^3$ read as follows:

$$\begin{cases} \partial_t \boldsymbol{\mathcal{D}} = \nabla \times \boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{J}}, \\ \partial_t \boldsymbol{\mathcal{B}} = -\nabla \times \boldsymbol{\mathcal{E}}, \\ \nabla \cdot \boldsymbol{\mathcal{D}} = \varrho, \\ \nabla \cdot \boldsymbol{\mathcal{B}} = 0. \end{cases} \tag{2.2.1}$$

Here $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{J}} : \mathbb{R}^3 \times (0, T') \to \mathbb{R}^3$ are the electric field, the electric displacement field, the magnetic field, the magnetic flux density and the current density, respectively, and $\varrho : \mathbb{R}^3 \times (0, T') \to \mathbb{R}$ is the charge density, for some $T' > 0$.
There are many different constitutive relations that describe $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{B}}$ in dependence of $\boldsymbol{\mathcal{E}}$

and $\boldsymbol{\mathcal{H}}$. We restrict ourselves to the case

$$\boldsymbol{\mathcal{D}}(\boldsymbol{x},t) := \epsilon_0(1+\chi_1(\boldsymbol{x}))\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) + \epsilon_0\chi_3(\boldsymbol{x})(\boldsymbol{\mathcal{E}}(\boldsymbol{x},t)\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{x},t))\boldsymbol{\mathcal{E}}(\boldsymbol{x},t), \tag{2.2.2}$$

$$\boldsymbol{\mathcal{B}}(\boldsymbol{x},t) := \mu_0\boldsymbol{\mathcal{H}}(\boldsymbol{x},t), \tag{2.2.3}$$

where $\epsilon_0 > 0$ is the permittivity of free space, $\mu_0 > 0$ is the permeability of free space and $\chi_1, \chi_3 : \mathbb{R}^3 \to \mathbb{R}$ being the linear and cubic susceptibilities of the media. Note that we assume that the materials are isotropic and that therefore χ_1, χ_3 are scalar quantities. Constitutive relation (2.2.2) is commonly used to model Kerr nonlinear dielectric media. The relation is local in time and contains a linear and a nonlinear part. The second relation (2.2.3) can be used for non-magnetic media where the magnetic permeability is close to the permeability of free space. This is the case for many diamagnetic or paramagnetic materials, see e.g. [40, Chapter 5].

Remark 2.2.1
There are also other models that are interesting for the analysis of SPPs. Especially intriguing are models that are non-local in time, e.g.

$$\widetilde{\boldsymbol{\mathcal{D}}}(\boldsymbol{x},t) = \epsilon_0\left(\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) + \int_{-\infty}^{\infty}\widetilde{\chi}_1(\boldsymbol{x},t-s)\boldsymbol{\mathcal{E}}(\boldsymbol{x},s)\,\mathrm{d}s + \widetilde{\chi}_3(\boldsymbol{x})(\boldsymbol{\mathcal{E}}(\boldsymbol{x},t)\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{x},t))\boldsymbol{\mathcal{E}}(\boldsymbol{x},t)\right).$$

We present the formal steps in the method of amplitude equations for Maxwell's equations with this type of displacement field in Appendix A, but the convolution term causes severe analytical difficulties in the rigorous analysis. In the main part of this thesis we therefore restrict ourselves to the local constitutive relation (2.2.2).
Other interesting models contain more complicated nonlinearities. Higher order nonlinearities or more involved cubic tensors are studied in the literature of nonlinear optics, see e.g. [8]. For example, one could replace the term $\epsilon_0\chi_3(\boldsymbol{x})(\boldsymbol{\mathcal{E}}(\boldsymbol{x},t)\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{x},t))\boldsymbol{\mathcal{E}}(\boldsymbol{x},t)$ *in* (2.2.2) *with the general cubic tensor* $\widetilde{\chi}_3$ *given by*

$$(\widetilde{\chi}_3(x_1,\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{E}}))_j := \sum_{k,l,m=1}^{3}\widetilde{\chi}_{3,jklm}(x_1)\mathcal{E}_k\mathcal{E}_l\mathcal{E}_m. \tag{2.2.4}$$

Our method is in general applicable to more complicated nonlinearities, but a careful analysis of the structure of the nonlinearity would be necessary. Such a structural assumption is discussed in Remark 4.0.1. We point out that certain symmetry properties of the nonlinearity will be needed for the analysis of Remark 6.1.1. Moreover, the construction of the asymptotic solution in Chapter 4 would change drastically since multiple terms in the approximative solution (4.2.1) *and the amplitude equation* (4.1.13) *depend on the structure of the nonlinearity.*
We think that the nonlinearity in (2.2.2) *is suitable to demonstrate many of the different techniques necessary to handle more complicated nonlinearities without hiding the ideas behind a convoluted notation. Therefore,* (2.2.2) *will be the main model used in this thesis.*

Combining the divergence of the first equation and the time-derivative of the third equation in (2.2.1) gives the relation

$$\partial_t \varrho = \partial_t (\nabla \cdot \boldsymbol{\mathcal{D}}) = \nabla \cdot (\partial_t \boldsymbol{\mathcal{D}}) = \nabla \cdot (\nabla \times \boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{J}}) = -\nabla \cdot \boldsymbol{\mathcal{J}}. \tag{2.2.5}$$

This can be interpreted as an equation for charge conservation. We want to study surface waves in the absence of free currents and therefore assume that $\boldsymbol{\mathcal{J}} \equiv 0$. From (2.2.5) it follows that ϱ has to be constant in time.
With these choices Maxwell's equations simplify to

$$\begin{cases} \partial_t \boldsymbol{\mathcal{D}} = \nabla \times \boldsymbol{\mathcal{H}}, \\ \mu_0 \partial_t \boldsymbol{\mathcal{H}} = -\nabla \times \boldsymbol{\mathcal{E}}, \\ \nabla \cdot \boldsymbol{\mathcal{D}} = \varrho_0, \\ \nabla \cdot \boldsymbol{\mathcal{H}} = 0, \end{cases} \tag{2.2.6}$$

with $\varrho_0 : \mathbb{R}^3 \times \mathbb{R}$ the initial charge density at time $t = 0$.
Since we are interested in an interface problem, we have to modify our problem even further. Let us assume that the interface is given by $\Gamma_3 = \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, x_1 = 0\}$. This hyperplane divides $\mathbb{R}^3$ into the two half-spaces $\mathbb{R}^3_\pm = \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, \pm x_1 > 0\}$. We now have to solve Maxwell's equations in these two half-spaces and make sure that certain interface conditions on the interface, that we now discuss, are satisfied.

Interface Conditions for Maxwell's Equations

Maxwell's equations for an interface problem imply under certain regularity assumptions additional interface conditions, see e.g. [31, 17, 40]. To derive them in a formal way we assume that Maxwell's equations in integral form are true:

$$\begin{cases} \partial_t \int_\Sigma \boldsymbol{\mathcal{D}} \cdot \boldsymbol{\nu} \, \mathrm{d}s = \int_{\partial\Sigma} \boldsymbol{\mathcal{H}} \cdot \boldsymbol{\tau} \, \mathrm{d}l, \\ \mu_0 \partial_t \int_\Sigma \boldsymbol{\mathcal{H}} \cdot \boldsymbol{\nu} \, \mathrm{d}s = -\int_{\partial\Sigma} \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\tau} \, \mathrm{d}l, \\ \int_{\partial\Omega} \boldsymbol{\mathcal{D}} \cdot \boldsymbol{\nu} \, \mathrm{d}s = \int_\Omega \varrho_0 \, \mathrm{d}\boldsymbol{x}, \\ \int_{\partial\Omega} \boldsymbol{\mathcal{H}} \cdot \boldsymbol{\nu} \, \mathrm{d}s = 0. \end{cases} \tag{2.2.7}$$

In the interface-free case the integral form can be derived from Gauss's and Stokes's theorems where Ω is an arbitrary (sufficiently regular) volume in $\mathbb{R}^3$, Σ an arbitrary (sufficiently regular) surface in $\mathbb{R}^2$, $\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{x})$ the outward unit normal to Σ or $\partial\Omega$ in $\boldsymbol{x}$ and $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{x})$ is the tangential unit vector to $\partial\Sigma$ in $\boldsymbol{x}$, where the orientation is fixed by the right-hand rule.
We now look at the first equation in (2.2.7). Let C be a curve in the interface Γ_3 and define

the surface

$$\Sigma_\vartheta := \{\boldsymbol{x} + \gamma \boldsymbol{e}_1 \,|\, \boldsymbol{x} \in C,\ \gamma \in (-\vartheta, \vartheta)\}\,,$$

with $\boldsymbol{e}_1 := (1,0,0)^\top$ the unit vector normal to Γ_3. An example for the integration area can be seen in Figure 2.2 (a). The integral $\int_{\Sigma_\vartheta} \boldsymbol{\mathcal{D}} \cdot \boldsymbol{\nu}\, \mathrm{d}s$ is finite and vanishes if ϑ tends to zero. On the other side of the equation we can split $\partial\Sigma_\vartheta$ into three parts $C_+ := \{\boldsymbol{x} + \vartheta\boldsymbol{e}_1 \,|\, \boldsymbol{x} \in C\}$, $C_- := \{\boldsymbol{x} - \vartheta\boldsymbol{e}_1 \,|\, \boldsymbol{x} \in C\}$ and $R_{\Sigma_\vartheta} := \partial\Sigma_\vartheta \backslash (C_- \cup C_+)$ and we get

$$0 = \lim_{\vartheta\to 0}\left(\partial_t \int_{\Sigma_\vartheta} \boldsymbol{\mathcal{D}} \cdot \boldsymbol{\nu}\, \mathrm{d}s\right) = \lim_{\vartheta\to 0}\left(\int_{C_-} \boldsymbol{\mathcal{H}} \cdot \boldsymbol{\tau}\, \mathrm{d}l + \int_{C_+} \boldsymbol{\mathcal{H}} \cdot \boldsymbol{\tau}\, \mathrm{d}l + \int_{R_{\Sigma_\vartheta}} \boldsymbol{\mathcal{H}} \cdot \boldsymbol{\tau}\, \mathrm{d}l\right).$$

For $\vartheta \to 0$ the last integral vanishes and we arrive at

$$0 = \int_C \lim_{\vartheta\to 0}(\boldsymbol{\mathcal{H}}(\boldsymbol{x} + \vartheta\boldsymbol{e}_1) - \boldsymbol{\mathcal{H}}(\boldsymbol{x} - \vartheta\boldsymbol{e}_1)) \cdot \boldsymbol{\tau}\, \mathrm{d}l = \int_C [\![\boldsymbol{\mathcal{H}} \cdot \boldsymbol{\tau}]\!]_{3\mathrm{D}}\, \mathrm{d}l.$$

Since this holds for all curves $C \in \Gamma_3$, the first interface condition is given by $[\![\boldsymbol{\mathcal{H}} \cdot \boldsymbol{\tau}]\!]_{3\mathrm{D}}(\boldsymbol{x}) = \boldsymbol{0}$ for all $\boldsymbol{x} \in \Gamma_3$ and every tangential vector $\boldsymbol{\tau}$, which means that in our case $\mathcal{H}_2$ and $\mathcal{H}_3$ have to be continuous at $x_1 = 0$. The second equation in (2.2.7) can be handled analogously and gives us continuity of $\mathcal{E}_2$ and $\mathcal{E}_3$ at the interface.
The idea for the remaining two Maxwell's equations is similar. For the third equation we choose a surface $S \subset \Gamma_3$, see Figure 2.2 (b), and define the volume

$$\Omega_\vartheta := \{\boldsymbol{x} + \gamma\boldsymbol{e}_1 \,|\, \boldsymbol{x} \in S,\ \gamma \in (-\vartheta, \vartheta)\}.$$

Now we split the boundary of Ω_ϑ in $S_+ := \{\boldsymbol{x} + \vartheta\boldsymbol{e}_1 \,|\, \boldsymbol{x} \in S\}$, $S_- := \{\boldsymbol{x} - \vartheta\boldsymbol{e}_1 \,|\, \boldsymbol{x} \in S\}$ and $R_{\Omega_\vartheta} := \partial\Omega_\vartheta \backslash (S_- \cup S_+)$. For $\vartheta \to 0$ we get

$$\int_S [\![\boldsymbol{\mathcal{D}} \cdot \boldsymbol{\nu}]\!]_{3\mathrm{D}}\, \mathrm{d}s = \int_S \varrho_\Gamma\, \mathrm{d}s,$$

where ϱ_Γ is the surface charge density, i.e. the restriction of ϱ_0 to the interface. We have selected S arbitrarily and therefore the interface condition $[\![\boldsymbol{\mathcal{D}} \cdot \boldsymbol{\nu}]\!]_{3\mathrm{D}}(\boldsymbol{x}) = \varrho_\Gamma(\boldsymbol{x})$ follows, which means that $\mathcal{D}_1$ has a jump at the interface depending on the initial charge density at the interface. Finally, we get in the same way from the last Maxwell equation that $\mathcal{H}_1$ has to be continuous at the interface.
Summarizing, the interface conditions are:

$$\begin{aligned}
[\![\mathcal{D}_1]\!]_{3\mathrm{D}}(\boldsymbol{x}) &= \varrho_\Gamma(\boldsymbol{x}), && \forall\, \boldsymbol{x} \in \Gamma_3,\\
[\![\mathcal{E}_2]\!]_{3\mathrm{D}}(\boldsymbol{x}) &= [\![\mathcal{E}_3]\!]_{3\mathrm{D}}(\boldsymbol{x}) = 0, && \forall\, \boldsymbol{x} \in \Gamma_3,\\
[\![\mathcal{H}_1]\!]_{3\mathrm{D}}(\boldsymbol{x}) &= [\![\mathcal{H}_2]\!]_{3\mathrm{D}}(\boldsymbol{x}) = [\![\mathcal{H}_3]\!]_{3\mathrm{D}}(\boldsymbol{x}) = 0, && \forall\, \boldsymbol{x} \in \Gamma_3.
\end{aligned} \tag{2.2.8}$$

Note that the interface conditions have to be satisfied for all time, so in particular we get conditions on the initial values.

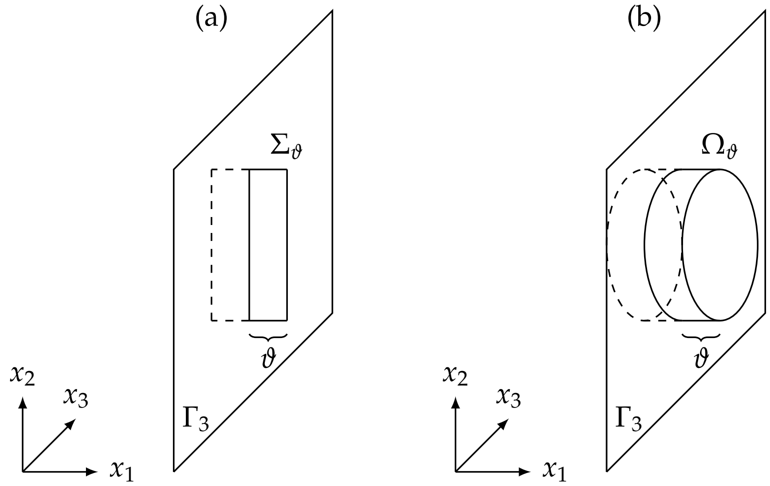

Figure 2.2.: Possible integration areas Σ_ϑ and Ω_ϑ.

Remark 2.2.2
When there are no surface charges, i.e. $\varrho_\Gamma = 0$*, the interface conditions* (2.2.8) *can be rigorously derived in the sense of traces for*

$$\begin{aligned} &\boldsymbol{\mathcal{E}}(\cdot,t) \in H_{\text{curl}}(\mathbb{R}^2) := \left\{ \boldsymbol{f} \in L^2(\mathbb{R}^2)^3 \,\middle|\, \nabla \cdot \boldsymbol{f} \in L^2(\mathbb{R}^2) \right\}, \\ &\boldsymbol{\mathcal{D}}(\cdot,t) \in H_{\text{div}}(\mathbb{R}^2) := \left\{ \boldsymbol{f} \in L^2(\mathbb{R}^2)^3 \,\middle|\, \nabla \times \boldsymbol{f} \in L^2(\mathbb{R}^2)^3 \right\}, \\ &\boldsymbol{\mathcal{H}}(\cdot,t) \in H^1(\mathbb{R}^2)^3, \end{aligned} \tag{2.2.9}$$

see e.g. [10]. Theorem 6.3.1 will give us a more regular solution such that (2.2.9) *is satisfied.*

Finally, we have to model the fact that the materials on the two sides of the interface are different. To simplify the problem, we also assume that the materials are homogeneous in the x_2- and x_3-direction. Therefore, the susceptibilities only depend on x_1 and are given by

$$\chi_1(x_1) := \begin{cases} \chi_1^-(x_1), & x_1 < 0, \\ \chi_1^+(x_1), & x_1 > 0, \end{cases} \qquad \chi_3(x_1) := \begin{cases} \chi_3^-(x_1), & x_1 < 0, \\ \chi_3^+(x_1), & x_1 > 0. \end{cases}$$

Note that χ_1, χ_3 are in general discontinuous at the interface.

Remark 2.2.3
Most of the time we will not specify the value of functions on the interface. Since we are interested in solving an interface problem, we will instead specify the function on both half-planes and add an interface condition to couple the two sides of the interface.

Since the materials are homogeneous in x_2- and x_3-direction and we are interested in wave packet solutions that travel in one constant direction along the interface, we can without loss of generality select the x_2-direction as the direction of propagation. We then assume that the solution is constant in x_3, which allows us to reduce (2.2.6) to the from x_3-independent 2D

Maxwell problem

$$\begin{cases} \partial_t \boldsymbol{\mathcal{D}} = \nabla \times \boldsymbol{\mathcal{H}}, \\ \mu_0 \partial_t \boldsymbol{\mathcal{H}} = -\nabla \times \boldsymbol{\mathcal{E}}, \\ \nabla \cdot \boldsymbol{\mathcal{D}} = \varrho_0, \\ \nabla \cdot \boldsymbol{\mathcal{H}} = 0 \end{cases} \tag{2.2.10}$$

on $\left(\mathbb{R}^2 \backslash \Gamma_2\right) \times (0, T')$, with

$$(\boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}) = (\boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}})(x_1, x_2, t), \qquad \varrho_0 = \varrho_0(x_1, x_2).$$

and the interface conditions

$$\begin{aligned} [\![\mathcal{D}_1]\!]_{2D}(\boldsymbol{x}) &= \varrho_\Gamma(\boldsymbol{x}), \\ [\![\mathcal{E}_2]\!]_{2D}(\boldsymbol{x}) &= [\![\mathcal{E}_3]\!]_{2D}(\boldsymbol{x}) = 0, \\ [\![\mathcal{H}_1]\!]_{2D}(\boldsymbol{x}) &= [\![\mathcal{H}_2]\!]_{2D}(\boldsymbol{x}) = [\![\mathcal{H}_3]\!]_{2D}(\boldsymbol{x}) = 0 \end{aligned} \tag{2.2.11}$$

on $\Gamma_2 \times [0, T')$. From now on we will only work in two spatial dimensions, i.e. $\boldsymbol{x} = (x_1, x_2)^\top$.

2.3. Method of Amplitude Equations

The main goal of this thesis is the derivation and justification of the nonlinear Schrödinger equation as a suitable amplitude equation for Maxwell's equations (2.2.10), (2.2.11). In this section we give a motivation and an introduction to the theory of amplitude equations, in particular we will outline when the theory can be applied and when it fails to give satisfactory results. More details can be found in [55, 84, 28, 71].

There are many problems in all fields of mathematics for which it is difficult or even impossible to calculate exact solutions. Mathematicians have therefore spent a lot of energy on finding methods that make it possible to calculate approximative solutions.
Dealing with partial differential equations, one often looks for solutions in function spaces that have infinite dimension and this causes many difficulties. To overcome these difficulties it is common to discretize the differential equation and to study a corresponding finite-dimensional problem. Most numerical methods for differential equations are based on this approach. However, the fundamental flaw of this approach is the discretization itself. To get a better approximation of the exact solution a finer discretization has to be selected, which results in bigger calculation costs and more memory space needed. For some type of partial differential equation this can get particularly troublesome when different scales are of interest. In optics for example it is important to understand very small wave packets over a long period of time. Without further improvements a very fine discretization in space over a long period of time would be necessary and would result in tremendous numerical costs. Another, more analytic, method is to start with a suitable ansatz to fundamentally change

the nature of the problem in the hope that the new one can be solved more easily. From separation of variables over splitting methods to finite element methods we see that this core concept is widely used. We note that under suitable assumptions the structure of the solution can drastically change the complexity of the problem. This can result in problems that are much easier to study from both an analytical and a numerical point of view. It also allows us to tackle problems with different scales, which is of utmost interest for us.
Perturbation theory, multiple-scale analysis and the theory of amplitude equations are often used to study approximative solutions and are based on the same idea. By introducing a (typically) small parameter $0 < \varepsilon \ll 1$ it is possible to study effects that are on different scales or of different sizes. Let us now illustrate this for a parameter dependent ordinary differential equation. This example can be found in [44, Chapter 12] and [84].

Example 1: The Damped Oscillator

We study approximative solutions of the initial value problem

$$\begin{cases} \partial_t^2 u(t) + 2\varepsilon\partial_t u(t) + u(t) = 0, & t \in (0, T'), \\ u(0) = a, & \\ \partial_t u(0) = 0 & \end{cases} \tag{2.3.1}$$

on the interval $(0, T')$ for $a, T' \in \mathbb{R}$, $T' > 0$. A typical first ansatz for an approximation of u in perturbation theory would be the first terms of a power series in ε:

$$u_{\text{ans},1}(t) = u_0(t) + \varepsilon u_1(t).$$

Inserting this ansatz into (2.3.1) and comparing the powers of ε gives us for the terms proportional to ε^0

$$\begin{cases} \partial_t^2 u_0(t) + u_0(t) = 0, & t \in (0, T'), \\ u_0(0) = a, & \\ \partial_t u_0(0) = 0. & \end{cases} \tag{2.3.2}$$

Note that for $\varepsilon = 0$ Problem (2.3.1) coincides with Problem (2.3.2).
For the terms proportional to ε^1 the comparison gives us

$$\begin{cases} \partial_t^2 u_1(t) + 2\partial_t u_0(t) + u_1(t) = 0, & t \in (0, T'), \\ u_1(0) = 0, & \\ \partial_t u_1(0) = 0. & \end{cases}$$

Since we are only looking for an approximative solution, we ignore the terms of order ε^2 and search for solutions u_0 and u_1.
For the simple Problem (2.3.1) we see interesting effects. On the one hand we note that it is slightly easier to find the solutions $u_0(t) = a\cos(t)$ and $u_1(t) = -at\cos(t) + a\sin(t)$ of the

perturbation problem than it is to find the exact solution

$$u(t) = \mathrm{e}^{-\varepsilon t}\left(a\cos(\omega t) + \varepsilon\frac{a}{\omega}\sin(\omega t)\right),$$

where $\omega = \sqrt{1-\varepsilon^2}$. On the other hand we note that the exponential term $\mathrm{e}^{-\varepsilon t}$ is missing in our approximative solution $u_{\text{ans},1}$ and that therefore our approximation is useless on large time scales, see Figure 2.3 (a). To fix this shortcoming a better ansatz is necessary.
For problems where we suspect a fast oscillating solution with slowly changing amplitude like Problem (2.3.1), which describes a damped oscillator, the theory of amplitude equations can often provide useful approximations. One typically uses an ansatz of the form

$$u_{\text{ans},2}(t) = A(\varepsilon t)\mathrm{e}^{\mathrm{i}\omega_2 t} + \text{c.c.}, \tag{2.3.3}$$

where $\omega_2 \in \mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$, $A : \mathbb{R} \to \mathbb{C}$ and c.c. stands for the complex conjugate of the previous term. We see that the exponential term describes the fast oscillating part and the amplitude is described by the function A that depends on the slow variable $T := \varepsilon t$. Again, comparing the ε-powers for this new ansatz gives us

$$\begin{cases} \left(-\omega_2^2 + 1\right) A(T) + \text{c.c.} = 0, & T \in (0, \varepsilon T') \\ A(0) + \text{c.c.} = a, & \\ -\mathrm{i}\omega_2 A(0) + \text{c.c.} = 0 & \end{cases}$$

and

$$\begin{cases} 2\mathrm{i}\omega_2 \partial_T A(T) + 2\mathrm{i}\omega_2 A(T) = 0, & T \in (0, \varepsilon T'), \\ \partial_T A(0) + \text{c.c.} = 0. & \end{cases} \tag{2.3.4}$$

Note that with the initial condition in (2.3.4) the problem of finding A and ω_2 is overdetermined, hence we have to compromise. We ignore this condition and search for a hopefully very good approximative solution $u_{\text{ans},2}$ of the differential equation that not necessarily satisfies all the initial conditions in (2.3.1) exactly.
For $a \neq 0$ the solution is given by $\omega_2 = 1$ and $A(T) = \frac{a}{2}\mathrm{e}^{-T}$ and we obtain

$$u_{\text{ans},2}(t) = \frac{a}{2}\mathrm{e}^{-\varepsilon t}\mathrm{e}^{\mathrm{i}\omega_2 t} + \frac{a}{2}\mathrm{e}^{-\varepsilon t}\mathrm{e}^{-\mathrm{i}\omega_2 t} = a\mathrm{e}^{-\varepsilon t}\cos(\omega_2 t).$$

In Figure 2.3 (a) and (b) we see that $u_{\text{ans},2}$ is a better approximation than $u_{\text{ans},1}$. Due to the exponential term in both u and $u_{\text{ans},2}$ we see that the error goes exponentially to zero in t. However, a closer look at the error

$$|u(t) - u_{\text{ans},2}(t)| = \mathrm{e}^{-\varepsilon t}\left|\varepsilon a\sin(t) + \frac{1}{2}\varepsilon^2 at\sin(t) + \mathcal{O}(\varepsilon^3)\right|$$

and Figure 2.3 (c) shows us that $u_{\text{ans},2}$ fails to predict the right oscillating behavior of u over very long time periods. Also note that, as expected, $u_{\text{ans},2}(0) = a$ but $\partial_t u_{\text{ans},2}(0) = -\varepsilon a$ and that therefore the initial conditions are only approximately satisfied.

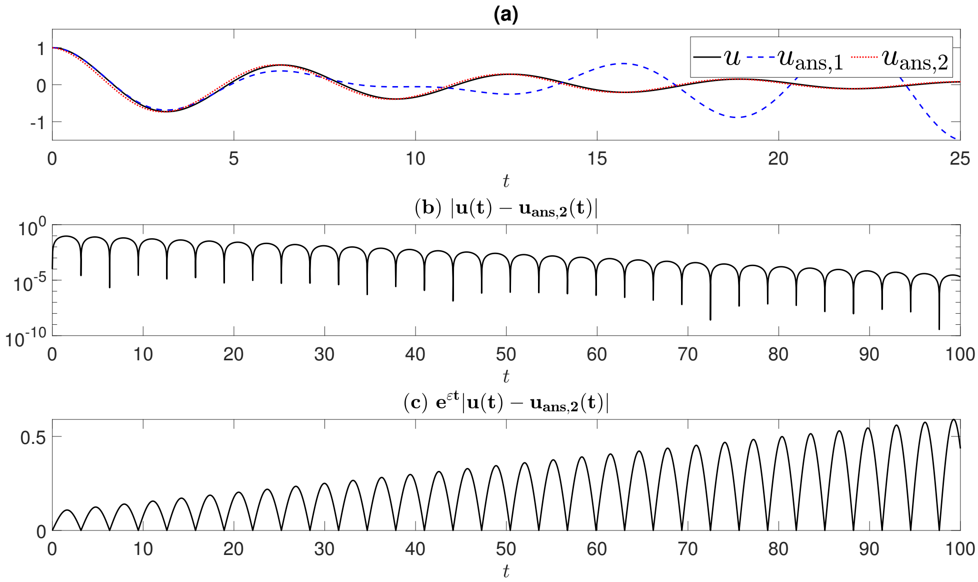

Figure 2.3.: (a) Plot of the exact solution u and the approximative solutions $u_{\text{ans},1}$, $u_{\text{ans},2}$ for $\varepsilon = 0.1$ and $a = 1$.
(b) Plot of the absolute error between $u(t)$ and $u_{\text{ans},2}(t)$ in time.
(c) Absolute error between $e^{\varepsilon t}u(t)$ and $e^{\varepsilon t}u_{\text{ans},2}(t)$ in time.

This short example shows us two fundamental facts. First, we see that an ansatz of the form (2.3.3) can be very successful for certain types of problems. Second, the failure of the first ansatz for long times scales makes it clear that the formal derivation of an approximative solution can fail to provide satisfying results and that a rigorous analysis is necessary to show that an ansatz delivers a sufficiently good approximation.

Example 2: The Cubic Klein-Gordon Equation

Let us now study a slightly more involved example that has many of the interesting features we also have to face when we analyze approximative solutions for Maxwell's equations. This example can be found in [73, Chapter 11].
We consider the one-dimensional (cubic) Klein-Gordon equation

$$\partial_t^2 u - \partial_x^2 u + u + u^3 = 0, \qquad x \in \mathbb{R}, \quad t \in (0, T') \tag{2.3.5}$$

for some $T' > 0$. The key steps of finding an approximative solution with the method of amplitude equations are the following:

Step 1: Solve the linear problem corresponding to (2.3.5).

Step 2: Make an asymptotic wave packet ansatz u_{ans} similar to (2.3.3) and choose A such that the residual

$$\text{Res}(u_{\text{ans}}) := \partial_t^2 u_{\text{ans}} - \partial_x^2 u_{\text{ans}} + u_{\text{ans}} + u_{\text{ans}}^3$$

is as small as possible.

Step 3: Prove that the ansatz provides a "good" approximative solution over a long time interval by means of suitable estimates of the error $u - u_{\text{ans}}$.

Note that A in general depends on space and time and since A determines the shape of the wave packet, it is called envelope of the wave packet. For applications one is often interested in solutions that are localized in space, therefore envelopes A that are localized are especially interesting.
Step 1: To solve the linear problem

$$\partial_t^2 u - \partial_x^2 u + u = 0, \qquad x \in \mathbb{R}, \quad t \in (0, T') \tag{2.3.6}$$

we take the ansatz

$$u_{\text{lin}} := \mathrm{e}^{\mathrm{i}(kx - \omega t)} + \text{c.c.}$$

for the wave number $k \in \mathbb{R}$ and the wave frequency $\omega \in \mathbb{R}$. The linear problem (2.3.6) then reduces to the algebraic problem

$$\omega^2 = k^2 + 1. \tag{2.3.7}$$

Note that we get the same algebraic equation when we apply the Fourier transformation in x and t to (2.3.6). Equation (2.3.7) is called dispersion relation and its solutions can be written as functions in k:

$$\omega(k) = \pm\sqrt{k^2 + 1}.$$

The dispersion relation plays a central role in the construction of suitable approximative solutions. In the sequel, we take a fixed k_0 and the corresponding $\nu_0 := \omega(k_0)$ to build our asymptotic solution around $\mathrm{e}^{\mathrm{i}(k_0 x - \nu_0 t)} + \text{c.c.}$, which solves the linear problem (2.3.6). Also, the derivatives

$$\nu_1 := \partial_k \omega(k_0) = \frac{k_0}{\nu_0}, \qquad \nu_2 := \partial_k^2 \omega(k_0) = \frac{1 - \nu_1^2}{\nu_0}$$

are essential. The so-called group velocity ν_1 determines the speed of the envelope A and the second derivative ν_2 is later used to determine the equation for A. It is easy to see that $\nu_2 \neq 0$, therefore Problem (2.3.6) is called dispersive, which means that waves with different wave numbers travel with different velocities. Hence, the different parts of a localized wave packet travel with different velocities and the wave packet disintegrates over time.

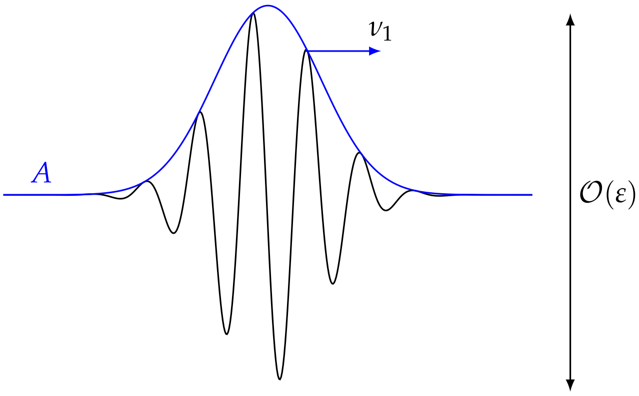

Figure 2.4.: Schematic representation of a small wave packet with localized envelope A traveling with speed ν_1.

For some equations this dispersive effect in the linear part can be mitigated or even canceled by the nonlinearity. Since we are interested in solutions that stay localized over time, we will use a special scaling in our ansatz to achieve a "balance" between the dispersive and nonlinear effects.

Step 2: We make the wave packet ansatz

$$u_{\text{ans}}(x,t) := \varepsilon A\left(\varepsilon(x-\nu_1 t), \varepsilon^2 t\right) \mathrm{e}^{\mathrm{i}(k_0 x - \nu_0 t)} + \text{c.c.}, \tag{2.3.8}$$

where we have $E(x,t) := \mathrm{e}^{\mathrm{i}(k_0 x - \nu_0 t)}$, a solution of (2.3.6), as the carrier wave and an envelope $A = A(X,T) : \mathbb{R} \times (0, \varepsilon^2 T') \to \mathbb{C}$ dependent on the slow variables $X := \varepsilon(x - \nu_1 t)$ and $T := \varepsilon^2 t$, see Figure 2.4 for a schematic representation of u_{ans}.

As before, one inserts (2.3.8) into (2.3.5) and computes the residual:

$$\begin{aligned}
\operatorname{Res}(u_{\text{ans}}) &:= \partial_t^2 u_{\text{ans}} - \partial_x^2 u_{\text{ans}} + u_{\text{ans}} + u_{\text{ans}}^3 \\
&= \varepsilon E\left(\left(k_0^2 - \nu_0^2 + 1\right) A\right) + \varepsilon^2 E(2\mathrm{i}(\nu_1\nu_0 - k_0)\partial_X A) \\
&+ \varepsilon^3 E\left(-2\mathrm{i}\nu_0 \partial_T A + \left(\nu_1^2 - 1\right)\partial_X^2 A + 3|A|^2 A\right) + \varepsilon^3 E^3 A^3 + \mathcal{O}(\varepsilon^4) + \text{c.c.}
\end{aligned}$$

At order ε^1 the terms vanish since k_0 and ν_0 satisfy the dispersion relation (2.3.7). The same happens at order ε^2 because of the choice of ν_1 as the group velocity. The terms which are proportional to $\varepsilon^3 E$ vanish provided A is a solution of the nonlinear Schrödinger equation

$$\mathrm{i}\partial_T A = -\frac{1}{2}\nu_2 \partial_X^2 A + \frac{3}{2\nu_0}|A|^2 A = 0. \tag{2.3.9}$$

One now sees what we meant by "balancing". The effects of the nonlinearity and the dispersion appear at the same order of ε and give us the nonlinear Schrödinger equation, which has localized solutions as we hoped for, see Section 2.4.

Since A^3 does not vanish for a non-trivial envelope, we have to improve our ansatz to eliminate the last term of order ε^3. When we repeat the calculation for

$$u_{\text{ext}}(x,t) := \varepsilon A\,(X,T)\,\mathrm{e}^{\mathrm{i}(k_0 x - \nu_0 t)} + \varepsilon^3 \left(9\nu_0^2 - 9k_0^2 - 1\right)^{-1} A^3\,(X,T)\,\mathrm{e}^{3\mathrm{i}(k_0 x - \nu_0 t)} + \text{c.c.},$$

we formally get $\operatorname{Res}(u_{\text{ext}}) = \mathcal{O}(\varepsilon^4)$. This example gives us interesting insight into the method of amplitude equations.
First, we see that the correction only works when the so-called non-resonance condition

$$9\nu_0^2 \neq 9k_0^2 + 1$$

is satisfied. For the dispersion relation (2.3.7) this condition is obviously fulfilled, for more complicated problems similar conditions appear and have to be checked. In particular, we see that this method does not work for a linear dispersion relation.
Second, the correction term in u_{ext} is of higher-order and its main purpose is to reduce the order of the residual. For our Maxwell problem we will use multiple corrections of higher-order to achieve the same goal. Often it is even possible to add more and more correction terms to make the residual arbitrarily small, see [73, Chapter 11].
Step 3: To understand what we mean by a "good" approximative solution, let us state the approximation result:

Theorem 2.3.1 (Approximation Theorem for the Klein-Gordon Equation)
Let $T_0 > 0$ and $A \in C\left([0,T_0], H^5(\mathbb{R})\right)$ be a solution of the nonlinear Schrödinger equation (2.3.9). *Then there exists an $\varepsilon_0 > 0$ and a constant $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there are solutions u of* (2.3.5) *such that for all $t \in \left[0, T_0\varepsilon^{-2}\right]$*

$$\left\|u(t) - u_{\text{ans}}(t)\right\|_{H^1(\mathbb{R})} \leqslant C\varepsilon^{3/2}. \tag{2.3.10}$$

PROOF: An extended version of this theorem and its proof can be found as Theorem 11.2.6 in [73]. Since some of the steps are similar to our approach for the Maxwell problem, we now describe the main steps.
First, we estimate the residual in the corresponding norm, here $H^1(\mathbb{R})$. This can be done easily since the residual is known explicitly and we assumed a highly regular envelope A.
Second, the problem is rewritten for the error

$$\varepsilon^{3/2} R := u - u_{\text{ans}}$$

and one aims to show that R is of size $\mathcal{O}(1)$ over the long time interval $\left[0, \varepsilon^{-2}T_0\right]$. This step contains the main difficulty of the proof. In [73] semigroup theory in combination with Gronwall's inequality is employed for this part. For the Maxwell problem we will use an existing local well-posedness result in combination with a bootstrapping argument and again Gronwall's inequality. □

A "good" approximation as in (2.3.10) has several important properties. The estimate has to be done in a suitable normed space. This space is often tied to the local existence and uniqueness of u. The constant C has to be independent of t and ε. Since the envelope A depends on the slow time variable $T = \varepsilon^2 t$, the corresponding long time interval has to be $[0, T_0\varepsilon^{-2}]$. Since

$$\begin{aligned}
\|u_{\text{ans}}(t)\|_{H^1(\mathbb{R})} &\leqslant C\varepsilon \left(\int_{\mathbb{R}} \left(\left| A\left(\varepsilon(x - \nu_1 t), \varepsilon^2 t\right)\right|^2 + \left|\partial_x A\left(\varepsilon(x - \nu_1 t), \varepsilon^2 t\right)\right|^2 \right) \mathrm{d}x \right)^{1/2} \\
&\leqslant C\varepsilon \left(\varepsilon^{-1} \int_{\mathbb{R}} \left(|A(X, \varepsilon^2 t)|^2 + \varepsilon^2 \left|\partial_X A(X, \varepsilon^2 t)\right|^2 \right) \mathrm{d}X \right)^{1/2} \\
&\leqslant C\varepsilon^{1/2} \left\| A(\cdot, \varepsilon^2 t)\right\|_{H^1(\mathbb{R})},
\end{aligned}$$

we see that the error of our approximation is much smaller than the size of our ansatz. This is fundamental in order to obtain a meaningful approximation. From the assumptions of Theorem 2.3.1 we can gather additional information about the method of amplitude equations. First, most approximations only work for ε small enough. On the one hand, this limits the possible applications since only very small solutions u can be approximated. On the other hand, we see that the approximation improves for smaller ε. Second, a high regularity of the envelope is often necessary to achieve the approximation result, but this causes no problem since the nonlinear Schrödinger equation allows for highly regular solutions, see Section 2.4.

Counterexamples

We have already seen in Example 1 that even for simple problems an asymptotic method can fail to predict the correct behavior of the exact solution. For the method of amplitude equation it was shown in [70] and [69] that the formal derivation of an amplitude equation is in general not sufficient. A rigorous proof of the approximation properties is therefore necessary and forms the main part of this thesis.

After a brief discussion of the nonlinear Schrödinger equation, see Section 2.4, the rest of the thesis is devoted to the application of the method of amplitude equations to the Maxwell problem (2.2.10), (2.2.11) and the proof of a result similar to Theorem 2.3.1. We will proceed as follows:

Step 1: In Chapter 3 we solve a linear problem corresponding to (2.2.10), (2.2.11).

Step 2: In Chapter 4 we construct a wave packet ansatz $\boldsymbol{U}_{\text{ans}}$ similar to (2.3.8) and choose a suitable A such that the residual is as small as possible.

Step 3: In Chapter 6 we prove that the ansatz provides a "good" approximative solution over a long time interval.

2.4. Nonlinear Schrödinger Equation

The nonlinear Schrödinger (NLS) equation plays an essential role in our construction of an asymptotic solution. In this section we therefore collect some known properties of the NLS equation. More information can be found in [80, 82, 73, 2]. That the NLS equation can be used to describe the behavior of wave packets in dispersive problems was for example shown in [41, 45]

The (cubic) nonlinear Schrödinger equation is given by

$$\mathrm{i}\partial_T A = \alpha_1 \partial_X^2 A + \alpha_2 |A|^2 A, \tag{2.4.1}$$

with $\alpha_1, \alpha_2 \in \mathbb{R}$ and $A : \mathbb{R} \times (0, T_0) \to \mathbb{C}$ for some $T_0 > 0$.

Theorem 2.4.1 (Existence and Regularity of Solutions of the NLS Equation)
Let $A^{(0)} \in H^m(\mathbb{R})$.

i) Let $m > \frac{1}{2}$. Then there exists a $T_0 > 0$ and a unique solution $A \in C([0, T_0], H^m(\mathbb{R}))$ of (2.4.1) *with $A(\cdot, 0) = A^{(0)}$.*

ii) Let $m \geqslant 2k$ for a positive $k \in \mathbb{N}$. Then $A \in \bigcap_{j=0}^{k} C^j\left([0, T_0], H^{m-2j}(\mathbb{R})\right)$.

PROOF: Part i) and its proof can be found in [73, Theorem 8.1.4] and in [82, Proposition 3.8]. The proof is based on semigroup theory and a fixed-point argument.
To prove ii) one can now use (2.4.1) to "trade" regularity in space for regularity in time. For $m \geqslant 2$ we have that $\alpha_1 \partial_X^2 A + \alpha_2 |A|^2 A \in C\left([0, T_0], H^{m-2}(\mathbb{R})\right)$ and with (2.4.1) it follows that $A \in C^1\left([0, T_0], H^{m-2}(\mathbb{R})\right)$. This process can now be iterated and proves the assertion.

□

Remark 2.4.2
Theorem 2.4.1 allows us to find arbitrary smooth solutions of the nonlinear Schrödinger equation as long as the initial value $A^{(0)}$ is smooth enough. For our approximative solution of Maxwell's equations $A \in \bigcap_{k=0}^{4} C^{4-k}\left([0, T_0], H^{3+k}(\mathbb{R})\right)$ will be necessary.

It is even possible to find localized, smooth solutions of (2.4.1) explicitly, see e.g. [73]. For $\tilde{\eta}, c, \gamma, x_0 \in \mathbb{R}$ the function

$$A(X, T) = \sqrt{2}\,\tilde{\eta}\,\mathrm{sech}(\tilde{\eta}(X - x_0 - cT))\mathrm{e}^{\mathrm{i}((c^2 - 4\tilde{\eta}^2)T - 2cX + \gamma)/4}$$

solves the NLS equation for $\alpha_1 = \alpha_2 = 1$. After a variable transformation one gets the general solution

$$A(X, T) = \beta_1^{-1}\sqrt{2}\,\tilde{\eta}\,\mathrm{sech}\left(\tilde{\eta}(\beta_2 X - x_0 - c\beta_3 T)\right)\mathrm{e}^{\mathrm{i}\left((c^2 - 4\tilde{\eta}^2)\beta_3 T - 2c\beta_2 X + \gamma\right)/4}, \tag{2.4.2}$$

with $\beta_1 = \beta_3 = \alpha_2^{1/3}$ and $\beta_2 = \left(\alpha_1 \alpha_2^{-1/3}\right)^{-1/2}$.

When no explicit solution is known, one can solve (2.4.1) numerically, see e.g. [81] for an overview of different methods. One could for example use a split-step Fourier method as follows:

1. Split (2.4.1) into the two problems

$$\mathrm{i}\partial_T A_F = \alpha_1 \partial_X^2 A_F, \qquad A_F(X, t_0) = A_{t_0}(X), \tag{2.4.3}$$

$$\mathrm{i}\partial_T A_N = \alpha_2 |A_N|^2 A_N, \qquad A_N(X, t_1) = A_{t_1}(X). \tag{2.4.4}$$

2. Calculate the Fourier transform of (2.4.3), construct the explicit solution

$$\widehat{A}_F(K, T) = \mathrm{e}^{\mathrm{i}\alpha_1 K^2 T} \widehat{A}_{t_0}(K)$$

   and apply the inverse Fourier transform.

3. Explicitly solve (2.4.4) with

$$A_N(X, T) = \mathrm{e}^{-\mathrm{i}\alpha_2 |A_{t_1}(X)|^2 T} A_{t_1}(X).$$

4. Solve the two problems alternately for small step sizes in time h_t and updated initial values. One could for example use the Strang splitting, this method goes as follows:
   a) Solve (2.4.3) with step size $\frac{1}{2}h_t$ and initial value $A_{t_0}(X)$;
   b) Solve (2.4.4) with step size h_t and initial $A_F\left(X, t_0 + \frac{1}{2}h_t\right)$;
   c) Solve (2.4.3) with step size $\frac{1}{2}h_t$ and initial value $A_N(X, t_0 + h_t)$.

The Matlab code for this method can be found in [2, Appendix B], for more about splitting methods see e.g. [35, Chapter II.5].

Remark 2.4.3
The nonlinear Schrödinger equation appears in the analysis of many different problems, e.g. Bose-Einstein condensates, nonlinear optics and water waves. It is possible to show that the NLS equation appears as the universal amplitude equation in the asymptotic analysis of wave packets for a large class of nonlinear dispersive equations, see e.g [73, Chapter 11*].*

3. Linear Problem

In this chapter we study Maxwell's equations (2.2.10) in the linear form, i.e. $\epsilon_3 := \epsilon_0\chi_3 \equiv 0$, both analytically and numerically. We will also analyze the corresponding inhomogeneous problem.

3.1. Linear Transverse Magnetic Modes

We want to find out what kind of localized solutions the linear 2D Maxwell equations possess and if they satisfy the interface conditions (2.2.11). As discussed in Section 2.3, we will use the solution of the linear Maxwell problem to construct an asymptotic solution of the nonlinear case. We start with a standard wave-ansatz for the electromagnetic field, see e.g. [64].

For $k \in \mathbb{R}$ and $\omega \in \mathbb{R}$ we are looking for solutions of the form

$$\boldsymbol{\mathcal{E}}(x_1, x_2, t) = \begin{pmatrix} \phi_1(x_1) \\ \phi_2(x_1) \\ \phi_3(x_1) \end{pmatrix} \mathrm{e}^{\mathrm{i}(kx_2 - \omega t)} + \text{c.c.}, \quad \boldsymbol{\mathcal{H}}(x_1, x_2, t) = \begin{pmatrix} \psi_1(x_1) \\ \psi_2(x_1) \\ \psi_3(x_1) \end{pmatrix} \mathrm{e}^{\mathrm{i}(kx_2 - \omega t)} + \text{c.c.}, \tag{3.1.1}$$

where we expect localized, integrable functions $\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \psi_3 : \mathbb{R} \to \mathbb{C}$.

Remark 3.1.1
We see that this ansatz has the main features we expect from our wave solution. For a fixed wave number k and a frequency ω our ansatz describes a wave traveling in x_2-direction that is localized at the interface Γ_2.

The linear part of the displacement field (2.2.2) is given by

$$\boldsymbol{\mathcal{D}}_{\text{lin}}(\boldsymbol{\mathcal{E}}) := \epsilon_1 \boldsymbol{\mathcal{E}},$$

with the abbreviation

$$\epsilon_1(x_1) := \begin{cases} \epsilon_1^-(x_1) := \epsilon_0\left(1 + \chi_1^-(x_1)\right), & x_1 < 0, \\ \epsilon_1^+(x_1) := \epsilon_0\left(1 + \chi_1^+(x_1)\right), & x_1 > 0. \end{cases}$$

The linear version of Maxwell's equations, i.e. (2.2.10) with the linear displacement field

$\boldsymbol{\mathcal{D}}_{\text{lin}}$, with $\varrho_0 \equiv 0$ and the ansatz (3.1.1) take the form

$$\begin{cases} \mathrm{i}\mu_0\omega\psi_1 = \mathrm{i}k\phi_3, & (3.1.2\text{a}) \qquad -\mathrm{i}\epsilon_1(x_1)\omega\phi_1 = \mathrm{i}k\psi_3, \quad (3.1.2\text{b}) \\ \mathrm{i}\mu_0\omega\psi_2 = -\partial_{x_1}\phi_3, & (3.1.2\text{c}) \qquad -\mathrm{i}\epsilon_1(x_1)\omega\phi_2 = -\partial_{x_1}\psi_3, \quad (3.1.2\text{d}) \\ \mathrm{i}\mu_0\omega\psi_3 = \partial_{x_1}\phi_2 - \mathrm{i}k\phi_1, & (3.1.2\text{e}) \qquad -\mathrm{i}\epsilon_1(x_1)\omega\phi_3 = \partial_{x_1}\psi_2 - \mathrm{i}k\psi_1 \quad (3.1.2\text{f}) \end{cases}$$

and

$$\begin{cases} \partial_{x_1}\epsilon_1(x_1)\phi_1 + \epsilon_1(x_1)(\partial_{x_1}\phi_1 + \mathrm{i}k\phi_2) = 0, & (3.1.3\text{a}) \\ \partial_{x_1}\psi_1 + \mathrm{i}k\psi_2 = 0, & (3.1.3\text{b}) \end{cases}$$

with the interface conditions

$$[\![\epsilon_1\phi_1]\!]_{1\mathrm{D}}(0) = [\![\phi_2]\!]_{1\mathrm{D}}(0) = [\![\phi_3]\!]_{1\mathrm{D}}(0) = [\![\psi_1]\!]_{1\mathrm{D}}(0) = [\![\psi_2]\!]_{1\mathrm{D}}(0) = [\![\psi_3]\!]_{1\mathrm{D}}(0) = 0.$$

Remark 3.1.2

Note that formally one also gets the complex conjugate versions of the eight equations (3.1.2a)–(3.1.3b), *e.g.* $\mu_0\partial_t\mathcal{H}_1 = -\partial_{x_2}\mathcal{E}_1$ *gives us the equation*

$$\mu_0\omega(\mathrm{i}\psi_1)\mathrm{e}^{\mathrm{i}(kx_2-\omega t)} + \mu_0\omega\overline{(\mathrm{i}\psi_1)}\mathrm{e}^{-\mathrm{i}(kx_2-\omega t)} = k(\mathrm{i}\phi_3)\mathrm{e}^{\mathrm{i}(kx_2-\omega t)} + k\overline{(\mathrm{i}\phi_3)}\mathrm{e}^{-\mathrm{i}(kx_2-\omega t)}.$$

Now a coefficient comparison of the exponential functions $\mathrm{e}^{\pm\mathrm{i}(kx_2-\omega t)}$ *gives us equation* (3.1.2a) *and its complex conjugate counterpart.*
These complex conjugate equations are trivially satisfied when (3.1.2a)–(3.1.3b) *are fulfilled.*

We now note that the problem decouples in two independent systems. Equations (3.1.2a), (3.1.2c), (3.1.2f) and (3.1.3b) only depend on ϕ_3, ψ_1, ψ_2 whereas the remaining equations only depend on ϕ_1, ϕ_2, ψ_3. We restrict ourselves to solutions where $\psi_1 = \psi_2 = \phi_3 = 0$. This corresponds to

$$\boldsymbol{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, 0)^\top \quad \text{and} \quad \boldsymbol{\mathcal{H}} = (0, 0, \mathcal{H}_3)^\top,$$

such solutions are often referred to as transverse magnetic modes (TM-modes).

Remark 3.1.3

Our restriction to TM-modes is motivated by two facts. First, we think that the TM-modes are of bigger mathematical interest. The other case where $\phi_1 = \phi_2 = \psi_3 = 0$, *so-called transverse electric mode, was already studied in [49] for the 2D setting with periodic material functions. There, a reduction to a scalar problem was possible. For the TM-modes such a restriction is not possible and we have to develop new techniques to handle this problem.*
Second, for the simple setting that ϵ_1 *is constant on both sides of the interface, only TM-modes are possible as we will show in the following example. A first analysis of SPPs also shows that SPPs are only possible for TM-modes, see e.g. [64, 51]. We therefore think that TM-modes are the "natural" solutions of our problem.*

Example 3.1.4 (Explicit Solution for the Linear Maxwell Problem)
Assume that ϵ_1^+ *and* ϵ_1^- *are constant, non-zero and satisfy* $\epsilon_1^+\epsilon_1^- < 0$.
In this special case we can solve system (3.1.2a)–(3.1.3b) *for* $x_1 > 0$ *and for* $x_1 < 0$ *explicitly since we only have to solve a system of ordinary differential equations with constant coefficients. For* $\omega \neq 0$ *we select the integrable solutions*

$$\left.\begin{aligned}\phi_1^-(x_1) &= -\frac{k}{\omega\epsilon_1^-}\,\psi_3^-(x_1),\\ \phi_2^-(x_1) &= C_1\sqrt{a^-}\,\mathrm{e}^{-\mathrm{i}\sqrt{c^-}\,x_1},\\ \phi_3^-(x_1) &= -\mathrm{i}C_2\sqrt{\mu_0\omega}\,\mathrm{e}^{-\mathrm{i}\sqrt{c^-}\,x_1},\\ \psi_1^-(x_1) &= \frac{k}{\mu_0\omega}\,\phi_3^-(x_1),\\ \psi_2^-(x_1) &= C_2\sqrt{b^-}\,\mathrm{e}^{-\mathrm{i}\sqrt{c^-}\,x_1},\\ \psi_3^-(x_1) &= -C_1\sqrt{\epsilon_1^-\omega}\,\mathrm{e}^{-\mathrm{i}\sqrt{c^-}\,x_1},\end{aligned}\right\}\ x_1<0,\qquad \left.\begin{aligned}\phi_1^+(x_1) &= -\frac{k}{\omega\epsilon_1^+}\,\psi_3^+(x_1),\\ \phi_2^+(x_1) &= C_3\sqrt{a^+}\,\mathrm{e}^{\mathrm{i}\sqrt{c^+}\,x_1},\\ \phi_3^+(x_1) &= \mathrm{i}C_4\sqrt{\mu_0\omega}\,\mathrm{e}^{\mathrm{i}\sqrt{c^+}\,x_1},\\ \psi_1^+(x_1) &= \frac{k}{\mu_0\omega}\,\phi_3^+(x_1),\\ \psi_2^+(x_1) &= C_4\sqrt{b^+}\,\mathrm{e}^{\mathrm{i}\sqrt{c^+}\,x_1},\\ \psi_3^+(x_1) &= C_3\sqrt{\epsilon_1^+\omega}\,\mathrm{e}^{\mathrm{i}\sqrt{c^+}\,x_1},\end{aligned}\right\}\ x_1>0, \tag{3.1.4}$$

with

$$a(x_1) := \begin{cases} a^- := \mu_0\omega - \frac{k^2}{\epsilon_1^-\omega}, & x_1<0,\\ a^+ := \mu_0\omega - \frac{k^2}{\epsilon_1^+\omega}, & x_1>0,\end{cases}\qquad b(x_1) := \begin{cases} b^- := -\epsilon_1^-\omega + \frac{k^2}{\mu_0\omega}, & x_1<0,\\ b^+ := -\epsilon_1^+\omega + \frac{k^2}{\mu_0\omega}, & x_1>0,\end{cases}$$

$$c(x_1) := \begin{cases} c^- := \mu_0\epsilon_1^-\omega^2 - k^2 = a^-\epsilon_1^-\omega = -b^-\mu_0\omega, & x_1<0,\\ c^+ := \mu_0\epsilon_1^+\omega^2 - k^2 = a^+\epsilon_1^+\omega = -b^+\mu_0\omega, & x_1>0.\end{cases}$$

For integrable solutions we have to check that we only allow such k and ω *for which we have*

$$\operatorname{Re}\left(-\mathrm{i}\sqrt{c^-}\right),\ \operatorname{Re}\left(-\mathrm{i}\sqrt{c^+}\right) > 0. \tag{3.1.5}$$

This is always possible when $\mu_0\epsilon_1^\pm\omega^2 - k^2 = c^\pm < 0$. *Here we use the usual convention that* $\sqrt{-x} = \mathrm{i}\sqrt{x}$ *for positive x.*
To satisfy the interface conditions, ϕ_2, ϕ_3, ψ_2 *and* ψ_3 *have to be continuous at* $x_1 = 0$. *This gives us* $\sqrt{a^-}\,C_1 = \sqrt{a^+}\,C_3$, $-C_2 = C_4$, $\sqrt{b^-}\,C_2 = \sqrt{b^+}\,C_4$ *and* $-\sqrt{\epsilon_1^-}\,C_1 = \sqrt{\epsilon_1^+}\,C_3$. *Note that the continuity of* $\epsilon_1\phi_1$ *and* ψ_1 *follows from the continuity of* ψ_3 *and* ϕ_3, *respectively.*
These equations imply

$$\begin{cases} -\dfrac{\sqrt{a^+}}{\sqrt{\epsilon_1^+}}\,C_1 = \dfrac{\sqrt{a^-}}{\sqrt{\epsilon_1^-}}\,C_1,\\ -\sqrt{b^+}\,C_2 = \sqrt{b^-}\,C_2.\end{cases} \tag{3.1.6}$$

From the first equation in (3.1.6) *we get for* $C_1 \neq 0$

$$\begin{aligned}
\frac{a^+}{\epsilon_1^+} = \frac{a^-}{\epsilon_1^-} \quad &\Longleftrightarrow \quad \frac{\mu_0\omega^2\epsilon_1^+ - k^2}{\left(\epsilon_1^+\right)^2\omega} = \frac{\mu_0\omega^2\epsilon_1^- - k^2}{\left(\epsilon_1^-\right)^2\omega} \\
&\Longleftrightarrow \quad k^2 = \omega^2\mu_0\frac{\left(\epsilon_1^+\right)^2\epsilon_1^- - \left(\epsilon_1^-\right)^2\epsilon_1^+}{\left(\epsilon_1^+\right)^2 - \left(\epsilon_1^-\right)^2} \\
&\Longleftrightarrow \quad k^2 = \omega^2\mu_0\frac{\epsilon_1^+\epsilon_1^-}{\epsilon_1^+ + \epsilon_1^-} \\
&\Longleftrightarrow \quad \omega^2 = k^2\frac{\epsilon_1^+ + \epsilon_1^-}{\mu_0\epsilon_1^+\epsilon_1^-}.
\end{aligned} \tag{3.1.7}$$

The last equation gives us an explicit relation between k *and* ω *for which integrable solutions can exist, the so-called dispersion relation.*
With this dispersion relation we can go back and check if (3.1.5) *is satisfied:*

$$\begin{aligned}
-\mathrm{i}\sqrt{c^-} &= -\mathrm{i}\sqrt{\mu_0\epsilon_1^+\omega^2 - k^2} = -\mathrm{i}\sqrt{k^2\frac{\epsilon_1^-}{\epsilon_1^+}}, \\
-\mathrm{i}\sqrt{c^+} &= -\mathrm{i}\sqrt{\mu_0\epsilon_1^-\omega^2 - k^2} = -\mathrm{i}\sqrt{k^2\frac{\epsilon_1^+}{\epsilon_1^-}}.
\end{aligned}$$

Therefore, a non-trivial integrable solution exists for all $k \neq 0$ *since* $\epsilon_1^+\epsilon_1^- < 0$ *per assumption.*
From the second equation in (3.1.6) *we get for* $C_2 \neq 0$

$$\begin{aligned}
b^+ = b^- \quad &\Longleftrightarrow \quad -\epsilon_1^+\omega + \frac{k^2}{\mu_0\omega} = -\epsilon_1^-\omega + \frac{k^2}{\mu_0\omega} \\
&\Longleftrightarrow \quad \epsilon_1^+ = \epsilon_1^-,
\end{aligned}$$

which cannot be true for our discontinuous ϵ_1 *and therefore* C_2 *has to be zero, hence the components* ϕ_3, ψ_1 *and* ψ_2 *have to vanish.*
In conclusion, the most general solution has the form $\boldsymbol{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, 0)^\top$ *and* $\boldsymbol{\mathcal{H}} = (0, 0, \mathcal{H}_3)^\top$. *This means that TM-modes are the only possible solutions for this special* ϵ_1.
Note that the dispersion relation (3.1.7) *gives a linear dependence between* $|\omega|$ *and* $|k|$. *Since we are mainly interested in dispersive equations where* $\partial_k^2\omega$ *has to be non-trivial, see Chapter 4, we have to study more complicated* ϵ_1.

For the TM-setting the number of equations in the Maxwell problem reduces to four:

$$\begin{cases}
-\mathrm{i}\epsilon_1(x_1)\omega\phi_1 = \mathrm{i}k\psi_3, \\
-\mathrm{i}\epsilon_1(x_1)\omega\phi_2 = -\partial_{x_1}\psi_3, \\
\mathrm{i}\mu_0\omega\psi_3 = \partial_{x_1}\phi_2 - \mathrm{i}k\phi_1, \\
0 = \partial_{x_1}\epsilon_1(x_1)\phi_1 + \epsilon_1(x_1)(\partial_{x_1}\phi_1 + \mathrm{i}k\phi_2),
\end{cases} \tag{3.1.8}$$

with three interface conditions

$$[\![\epsilon_1\phi_1]\!]_{1\mathrm{D}}(0) = [\![\phi_2]\!]_{1\mathrm{D}}(0) = [\![\psi_3]\!]_{1\mathrm{D}}(0) = 0.$$

To solve (3.1.8) for a more general ϵ_1 we first note that the last equation is automatically satisfied when the other three equations are fulfilled and $\omega \neq 0$. To see this, we differentiate the first equation of (3.1.8) in x_1 and replace $\partial_{x_1}\psi_3$ with the help of the second equation in (3.1.8):

$$\begin{aligned} & -\mathrm{i}\partial_{x_1}\epsilon_1(x_1)\omega\phi_1 - \mathrm{i}\epsilon_1(x_1)\omega\partial_{x_1}\phi_1 = \mathrm{i}k\partial_{x_1}\psi_3 \\ \iff & -\mathrm{i}\partial_{x_1}\epsilon_1(x_1)\omega\phi_1 - \mathrm{i}\epsilon_1(x_1)\omega\partial_{x_1}\phi_1 = \mathrm{i}k(\mathrm{i}\epsilon_1(x_1)\omega\phi_2) \\ \iff & \partial_{x_1}\epsilon_1(x_1)\phi_1 + \epsilon_1(x_1)\partial_{x_1}\phi_1 = -\mathrm{i}\epsilon_1(x_1)k\phi_2. \end{aligned} \tag{3.1.9}$$

Additionally, we notice that from the first equation in (3.1.8) and $[\![\psi_3]\!]_{1\mathrm{D}}(0) = 0$ it follows that $[\![\epsilon_1\phi_1]\!]_{1\mathrm{D}}(0) = 0$.

Remark 3.1.5
Note that (3.1.9) *implies that ansatz* (3.1.1) *can only solve the linear Maxwell equations for* $\varrho_0 \equiv 0$.

All in all, the TM-setting reduces the linear Maxwell equations to

$$\begin{cases} L(k)\boldsymbol{w} + \omega\Lambda(x_1)\boldsymbol{w} = \boldsymbol{0} \qquad \text{in} \quad \mathbb{R}\backslash\{0\}, \\ [\![w_2]\!]_{1\mathrm{D}}(0) = [\![w_3]\!]_{1\mathrm{D}}(0) = 0, \end{cases} \tag{3.1.10}$$

where $\boldsymbol{w} := (\phi_1, \phi_2, \psi_3)^\top$ and

$$L(k)\boldsymbol{w} := \begin{pmatrix} kw_3 \\ \mathrm{i}\partial_{x_1}w_3 \\ kw_1 + \mathrm{i}\partial_{x_1}w_2 \end{pmatrix}, \qquad \Lambda(x_1)\boldsymbol{w} := \begin{pmatrix} \epsilon_1(x_1)w_1 \\ \epsilon_1(x_1)w_2 \\ \mu_0 w_3 \end{pmatrix}. \tag{3.1.11}$$

3.2. Analysis of the Linear Eigenvalue Problem

In the last section we derived (3.1.10) in a formal way. In this section we will rigorously study the properties of (3.1.10).

We assume that ϵ_1 is sufficiently smooth on $\mathbb{R}\backslash\{0\}$ and bounded from below by a positive constant, i.e. there are constants $\epsilon_{1,m}^{\pm} > 0$ such that

$$\epsilon_1^{\pm} \in C^3(\mathbb{R}_\pm) \cap W^{3,\infty}(\mathbb{R}_\pm), \quad \epsilon_1^{\pm}(x_1) \geqslant \epsilon_{1,m}^{\pm}, \qquad \forall\, x_1 \in \mathbb{R}\backslash\{0\}. \tag{A1}$$

For $k \in \mathbb{R}$ we define the operators $L(k) : \mathrm{D}(L(k)) \to L^2(\mathbb{R})^3$ and $\Lambda : \mathrm{D}(\Lambda) \to L^2(\mathbb{R})^3$ via (3.1.11) and with the domains

$$\begin{aligned} \mathrm{D}(L(k)) &:= \left\{ \boldsymbol{w} : \mathbb{R} \to \mathbb{C}^3 \,\middle|\, w_1 \in L^2(\mathbb{R}),\, w_2, w_3 \in H^1(\mathbb{R}) \right\}, \\ \mathrm{D}(\Lambda) &:= L^2(\mathbb{R})^3. \end{aligned} \tag{3.2.1}$$

Note that the domains are dense in $L^2(\mathbb{R})^3$ and that we incorporated the interface conditions into the domain of $L(k)$, indeed $w_2, w_3 \in H^1(\mathbb{R})$ and Sobolev embeddings imply that w_2, w_3 can be chosen continuous on $\mathbb{R}$. Therefore, we can from now on ignore the interface conditions in (3.1.10) and have to solve

$$L(k)\boldsymbol{w}(x_1) + \omega\Lambda(x_1)\boldsymbol{w}(x_1) = \boldsymbol{0}, \qquad x_1 \in \mathbb{R}\backslash\{0\}. \tag{3.2.2}$$

Since we are looking for non-trivial solutions, the remaining equation can be interpreted as a generalized eigenvalue problem with an eigenvalue $\omega = \omega(k) \in \mathbb{C}$ and an eigenfunction $\boldsymbol{w} = \boldsymbol{w}(k) \in \mathrm{D}(L(k)) \backslash \{\boldsymbol{0}\}$.

Remark 3.2.1
The rather high regularity assumption on ϵ_1 will be used in Chapter 6 to prove the existence of solutions of Maxwell's equations with $\mathcal{H}^3$-regularity in space.
From the positivity of ϵ_1 it follows that Λ is positive definite, which will be essential in our analysis. It also removes the case that gradient fields are eigenfunctions, which would result in the fact that each $\omega \in \mathbb{C}$ is an eigenvalue of infinite multiplicity. Let for example $\epsilon_1^+ = 0$, then $\boldsymbol{w}$, with $\boldsymbol{w}^- = 0$ and $\boldsymbol{w}^+ = (\partial_{x_1} f, \mathrm{i}kf, 0)^\top$, would be a non-trivial solution of (3.1.10) *for all $f \in C_c^\infty(\mathbb{R}_+)\backslash\{0\}$ and $\omega \in \mathbb{C}$.*

Let us state some properties of the operator $L(k)$ that will be used later on.

Lemma 3.2.2 (Properties of the Operator $L(k)$)
Let $k \in \mathbb{R}$. The operator $L(k) : \mathrm{D}(L(k)) \to L^2(\mathbb{R})^3$ as defined in (3.1.11) *and* (3.2.1) *is linear, closed and self-adjoint.*

PROOF: It is immediately clear that $L(k)$ is linear.
To show that $L(k)$ is closed, we take a sequence $(\boldsymbol{w}_n)_n \subset \mathrm{D}(L(k))$ with $\boldsymbol{w}_n \to \boldsymbol{w}$ and $L(k)\boldsymbol{w}_n \to \boldsymbol{v}$ in $L^2(\mathbb{R})^3$ and show that $\boldsymbol{w} \in \mathrm{D}(L(k))$ and $L(k)\boldsymbol{w} = \boldsymbol{v}$.
First, we have to show $w_3 \in H^1(\mathbb{R})$. Since $\mathrm{i}\partial_{x_1} w_{n,3} \to v_2$ and $w_{n,3} \to w_3$ in $L^2(\mathbb{R})$, it follows that there exists a function $\widetilde{w}_3 = -\mathrm{i}v_2 \in L^2(\mathbb{R})$ such that $\partial_{x_1} w_{n,3} \to \widetilde{w}_3$ in $L^2(\mathbb{R})$. With the definition of weak derivatives it follows

$$\begin{aligned} \left| \int_{\mathbb{R}} (\widetilde{w}_3 \varphi + w_3 \partial_{x_1} \varphi) \mathrm{d}x_1 \right| &= \left| \int_{\mathbb{R}} ((\widetilde{w}_3 - \partial_{x_1} w_{n,3})\varphi + \partial_{x_1} w_{n,3}\varphi + w_{n,3}\partial_{x_1}\varphi + (w_3 - w_{n,3})\partial_{x_1}\varphi) \mathrm{d}x_1 \right| \\ &\leqslant \left| \int_{\mathbb{R}} (\partial_{x_1} w_{n,3}\varphi + w_{n,3}\partial_{x_1}\varphi) \mathrm{d}x_1 \right| + \|\widetilde{w}_3 - \partial_{x_1} w_{n,3}\|_{L^2(\mathbb{R})} \|\varphi\|_{L^2(\mathbb{R})} \\ &\quad + \|w_3 - \partial_{x_1} w_{n,3}\|_{L^2(\mathbb{R})} \|\partial_{x_1}\varphi\|_{L^2(\mathbb{R})} \to 0 \qquad (n \to \infty) \end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbb{R})$ and therefore $\partial_{x_1} w_3 = \widetilde{w}_3$ and $w_3 \in H^1(\mathbb{R})$.
Analogously, it follows that $w_2 \in H^1(\mathbb{R})$.
From the definition of $L(k)$, the convergence of $w_{n,1}$ in $L^2(\mathbb{R})$ and $w_{n,2}$, $w_{n,3}$ in $H^1(\mathbb{R})$ we have

$$\begin{aligned}\|L(k)\boldsymbol{w} - \boldsymbol{v}\|_{L^2(\mathbb{R})^3} &\leqslant \|L(k)(\boldsymbol{w} - \boldsymbol{w}_n)\|_{L^2(\mathbb{R})^3} + \|L(k)\boldsymbol{w}_n - \boldsymbol{v}\|_{L^2(\mathbb{R})^3} \\ &\leqslant C\,\|\boldsymbol{w} - \boldsymbol{w}_n\|_{L^2(\mathbb{R})^3} + \sum_{j=2}^{3} \left\|\partial_{x_1} w_j - \partial_{x_1} w_{n,j}\right\|_{L^2(\mathbb{R})} + \|L(k)\boldsymbol{w}_n - \boldsymbol{v}\|_{L^2(\mathbb{R})^3} \\ &\to 0 \qquad (n \to \infty).\end{aligned}$$

To see that $L(k)$ is symmetric we use partial integration and have for all $\boldsymbol{w}, \boldsymbol{v} \in \mathrm{D}\,(L(k))$ that

$$\begin{aligned}\langle L(k)\boldsymbol{w}, \boldsymbol{v}\rangle_{L^2(\mathbb{R})^3} &= \int_{\mathbb{R}} (k w_3 \overline{v}_1 + \mathrm{i}\partial_{x_1} w_3 \overline{v}_2 + k w_1 \overline{v}_3 + \mathrm{i}\partial_{x_1} w_2 \overline{v}_3)\,\mathrm{d}x_1 \\ &= \int_{\mathbb{R}} (k w_1 \overline{v}_3 - \mathrm{i} w_2 \partial_{x_1} \overline{v}_3 + k w_3 \overline{v}_1 - \mathrm{i} w_3 \partial_{x_1} \overline{v}_2)\,\mathrm{d}x_1 \\ &= \langle \boldsymbol{w}, L(k)\boldsymbol{v}\rangle_{L^2(\mathbb{R})^3}.\end{aligned}$$

Finally, we have to prove that $\mathrm{D}\,(L(k)) = \mathrm{D}\,(L^*(k))$. Let $\boldsymbol{v} \in \mathrm{D}\,(L(k))$. From the symmetry it follows that

$$\langle L(k)\boldsymbol{w}, \boldsymbol{v}\rangle_{L^2(\mathbb{R})^3} = \langle \boldsymbol{w}, L(k)\boldsymbol{v}\rangle_{L^2(\mathbb{R})^3} = \langle \boldsymbol{w}, L^*(k)\boldsymbol{v}\rangle_{L^2(\mathbb{R})^3}$$

for all $\boldsymbol{w} \in \mathrm{D}\,(L(k))$ and hence $\mathrm{D}\,(L(k)) \subset \mathrm{D}\,(L^*(k))$. Now let $\boldsymbol{v} \in \mathrm{D}\,(L^*(k)) \subset L^2(\mathbb{R})^3$ and $L^*(k)\boldsymbol{v} =: \boldsymbol{\phi} \in L^2(\mathbb{R})^3$. We again use that

$$\langle L(k)\boldsymbol{w}, \boldsymbol{v}\rangle_{L^2(\mathbb{R})^3} = \langle \boldsymbol{w}, L^*(k)\boldsymbol{v}\rangle_{L^2(\mathbb{R})^3} = \langle \boldsymbol{w}, \boldsymbol{\phi}\rangle_{L^2(\mathbb{R})^3} \tag{3.2.3}$$

for all $\boldsymbol{w} \in \mathrm{D}\,(L(k))$. We now look at special functions $\boldsymbol{w}$ where the first and the third component are zero. From (3.2.3) we get

$$\int_{\mathbb{R}} \mathrm{i}\partial_{x_1} w_2 \overline{v}_3 \mathrm{d}x_1 = \int_R w_2 \overline{\phi}_2\,\mathrm{d}x_1$$

and therefore

$$\int_{\mathbb{R}} \partial_{x_1} w_2 \overline{v}_3\,\mathrm{d}x_1 = -\int_R w_3 \mathrm{i}\overline{\phi}_2 \mathrm{d}x_1$$

for all $w_2 \in H^1(\mathbb{R})$. By the definition of weak derivatives it follows that $\overline{v}_3 \in H^1(\mathbb{R})$ with the weak derivative $\partial_{x_1} \overline{v}_3 = \mathrm{i}\overline{\phi}_2 \in L^2(\mathbb{R})$.
Analogously, we get that $\overline{v}_2 \in H^1(\mathbb{R})$ with $\partial_{x_1} \overline{v}_2 = \mathrm{i}\overline{\phi}_3 - \mathrm{i}k\overline{v}_1 \in L^2(\mathbb{R})$ when we insert $\boldsymbol{w} = (0, 0, w_3)^\top$ into (3.2.3). All in all, we get $\boldsymbol{v} \in \mathrm{D}\,(L(k))$ and hence that $L(k)$ is self-adjoint.

□

Remark 3.2.3
Since $L(k)$ is self-adjoint, it follows that the spectrum of $L(k)$ is real. Since Λ is real and diagonal, this implies that the generalized eigenvalue problem $L(k)\boldsymbol{w} = -\omega\Lambda\boldsymbol{w}$ has only real eigenvalues $\omega = \omega(k) \in \mathbb{R}$. Indeed, testing the eigenvalue equation with $\boldsymbol{w}$ gives us:

$$-\omega\langle\Lambda\boldsymbol{w},\boldsymbol{w}\rangle_{L^2(\mathbb{R})^3} = \langle L(k)\boldsymbol{w},\boldsymbol{w}\rangle_{L^2(\mathbb{R})^3} = \langle\boldsymbol{w},L(k)\boldsymbol{w}\rangle_{L^2(\mathbb{R})^3} = -\overline{\omega}\langle\boldsymbol{w},\Lambda\boldsymbol{w}\rangle_{L^2(\mathbb{R})^3}.$$

Since $\langle\Lambda\boldsymbol{w},\boldsymbol{w}\rangle_{L^2(\mathbb{R})^3} = \langle\boldsymbol{w},\Lambda\boldsymbol{w}\rangle_{L^2(\mathbb{R})^3}$, it follows that $\omega \in \mathbb{R}$.

Let us now impose some assumptions on solutions of (3.2.2). Assume that in a neighborhood of a fixed $k = k_0 \in \mathbb{R}$ there is a unique smooth eigenvalue curve

$$k \mapsto \omega(k).$$

This eigenvalue curve defines the dispersion relation for our Maxwell problem.

Remark 3.2.4
In Corollary 3.3.9 we will prove the existence of a C^∞-eigenvalue curve under the following Assumptions (A1)–(A4)*. We will use this smoothness in Chapter 4 to calculate the Taylor expansion of $\omega(k)$.*

For our analysis it will be necessary to additionally assume that

$$\nu_0 := \omega(k_0) \text{ is a simple eigenvalue of (3.2.2) isolated from all other eigenvalues at } k = k_0. \tag{A2}$$

Let $B \subset \mathbb{R}^2$ be a small neighborhood of $(k_0, \omega(k_0))$. These assumptions guarantee that for $(k,\omega) \in B$ the eigenvalue problem (3.2.2) has a solution if and only if the dispersion relation is satisfied.

Remark 3.2.5
Since ϵ_1 depends on x_1, it is in general not possible to solve (3.2.2) *explicitly. We refer to Example 3.1.4 where an explicit solution and a dispersion relation was calculated for a special choice of ϵ_1. In Section 3.4 we will also present a method to calculate solutions numerically and check if the eigenvalues are isolated.*

Remark 3.2.6
When there is a solution of (3.2.2)*, we will always select a solution $\boldsymbol{w}$ with w_1, w_3 real-valued and w_2 imaginary-valued, to shorten some of the computations later on. This can always be done since the problem is linear. To prove this we start by assuming that $\boldsymbol{v}$ is a non-trivial solution of* (3.2.2)*. If* $\mathrm{Re}\,(v_1) = \mathrm{i}\,\mathrm{Im}\,(v_2) = \mathrm{Re}\,(v_3) = 0$ *we simply select the solution $\boldsymbol{w} = \mathrm{i}\boldsymbol{v}$. If not, write $\overline{L(k)\boldsymbol{v}} = \boldsymbol{0}$ as*

$$\begin{cases} \epsilon_1(x_1)\omega\overline{v}_1 + k\overline{v}_3 = 0, \\ \epsilon_1(x_1)\omega\overline{v}_2 - \mathrm{i}\partial_{x_1}\overline{v}_3 = 0, \\ k\overline{v}_1 - \mathrm{i}\partial_{x_1}\overline{v}_2 + \mu_0\nu_0\overline{v}_3 = 0, \end{cases}$$

which shows us that $\widetilde{v} := (\overline{v}_1, -\overline{v}_2, \overline{v}_3)^\top$ *is also a solution of* (3.2.2). *Now*

$$w := \frac{1}{2}(v + \widetilde{v}) = (\mathrm{Re}\,(v_1)\,, \mathrm{i}\,\mathrm{Im}\,(v_2)\,, \mathrm{Re}\,(v_3))^\top$$

has the desired properties.

For the eigenfunction at $k = k_0$ we will write $m := w(k_0)$ and we choose the normalization

$$\langle \Lambda m, m \rangle_{L^2(\mathbb{R})^3} = \int_{\mathbb{R}} \left(\epsilon_1 \left(m_1^2 - m_2^2\right) + \mu_0 m_3^2\right) \, \mathrm{d}x_1 = 1, \tag{3.2.4}$$

which will simplify the calculation later on.

3.3. Solution of the Inhomogeneous Problem

In this section we want to study the inhomogeneous version of the eigenvalue problem

$$T_{k,\omega} v := L(k) v + \omega \Lambda v = \begin{pmatrix} \omega\epsilon_1 & 0 & k \\ 0 & \omega\epsilon_1 & \mathrm{i}\partial_{x_1} \\ k & \mathrm{i}\partial_{x_1} & \omega\mu_0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = f \tag{3.3.1}$$

with $k, \omega \in \mathbb{R}$ and $f \in \mathrm{N}\,(T_{k,\omega})^\perp$, where

$$\mathrm{N}\,(T_{k,\omega}) \subset \mathrm{D}\,(T_{k,\omega}) := \left\{ w : \mathbb{R} \to \mathbb{C}^3 \,\middle|\, w_1 \in L^2(\mathbb{R}),\, w_2, w_3 \in H^1(\mathbb{R}) \right\}.$$

Remark 3.3.1
The solutions of certain inhomogeneous problems will play an important role in the construction of higher order correction terms for our asymptotic solution of Maxwell's equations. Note that for the approximation result of Chapter 6 it will not be necessary to calculate the solutions of the inhomogeneous problems explicitly.

Before we can show the existence of solutions of (3.3.1) we have to collect some helpful results.
Let us first state some properties of $T_{k,\omega}$.

Corollary 3.3.2 (Properties of the Operator $T_{k,\omega}$)
Let $k, \omega \in \mathbb{R}$. *The operator* $T_{k,\omega} : \mathrm{D}\,(T_{k,\omega}) \to L^2(\mathbb{R})^3$ *is linear, closed and self-adjoint.*

PROOF: The proof can be done analogously to the proof of Lemma 3.2.2.

□

This corollary will allow us to apply the well-known closed range theorem, see e.g. [86]:

Theorem 3.3.3 (Closed Range Theorem)
Let X, Y be Banach spaces and $T : \mathrm{D}(T) \to Y$ a linear, closed and self-adjoint operator with $\mathrm{D}(T)$ *dense in X. Then*

$$\mathrm{N}(T)^{\perp} = \mathrm{R}(T)$$

if and only if $\mathrm{R}(T)$ *is closed.*

Before we can prove the existence of solutions of (3.3.1) we have to add an additional assumption on ϵ_1. Assume that there are constants $\epsilon_1^{\pm\infty} > 0$ such that

$$\epsilon_1^{\pm}(x_1) \to \epsilon_1^{\pm\infty} \quad \text{as} \quad x_1 \to \pm\infty. \tag{A3}$$

Under this assumption it will be possible to use the theory of exponential dichotomy, see [16] for more details.

Definition 3.3.4 (Exponential Dichotomy)
Let $J \subset \mathbb{R}$ be an interval and $A : J \to \mathbb{C}^{n\times n}$ be a continuous coefficient matrix. Let $Y(t)$ be a fundamental matrix for the linear differential equation

$$\partial_t y = A(t)y. \tag{3.3.2}$$

Equation (3.3.2) *possesses exponential dichotomy if there exist constants $K, L, \alpha, \beta > 0$ and a projection P such that*

$$\begin{aligned} |Y(t)PY^{-1}(s)| &\leqslant K\mathrm{e}^{-\alpha(t-s)}, \qquad t \geqslant s,\\ |Y(t)(I-P)Y^{-1}(s)| &\leqslant L\mathrm{e}^{-\beta(s-t)}, \qquad s \geqslant t. \end{aligned}$$

It can now be shown that exponential dichotomy is related to certain Fredholm operators.

Theorem 3.3.5 (Exponential Dichotomy and Fredholm Property)
Let $A : (-\infty, \infty) \to \mathbb{C}^{n\times n}$ be bounded and measurable. The operator $T : \mathrm{D}(T) \to L^2(\mathbb{R})^n$ defined by

$$(Ty)(t) = \partial_t x(t) - A(t)y(t)$$

on $\mathrm{D}(T) = H^1(\mathbb{R})^n$ is a Fredholm operator if and only if the ordinary differential equations

$$\partial_t y = A(t)y, \qquad t \geqslant 0$$

and

$$\partial_t y = -A(-t)y, \qquad t \geqslant 0$$

possess exponential dichotomy.

Proof: This theorem and its proof can be found as Theorem 1.2 in [5].

□

Furthermore, one can show that small perturbations do not influence the dichotomy.

Lemma 3.3.6 (Perturbation of Exponential Dichotomy)
Let $J = \mathbb{R}^+$ and $B : J \to \mathbb{C}^{n\times n}$ be a perturbation with $\lim_{t\to\infty} |B(t)| = 0$. If (3.3.2) *possesses exponential dichotomy then*

$$\partial_t y = (A(t) + B(t))y$$

also possesses exponential dichotomy.

PROOF: This follows from Proposition 1 in Chapter 4 of [16]. There it is stated that exponential dichotomy is preserved on an interval $[t^*, \infty)$ when $|B(t)| \leqslant C^*$ for all $t \in [t^*, \infty)$, where $C^* > 0$ only depends on the constants K, L, α, β from the definition of the exponential dichotomy. Since $\lim_{t\to\infty} |B(t)| = 0$, this condition is satisfied for t big enough.
The discussion starting on page 13 of [16] allows us to extend the exponential dichotomy to the whole half-line.

□

We can now combine Theorem 3.3.5 with the closed range theorem since Fredholm operators have closed range by definition.

Lemma 3.3.7 (Solutions of the Inhomogeneous Problem)
Let $\epsilon_1 \in \mathcal{W}^{1,\infty}(\mathbb{R})$ satisfy (A3) *and let $k, \omega \in \mathbb{R}$ be such that $k^2 > \omega^2 \epsilon_1^{\pm\infty} \mu_0$ and $\omega\epsilon_1 \neq 0$. Assume that we are in one of the cases*

i) 0 is a simple eigenvalue of $T_{k,\omega}$ isolated from all other eigenvalues;

ii) 0 is not an eigenvalue of $T_{k,\omega}$.

If $\boldsymbol{f} \in \mathrm{N}(T_{k,\omega})^{\perp} \subset L^2(\mathbb{R})^3$ ($\boldsymbol{f} \in L^2(\mathbb{R})^3$ if $\mathrm{N}(T_{k,\omega}) = \{0\}$), then (3.3.1) *has a solution $\boldsymbol{v} \in \mathrm{D}(T_{k,\omega})$.*

PROOF: Equation (3.3.1) splits for $\omega\epsilon_1 \neq 0$ into the scalar equation

$$v_1 = \frac{1}{\omega\epsilon_1}(f_1 - kv_3) \tag{3.3.3}$$

and the reduced problem

$$\widetilde{T}_{k,\omega}\widetilde{\boldsymbol{v}} = \widetilde{\boldsymbol{f}} \tag{3.3.4}$$

with

$$\widetilde{T}_{k,\omega} := \begin{pmatrix} \omega\epsilon_1 & \mathrm{i}\partial_{x_1} \\ \mathrm{i}\partial_{x_1} & \omega\mu_0 - \frac{k^2}{\omega\epsilon_1} \end{pmatrix}, \qquad \widetilde{\boldsymbol{v}} := \begin{pmatrix} v_2 \\ v_3 \end{pmatrix}, \qquad \widetilde{\boldsymbol{f}} := \begin{pmatrix} f_2 \\ f_3 - \frac{k}{\omega\epsilon_1} f_1 \end{pmatrix}$$

and

$$\mathrm{D}\left(\widetilde{T}_{k,\omega}\right) := H^1(\mathbb{R})^2.$$

Note that

$$(v_2, v_3)^\top \in \mathrm{N}\left(\widetilde{T}_{k,\omega}\right) \iff \left(-\frac{k}{\omega\epsilon_1} v_3, v_2, v_3\right)^\top \in \mathrm{N}(T_{k,\omega})$$

and hence

$$\widetilde{\boldsymbol{f}} \in \mathrm{N}\left(\widetilde{T}_{k,\omega}\right)^{\perp} \iff \boldsymbol{f} \in \mathrm{N}\left(T_{k,\omega}\right)^{\perp}.$$

We also obtain that $0 \notin \rho(\widetilde{T}_{k,\omega})$ if and only if $0 \notin \rho(T_{k,\omega})$, where $\rho(\widetilde{T}_{k,\omega})$ as usual denotes the resolvent set of the operator $\widetilde{T}_{k,\omega}$. Indeed, we see that $T_{k,\omega}$ is invertible if and only if $\widetilde{T}_{k,\omega}$ is invertible since (3.3.3) is just an algebraic equation.
As a direct consequence of Corollary 3.3.2 we have that $\widetilde{T}_{k,\omega}$ is linear, closed and self-adjoint. The result will follow from the closed range theorem when we can show that $\widetilde{T}_{k,\omega}$ is a Fredholm operator. With this aim, we rewrite the problem as the linear ordinary differential equation

$$\partial_{x_1}\widetilde{\boldsymbol{v}} = A(x_1)\widetilde{\boldsymbol{v}} + \boldsymbol{g}$$

with

$$A(x_1) := \begin{pmatrix} 0 & \mathrm{i}\left(\omega\mu_0 - \frac{k^2}{\omega\epsilon_1(x_1)}\right) \\ \mathrm{i}\omega\epsilon_1(x_1) & 0 \end{pmatrix} =: \begin{cases} A_-(x_1), & x_1 < 0, \\ A_+(x_1), & x_1 > 0, \end{cases}$$

$$\boldsymbol{g} := -\mathrm{i}\begin{pmatrix} f_3 - \frac{k}{\omega\epsilon_1} f_1 \\ f_2 \end{pmatrix}.$$

Theorem 3.3.5 states that $\widetilde{T}_{k,\omega}$ is Fredholm if and only if the ODEs

$$\partial_{x_1}\widetilde{\boldsymbol{v}}^- = A_-(x_1)\widetilde{\boldsymbol{v}}^-, \qquad x_1 < 0, \tag{3.3.5}$$

and

$$\partial_{x_1}\widetilde{\boldsymbol{v}}^+ = A_+(x_1)\widetilde{\boldsymbol{v}}^+, \qquad x_1 > 0, \tag{3.3.6}$$

have exponential dichotomies. We only show the dichotomy for (3.3.6) as (3.3.5) can be treated analogously.
We now want to use Lemma 3.3.6, hence we introduce the problem

$$\partial_{x_1}\boldsymbol{w} = A_{+\infty}\boldsymbol{w} \tag{3.3.7}$$

with the constant coefficient matrix

$$A_{+\infty} := \lim_{x_1\to\infty} A(x_1) = \begin{pmatrix} 0 & \mathrm{i}\omega\mu_0 - \frac{\mathrm{i}k^2}{\omega\epsilon_1^{+\infty}} \\ \mathrm{i}\omega\epsilon_1^{+\infty} & 0 \end{pmatrix}.$$

Now it is easy to see that (3.3.7) possesses an exponential dichotomy since the eigenvalues

$$\lambda_{1,2} = \pm\sqrt{k^2 - \omega^2\epsilon_1^{+\infty}\mu_0}$$

of $A_{+\infty}$ are real and have different signs for $k^2 > \omega^2\epsilon_1^{+\infty}\mu_0$, see [16, Chapter 2]. Note that this corresponds to (3.1.5) in Example 3.1.4.

Now Lemma 3.3.6 also implies that

$$\partial_{x_1}\widetilde{\boldsymbol{v}}^+ = A_+(x_1)\widetilde{\boldsymbol{v}}^+ = (A_{+\infty} + (A_+(x_1) - A_{+\infty}))\widetilde{\boldsymbol{v}}^+$$

possesses exponential dichotomy because $|A_+(x_1) - A_{+\infty}| \to 0$ for $x_1 \to \infty$.

□

Remark 3.3.8

In Chapter 4 we will use Lemma 3.3.7 for two different pairs of parameters (k, ω). *We will use case i) to analyze* T_{k_0,ν_0} *and we will use case ii) to analyze* $T_{3k_0,3\nu_0}$, *where* k_0, ν_0 *are fixed as in* (A2). *Hence, it will be necessary to assume that*

$$\omega(k_0)\epsilon_1 \neq 0, \quad \omega(3k_0)\epsilon_1 \neq 0 \quad \textit{and} \quad k_0^2 > \omega(k_0)^2\mu_0\epsilon_1^{\pm\infty} \tag{A4}$$

and

$$3\nu_0 \neq \omega(3k_0), \ \textit{i.e. } 3\nu_0 \textit{ is not an eigenvalue of } (3.2.2) \textit{ at } k = 3k_0. \tag{A5}$$

Assumption (A5) *is called non-resonance condition.*

The reduction to (3.3.4) can of course also be done for $\boldsymbol{f} = \boldsymbol{0}$. We will use this reduction to show some useful properties of solutions of the homogeneous problem (3.2.2).

Corollary 3.3.9 (Smooth Eigenvalue/Eigenfunction Curve)

Let (A1)–(A4) *be true. Then for some* $\delta > 0$ *the eigenvalues and the corresponding eigenfunctions of problem* (3.2.2) *satisfy*

$$\omega \in C^\infty((k_0 - \delta, k_0 + \delta), \mathbb{R}) \quad \textit{and} \quad \boldsymbol{w} \in C^\infty\left((k_0 - \delta, k_0 + \delta), L^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})\right).$$

PROOF: We translate our problem into standard perturbation theory of spectra, as discussed in [42]. We rewrite (3.2.2) as

$$-\Lambda^{-1}L(k)\boldsymbol{w}(x_1) = \omega\boldsymbol{w}(x_1), \qquad x_1 \in \mathbb{R}\backslash\{0\}.$$

By the assumptions, ν_0 is a simple eigenvalue of $-\Lambda^{-1}L(k_0)$ with eigenfunction $\boldsymbol{w}(k_0)$, and there are no other eigenvalues nearby. As shown in the proof of Lemma 3.3.7, ω belongs to the resolvent set of $\Lambda^{-1}L(k_0)$ if and only if $\widetilde{T}_{k_0,\omega}$ is invertible. For $\omega \approx \nu_0$ we can write $\widetilde{T}_{k_0,\omega} = \widetilde{T}_{k_0,\nu_0} + R$ with a perturbation $R : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, the norm of which is bounded by $C\,|\omega - \nu_0|$.

In the proof of Lemma 3.3.7 we have seen that $\widetilde{T}_{k_0,\nu_0}$ is a Fredholm operator, therefore $0 \notin \sigma_{\text{ess}}\left(\widetilde{T}_{k_0,\nu_0}\right)$, where $\sigma_{\text{ess}}\left(\widetilde{T}_{k_0,\nu_0}\right)$ denotes the essential spectrum of $\widetilde{T}_{k_0,\nu_0}$, and the same is true for $\widetilde{T}_{k_0,\omega}$ if ω is close to ν_0. If 0 was an eigenvalue of $\widetilde{T}_{k_0,\omega}$, the number $\omega \neq \nu_0$ would be an eigenvalue of $-\Lambda^{-1}L(k_0)$ which is impossible in a small enough neighborhood of ν_0 by Assumption (A2). As a result, 0 is contained in $\rho\left(\widetilde{T}_{k_0,\omega}\right)$ and thus ν_0 is an isolated simple eigenvalue of $-\Lambda^{-1}L(k_0)$.

For $k \approx k_0$, Theorem 1.8 in §VII.1 of [42] now shows that $-\Lambda^{-1}L(k)$ has a simple eigenvalue $\omega(k)$ smoothly depending on k. Also, the projection $P(k)$ onto the eigenspace is smooth in k. Hence, the mapping $k \mapsto P(k)\boldsymbol{w}(k_0)$ is a smooth family of eigenfunctions of (3.2.2) if k is close to k_0.

□

Remark 3.3.10
The reduction to two systems of ordinary differential equations (3.3.5) *and* (3.3.6) *can also be helpful to solve the eigenvalue problem* (3.2.2) *explicitly, for more on ODEs see e.g. [63].*
First, we have to determine the corresponding fundamental matrices on both sides of the interface. To make sure that the solutions are integrable we then have to select a solution from the stable subspace $E^{\mathrm{s}}(A_+)$ *on the right of the interface and a solution from the unstable subspace* $E^{\mathrm{u}}(A_-)$ *on the left of the interface. Finally, we have to match the solutions at the interface to get continuous functions. This implies that solutions can only exist when the intersection* $E^{\mathrm{s}}(A_+) \cap E^{\mathrm{u}}(A_-)$ *is non-trivial at the interface.*
For a general ϵ_1 *it is difficult to determine the fundamental matrices, but for some special choices of* ϵ_1 *it is doable, see e.g. Example 3.1.4.*

The next lemma allows us to improve the regularity of solutions of (3.3.1) if the right-hand side is smooth enough.

Lemma 3.3.11 (Higher Regularity Solutions of the Inhomogeneous Problem)
Assume (A1) *and* (A4). *Let* $k, \omega \in \mathbb{R}$, $\boldsymbol{f} := (f_1, f_2, f_3)^\top$ *with* $f_1 \in \mathcal{H}^3(\mathbb{R})$ *and* $f_2, f_3 \in \mathcal{H}^2(\mathbb{R})$. *If* $\boldsymbol{v} \in L^2(\mathbb{R})^3$ *is a solution of* (3.3.1), *then* $\boldsymbol{v} \in \mathcal{H}^3(\mathbb{R})^3$.

Proof: We start by showing that $\boldsymbol{v} \in \mathcal{H}^1(\mathbb{R})$. From (3.3.1) we know that

$$\begin{cases} \mathrm{i}\partial_{x_1} v_2 = f_3 - k v_1 - \mu_0 \omega v_3, \\ \mathrm{i}\partial_{x_1} v_3 = f_2 - \epsilon_1 \omega v_2. \end{cases} \tag{3.3.8}$$

The right-hand sides in (3.3.8) belong to $L^2(\mathbb{R})$ and therefore $v_2, v_3 \in \mathcal{H}^1(\mathbb{R})$. The assumptions on ϵ_1 imply that $\partial_{x_1}\left(\epsilon_1^{-1}\right) = -\epsilon_1^{-2}\partial_{x_1}\epsilon_1 \in L^\infty(\mathbb{R})$. Now $v_1 \in \mathcal{H}^1(\mathbb{R})$ is a direct consequence of

$$v_1 = \frac{1}{\epsilon_1 \omega}(f_1 - k v_3). \tag{3.3.9}$$

We can now iterate this process since $\epsilon_1^{-1} \in \mathcal{W}^{3,\infty}(\mathbb{R})$:

$$\begin{aligned} \partial_{x_1}^2\left(\epsilon_1^{-1}\right) &= 2\epsilon_1^{-3}(\partial_{x_1}\epsilon_1)^2 - \epsilon_1^{-2}\partial_{x_1}^2\epsilon_1 \in L^\infty(\mathbb{R}), \\ \partial_{x_1}^3\left(\epsilon_1^{-1}\right) &= -6\epsilon_1^{-4}(\partial_{x_1}\epsilon_1)^3 + 6\epsilon_1^{-3}\partial_{x_1}\epsilon_1\partial_{x_1}^2\epsilon_1 - \epsilon_1^{-2}\partial_{x_1}^3\epsilon_1 \in L^\infty(\mathbb{R}). \end{aligned}$$

Equations (3.3.8) and (3.3.9) yield that $\boldsymbol{v} \in \mathcal{H}^2(\mathbb{R})^3$ if one knows that $\boldsymbol{v} \in \mathcal{H}^1(\mathbb{R})^3$. This fact then implies that $\boldsymbol{v} \in \mathcal{H}^3(\mathbb{R})^3$.

□

3.4. Numerical Calculation of the Eigenfunctions

Let us now describe one way to calculate the eigenfunctions and the dispersion relation numerically.

Since we are interested in sufficiently regular solutions of (3.2.2), we rewrite the problem as a second-order ordinary differential equation for w_3. To this end, we differentiate the second equation in (3.2.2) and use (3.2.2) itself on $\mathbb{R}\backslash\{0\}$ to get

$$\begin{aligned}\partial_{x_1}^2 w_3 &= \mathrm{i}\partial_{x_1}\epsilon_1(x_1)\omega w_2 + \mathrm{i}\epsilon_1(x_1)\omega\partial_{x_1} w_2 \\ &= \frac{\partial_{x_1}\epsilon_1(x_1)}{\epsilon_1(x_1)}\,\partial_{x_1} w_3 - \epsilon_1(x_1)\omega(\mu_0\omega w_3 + k w_1) \\ &= \frac{\partial_{x_1}\epsilon_1(x_1)}{\epsilon_1(x_1)}\,\partial_{x_1} w_3 - \epsilon_1(x_1)\mu_0\omega^2 w_3 + k^2 w_3.\end{aligned}$$

From the interface condition $[\![w_2]\!]_{1\mathrm{D}}(0) = 0$ we deduce the condition $\left[\!\left[\frac{\partial_{x_1} w_3}{\epsilon_1}\right]\!\right]_{1\mathrm{D}}(0) = 0$. Now we have to solve the eigenvalue problem

$$\begin{cases} -\partial_{x_1}^2 w_3(x_1) + \dfrac{\partial_{x_1}\epsilon_1(x_1)}{\epsilon_1(x_1)}\,\partial_{x_1} w_3(x_1) + k^2 w_3(x_1) = \epsilon_1(x_1)\mu_0\omega^2 w_3(x_1), & x_1 \in \mathbb{R}\backslash\{0\}, \\ [\![w_3]\!]_{1\mathrm{D}}(0) = \left[\!\left[\dfrac{\partial_{x_1} w_3}{\epsilon_1}\right]\!\right]_{1\mathrm{D}}(0) = 0. \end{cases} \tag{3.4.1}$$

Note that we can use $w_1 = -\frac{k}{\epsilon_1\omega}\,w_3$ and $w_2 = -\frac{\mathrm{i}}{\epsilon_1\omega}\,\partial_{x_1} w_3$ to calculate the remaining components of $\boldsymbol{w}$. We also see that the interface conditions $[\![\epsilon_1 w_1]\!]_{1\mathrm{D}}(0) = [\![w_2]\!]_{1\mathrm{D}}(0) = 0$ are satisfied if w_3 solves (3.4.1).

To simplify the numerics we assume that we can write $w_3 = w_{3,\mathrm{r}} + w_{3,\mathrm{s}}$, with an at least two times differentiable function $w_{3,\mathrm{r}}$ and a function $w_{3,\mathrm{s}}$ that has a discontinuous first derivative at $x_1 = 0$, e.g.

$$w_{3,\mathrm{s}}(x_1) = \begin{cases} w_{3,\mathrm{s}}^- = \text{const.}, & x_1 \leqslant 0, \\ w_{3,\mathrm{s}}^+(x_1), & x_1 > 0 \end{cases}$$

and choose $w_{3,\mathrm{s}}^+(0) = w_{3,\mathrm{s}}^-$ such that $w_{3,\mathrm{s}}$ is continuous. Note that with this choice w_3 satisfies the first interface condition.

For the second interface condition we calculate $\partial_{x_1} w_3$ and get that

$$\begin{aligned} & \left[\!\left[\frac{\partial_{x_1} w_3}{\epsilon_1\omega}\right]\!\right]_{1\mathrm{D}}(0) = 0 \\ \iff \quad & \epsilon_1^-(0)\left(\partial_{x_1} w_{3,\mathrm{r}}(0) + \partial_{x_1} w_{3,\mathrm{s}}^+(0)\right) = \epsilon_1^+(0)\partial_{x_1} w_{3,\mathrm{r}}(0) \\ \iff \quad & \partial_{x_1} w_{3,\mathrm{s}}^+(0) = \frac{\epsilon_1^+(0) - \epsilon_1^-(0)}{\epsilon_1^-(0)}\,\partial_{x_1} w_{3,\mathrm{r}}(0). \end{aligned}$$

With this we set $\widetilde{\epsilon} := \frac{\epsilon_1^+(0)-\epsilon_1^-(0)}{\epsilon_1^-(0)}$ and define the linear operator $\mathcal{L}$ as follows:

$$(\mathcal{L}w_{3,\mathrm{r}})(x_1) := \begin{cases} -\operatorname{sgn}(\widetilde{\epsilon})\,\partial_{x_1} w_{3,\mathrm{r}}(0), & x_1 < 0, \\ -\operatorname{sgn}(\widetilde{\epsilon})\,\partial_{x_1} w_{3,\mathrm{r}}(0)\mathrm{e}^{-|\widetilde{\epsilon}|x_1}, & x_1 \geqslant 0. \end{cases}$$

Note that with $w_{3,\mathrm{s}} := \mathcal{L}w_{3,\mathrm{r}}$ the second interface condition is satisfied and $\lim_{x_1\to\infty} w_{3,\mathrm{s}}(x_1) = 0$, $\lim_{x_1\to-\infty} w_{3,\mathrm{s}}(x_1) = -\operatorname{sgn}(\widetilde{\epsilon})\,\partial_{x_1} w_{3,\mathrm{r}}(0)$.
Thus, $w_{3,\mathrm{r}}$ has to solve

$$\begin{cases} \left(-\partial_{x_1}^2 + \frac{\partial_{x_1}\epsilon_1(x_1)}{\epsilon_1(x_1)}\,\partial_{x_1} + k^2\right)(I+\mathcal{L})w_{3,\mathrm{r}}(x_1) = \epsilon_1(x_1)\mu_0\omega^2(I+\mathcal{L})w_{3,\mathrm{r}}(x_1), & x_1 \in \mathbb{R}\backslash\{0\}, \\ [\![w_{3,\mathrm{r}}]\!]_{1\mathrm{D}}(0) = [\![\partial_{x_1} w_{3,\mathrm{r}}]\!]_{1\mathrm{D}}(0) = 0. \end{cases} \tag{3.4.2}$$

We are interested in $H^1(\mathbb{R})$-solutions, therefore we have at least the boundary conditions

$$\lim_{x_0\to-\infty} w_{3,\mathrm{r}}(x_1) = \operatorname{sgn}(\widetilde{\epsilon})\,\partial_{x_1} w_{3,\mathrm{r}}(0), \qquad \lim_{x_1\to\infty} w_{3,\mathrm{r}}(x_1) = 0.$$

To solve (3.4.2) numerically for a fixed $k \in \mathbb{R}$ we discretize the problem over a finite interval $[-d, d] \subset \mathbb{R}$ and add the numerical boundary conditions

$$\begin{cases} w_{3,\mathrm{r}}(-d) = \operatorname{sgn}(\widetilde{\epsilon})\,\partial_{x_1} w_{3,\mathrm{r}}(0), \\ w_{3,\mathrm{r}}(d) = 0 \end{cases}$$

to ensure that the solution w_3 is zero at the boundary of the interval. We now apply a solver for a generalized eigenvalue problem, for example a solver based on a Krylov-Schur algorithm, see e.g. [78].
Let us now calculate the solution of the eigenvalue problem for a non-trivial ϵ_1 and let us check if the Assumptions (A2), (A4) and (A5) are satisfied.

Example 3.4.1 (Numerical Solution of the Linear Eigenvalue Problem)
For $\epsilon_1(x_1) = 1\chi_{\mathbb{R}_-} + (1+\mathrm{e}^{-x_1})\chi_{\mathbb{R}_+}$, see Figure 3.1 (a), *and $\mu_0 = 1$, we compute a numerical solution of* (3.2.2) *with the help of* (3.4.2) *and the method described above. Note that for this choice of ϵ_1 the Assumptions* (A1) *and* (A3) *are satisfied with $\epsilon_{1,m}^{\pm} = \epsilon_1^{\pm\infty} = 1$.*
First, we determine the dispersion relation by solving (3.4.2) *for $k \in [0.4, 1.6]$. See Figure 3.1* (b) *for a plot of the nonlinear dispersion relation. We now fix $k_0 = 0.5$ and check numerically that Assumption* (A5) *is satisfied, indeed $\omega(k_0) = \nu_0 \approx 0.494$, $3\nu_0 \approx 1.481$ and $\omega(3k_0) \approx 1.404$. Since $k_0 > \omega(k_0)$, it also follows that Assumption* (A4) *is satisfied.*
For the fixed k_0 the calculated eigenfunction $\boldsymbol{m}$ can be seen in Figure 3.2. We note that $\epsilon_1 m_1$, m_2, m_3 are continuous functions as demanded by the interface conditions. The eigenfunction $\boldsymbol{m}$ is also exponentially decaying for $|x_1| \to \infty$, which allows us to construct a localized wave packet for the nonlinear Maxwell problem, see Chapter 4.

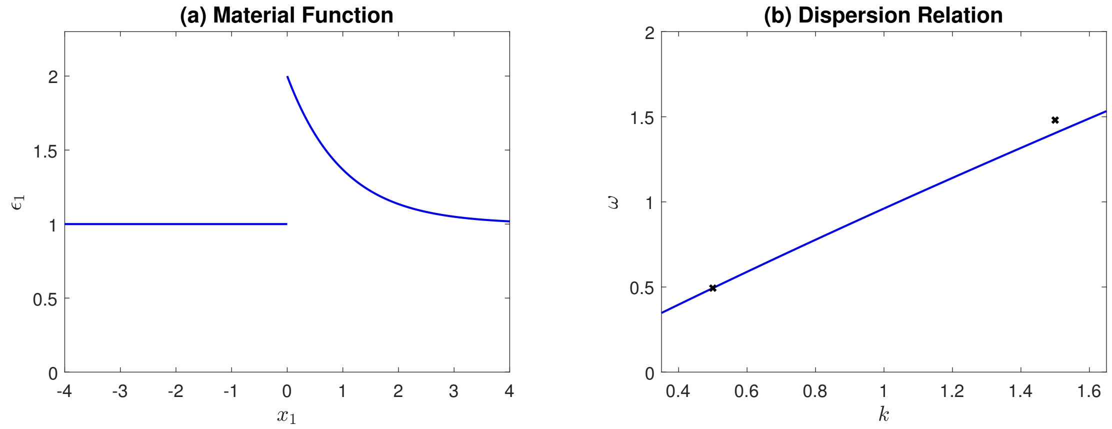

Figure 3.1.: (a) The graph of $x_1 \mapsto \epsilon_1(x_1)$ for the chosen potential $\epsilon_1(x_1) = 1\chi_{\mathbb{R}_-} + (1 + \mathrm{e}^{-x_1})\chi_{\mathbb{R}_+}$.
(b) The dispersion relation $k \mapsto \omega(k)$ for the problem. Marked are the points $(0.5, \omega(0.5))$ and $(1.5, 3\omega(0.5))$ to illustrate that (A5) is satisfied for $k_0 = 0.5$.

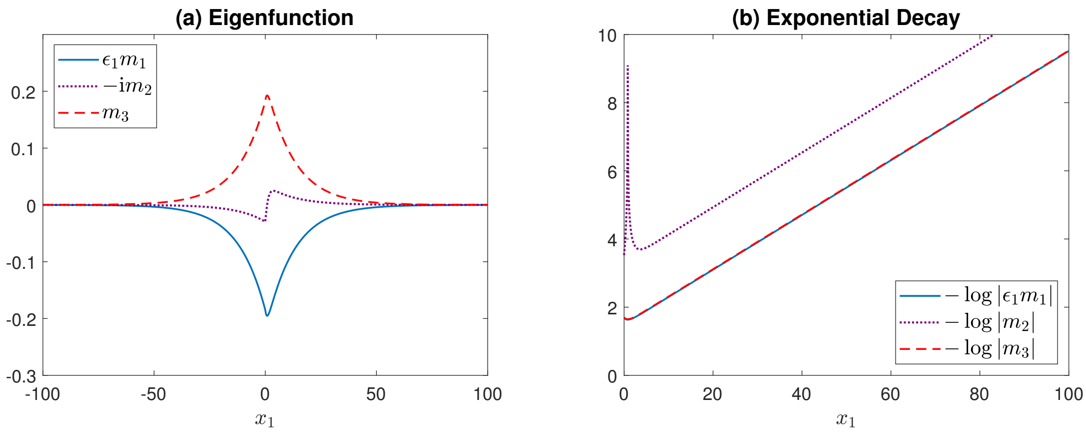

Figure 3.2.: (a) The eigenfunction $\boldsymbol{m}$ of the linear problem (3.2.2) for $k_0 = 0.5$. (We plot $\epsilon_1 m_1$ to show that the linear interface conditions are satisfied.)
(b) Logarithmic plot of the eigenfunctions to illustrate that the solution is exponentially decaying.

To check the effects of the boundary, we repeat the calculation for different intervals $[-d, d]$ *and get the eigenvalue* $\omega(d)$ *closest to* ν_0 *depending on d. Figure 3.3* (a) *shows that the error in the calculation of* $\omega(d)$ *converges to zero for increasing d.*
To be more precise, we used step size $h_x = 0.001$ *in space and step size* $h_k = 0.001$ *in k and interval length* $d = 5 \cdot 10^3$ *to calculate the dispersion relation and we used* $h_x = 0.01$ *and interval length* $d = 5 \cdot 10^4$ *to calculate the eigenfunction. For the calculation of Figure 3.3* (a) *we used d ranging from* 10^2 *to* 10^4 *with the step size* $h_x = 0.01$.
To solve the generalized eigenvalue problem, we used the second-order difference quotients with zero Dirichlet boundary conditions to discretize the derivatives. Then we utilized the Matlab functions "eigs", where we calculated the first 10 *eigenvalues closest to* ν_0 *with a convergence tolerance of* 10^{-10}. *We then only selected solutions where the corresponding eigenfunctions were almost zero in a small neighborhood of the boundary, i.e. the norm of* w_3 *on* $[-d, -d + 100h] \cup [d - 100h, d]$ *is*

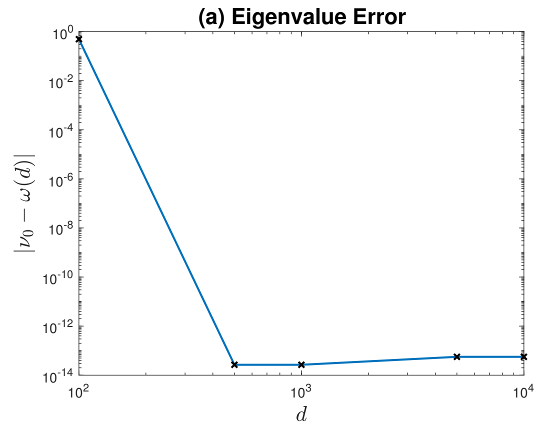

Figure 3.3.: (a) Numerical convergence test for the eigenvalue $\omega = \nu_0 \approx 0.494$ of $T_{k_0,\omega} := L(k_0) + \omega\Lambda$ for $k_0 = 0.5$ in dependence on the computational box size d.

smaller than 10^{-6}. *Our computations suggest that there are no other eigenvalues in a neighborhood of* ν_0 *and that therefore* ν_0 *is an isolated eigenvalue, as demanded by Assumption* (A2).

Remark 3.4.2

For the approximative solution in Chapter 4 it will be necessary to determine $\partial_k\omega(k_0)$ *and* $\partial_k^2\omega(k_0)$. *Let us now present two different methods to calculate these values.*

First method: *We solve the linear eigenvalue problem* (3.2.2) *not only for* k_0, *but instead for multiple values* k *in a small neighborhood of* k_0, *i.e. we determine the dispersion relation in a neighborhood of* k_0. *Then we use finite differences to approximate* $\partial_k\omega(k_0)$ *and* $\partial_k^2\omega(k_0)$.

For our calculations we will always use this method since we already determined the dispersion relation in a neighborhood of k_0.

Second method: *We use* (3.2.2) *to derive an explicit expression for* $\partial_k\omega(k_0)$ *and* $\partial_k^2\omega(k_0)$. *Note that we will suppress the* x_1*-dependency and explicitly write the* k*-dependence in the following calculation. By differentiating* (3.2.2) *in* k *and testing it with the solution* $\boldsymbol{m} = \boldsymbol{w}(k_0)$ *we get in* $k = k_0$

$$\partial_k\omega(k_0)\langle\Lambda\boldsymbol{m},\boldsymbol{m}\rangle_{L^2(\mathbb{R})^3} = -\langle\partial_k L(k_0)\boldsymbol{m},\boldsymbol{m}\rangle_{L^2(\mathbb{R})^3} - \langle(L(k_0)+\omega(k_0)\Lambda)\partial_k\boldsymbol{w}(k_0),\boldsymbol{m}\rangle_{L^2(\mathbb{R})^3}, \tag{3.4.3}$$

where $\partial_k L(k_0)\boldsymbol{m} = (m_3, 0, m_1)^\top$. *Note that by Corollary 3.3.9 the differentiation in* k *is possible. Since* $L(k_0) + \omega(k_0)\Lambda$ *is a self-adjoint operator, see Corollary 3.3.2, it follows that*

$$\langle(L(k_0)+\omega(k_0)\Lambda)\partial_k\boldsymbol{w}(k_0),\boldsymbol{m}\rangle_{L^2(\mathbb{R})^3} = \langle\partial_k\boldsymbol{w}(k_0),(L(k_0)+\omega(k_0)\Lambda)\boldsymbol{m}\rangle_{L^2(\mathbb{R})^3} = 0,$$

and with the normalization (3.2.4) *we deduce from* (3.4.3) *that*

$$\partial_k\omega(k_0) = -2\int_{\mathbb{R}} m_1 m_3 \,\mathrm{d}x_1.$$

Note that we used Remark 3.2.6 and that therefore m_1, m_3 *are real-valued functions.*

To derive a formula for $\partial_k^2\omega(k_0)$ we differentiate (3.2.2) *two times in k. With the same arguments as before one arrives at*

$$\partial_k^2\omega(k_0) = -2\langle(\partial_k L(k_0) + \partial_k\omega(k_0)\Lambda)\partial_k \boldsymbol{w}(k_0), \boldsymbol{m}\rangle_{L^2(\mathbb{R})^3}.$$

To determine $\partial_k \boldsymbol{w}(k_0)$ we now have to solve

$$(L(k_0) + \omega(k_0)\Lambda)\partial_k \boldsymbol{w}(k_0) = (\partial_k L(k_0) + \partial_k\omega(k_0)\Lambda)\boldsymbol{m} =: \boldsymbol{f}$$

with $[\![\partial_k w_2(k_0)]\!]_{1\mathrm{D}}(0) = [\![\partial_k w_3(k_0)]\!]_{1\mathrm{D}}(0) = 0$.
This can again be transformed to a second order ODE for $\partial_k w_3(k_0)$:

$$\begin{cases} \mathcal{L}_{k_0,\nu_0}\partial_k w_3(k_0) := \left(-\partial_{x_1}^2 + \dfrac{\partial_{x_1}\epsilon_1(x_1)}{\epsilon_1(x_1)}\,\partial_{x_1} + k_0^2 - \epsilon_1(x_1)\mu_0\nu_0^2\right)\partial_k w_3(k_0) \\ \qquad = k_0 f_1 - \nu_0\epsilon_1(x_1) f_3 - \mathrm{i}\dfrac{\partial_{x_1}\epsilon_1(x_1)}{\epsilon_1(x_1)} f_2 + \mathrm{i}\partial_{x_1} f_2, \qquad x_1 \in \mathbb{R}\backslash\{0\}, \\ [\![\partial_k w_3(k_0)]\!]_{1\mathrm{D}}(0) = \left[\!\!\left[\dfrac{\partial_{x_1}\partial_k w_3(k_0)}{\epsilon_1}\right]\!\!\right]_{1\mathrm{D}}(0) = 0. \end{cases} \tag{3.4.4}$$

For the transformation of the interface conditions we used that

$$\left[\!\!\left[\frac{f_2}{\nu_0\epsilon_1}\right]\!\!\right]_{1\mathrm{D}}(0) = \left[\!\!\left[\frac{\nu_0\epsilon_1 m_2}{\nu_0\epsilon_1}\right]\!\!\right]_{1\mathrm{D}}(0) = 0.$$

Now one can proceed similarly to the calculation of w_3, but instead of a generalized eigenvalue problem, one has to solve an inhomogeneous ordinary differential equation. Note that the solution of (3.4.4) *is not unique, since $w_3(k_0)$ is a solution of $\mathcal{L}_{k_0,\nu_0} w_3(k_0) = 0$. To circumvent this problem one has to find solutions in the orthogonal complement of* $\operatorname{span}\{w_3(k_0)\}$*, e.g. by employing a biconjugate gradient method or other iterative methods in combination with projections, see e.g [54, Chapter 4]. Note that for the first method* (3.2.2) *has to be solved multiple times. For the second method* (3.4.4) *has to be solved only once.*

All in all, we see that it is possible to solve the linear Maxwell problem, at least numerically. With Lemma 3.3.7, Corollary 3.3.9 and Lemma 3.3.11 we have also established the necessary tools to analyze the nonlinear Maxwell problem in the following chapters.

4. Formal Asymptotic Solution of the Nonlinear Problem

Let us formally derive an asymptotic solution of Maxwell's equations with the help of the method of amplitude equation. But first, we state the exact problem we want to solve.

In Section 3.1 we introduced transverse magnetic modes for solutions of the linear Maxwell problem. Since our asymptotic ansatz is based on the solutions of the linear problem, we therefore also use the reduction to TM-modes for the nonlinear problem and set

$$\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) = (\mathcal{E}_1(\boldsymbol{x},t), \mathcal{E}_2(\boldsymbol{x},t), 0)^\top, \qquad \boldsymbol{\mathcal{H}}(\boldsymbol{x},t) = (0, 0, \mathcal{H}_3(\boldsymbol{x},t))^\top. \tag{4.0.1}$$

As before we use the displacement field

$$\boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{E}}) = \epsilon_1 \boldsymbol{\mathcal{E}} + \epsilon_3 (\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{E}}) \boldsymbol{\mathcal{E}}$$

with $\epsilon_1 = \epsilon_0(1 + \chi_1)$ and $\epsilon_3 = \epsilon_0 \chi_3$.

Remark 4.0.1

Reduction (4.0.1) *to TM-modes is consistent with the chosen nonlinearity in the displacement field since Ampère's circuital law* $\partial_t \boldsymbol{\mathcal{D}} = \nabla \times \boldsymbol{\mathcal{H}}$ *can be written as*

$$\partial_t \boldsymbol{\mathcal{D}} = \epsilon_0 \partial_t \left((1+\chi_1) \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ 0 \end{pmatrix} + \chi_3 \left(\mathcal{E}_1^2 + \mathcal{E}_2^2 \right) \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} \partial_{x_2} \mathcal{H}_3 \\ -\partial_{x_1} \mathcal{H}_3 \\ 0 \end{pmatrix} = \nabla \times \boldsymbol{\mathcal{H}}.$$

For a more general nonlinearity as in (2.2.4) *we have to guarantee that*

$$\partial_t (\chi_3(x_1, \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}))_3 = \partial_t \left(\sum_{k,l,m=1}^{2} \chi_{3,3klm}(x_1) \mathcal{E}_k \mathcal{E}_l \mathcal{E}_m \right) = 0$$

to be consistent with TM-modes. This is for example satisfied if

$$\sum_{k,l,m=1}^{2} \chi_{3,3klm}(x_1) \mathcal{E}_k \mathcal{E}_l \mathcal{E}_m = 0.$$

Let $T' > 0$. With (4.0.1) we can reduce (2.2.10) and (2.2.11) to the following TM-Maxwell's equations

$$\begin{cases} \partial_t \mathcal{D}_1 - \partial_{x_2} \mathcal{H}_3 = 0, \\ \partial_t \mathcal{D}_2 + \partial_{x_1} \mathcal{H}_3 = 0, \\ -\partial_{x_2} \mathcal{E}_1 + \partial_{x_1} \mathcal{E}_2 + \mu_0 \partial_t \mathcal{H}_3 = 0, \\ \nabla \cdot \boldsymbol{\mathcal{D}} = \varrho_0, \\ \nabla \cdot \boldsymbol{\mathcal{H}} = 0 \end{cases}$$

on $\left(\mathbb{R}^2 \backslash \Gamma_2\right) \times (0, T')$ with the interface conditions

$$[\![\mathcal{D}_1]\!]_{2D} = \varrho_\Gamma, \quad [\![\mathcal{E}_2]\!]_{2D} = [\![\mathcal{H}_3]\!]_{2D} = 0$$

on $\Gamma_2 \times [0, T')$.

This can be reduced even further. First, note that the divergence equation for $\boldsymbol{\mathcal{H}}$ is always satisfied since the only non-trivial component $\mathcal{H}_3$ is independent of x_3. For the other divergence equation we use that solutions of Maxwell's equations satisfy $\partial_t \boldsymbol{\mathcal{D}} = \nabla \times \boldsymbol{\mathcal{H}}$. This implies that

$$\partial_t (\nabla \cdot \boldsymbol{\mathcal{D}}) = \nabla \cdot (\partial_t \boldsymbol{\mathcal{D}}) = \nabla \cdot (\nabla \times \boldsymbol{\mathcal{H}}) = 0.$$

Therefore, $\nabla \cdot \boldsymbol{\mathcal{D}} = \varrho_0$ is satisfied for all time $t > 0$, if the initial value for $\boldsymbol{\mathcal{D}}$ satisfies this divergence condition. Similarly, we get from $\partial_t \mathcal{D}_1 = \partial_{x_2} \mathcal{H}_3$ that

$$\partial_t [\![\mathcal{D}_1]\!]_{2D} = [\![\partial_{x_2} \mathcal{H}_3]\!]_{2D} = \partial_{x_2} [\![\mathcal{H}_3]\!]_{2D} = 0.$$

Hence, we only have to check the interface condition for $\mathcal{D}_1$ at the initial time $t = 0$.

Remark 4.0.2
Note that the tangential derivative of a jump is the jump of the tangential derivatives. This follows by Hadamard's lemma, see e.g. [83, Section 173-175].

We will combine the three non-trivial components of the electromagnetic field into the vector

$$\boldsymbol{U}(\boldsymbol{x}, t) := (\mathcal{E}_1(\boldsymbol{x}, t), \mathcal{E}_2(\boldsymbol{x}, t), \mathcal{H}_3(\boldsymbol{x}, t))^\top$$

and also write

$$\boldsymbol{U}_E(\boldsymbol{x}, t) := (\mathcal{E}_1(\boldsymbol{x}, t), \mathcal{E}_2(\boldsymbol{x}, t), 0)^\top$$

to denote the part of $\boldsymbol{U}$ that corresponds to the electric field.

For initial data $\boldsymbol{U}^{(0)} : \mathbb{R}^2 \to \mathbb{R}^3$, our reduced Maxwell problem is now given by

$$\begin{pmatrix} \epsilon_1^{\pm} & 0 & 0 \\ 0 & \epsilon_1^{\pm} & 0 \\ 0 & 0 & \mu_0 \end{pmatrix} \partial_t \boldsymbol{U}^{\pm} + \epsilon_3^{\pm} \partial_t \begin{pmatrix} \left(U_1^{\pm^2} + U_2^{\pm^2}\right) U_1^{\pm} \\ \left(U_1^{\pm^2} + U_2^{\pm^2}\right) U_2^{\pm} \\ 0 \end{pmatrix} + \begin{pmatrix} -\partial_{x_2} U_3^{\pm} \\ \partial_{x_1} U_3^{\pm} \\ \partial_{x_1} U_2^{\pm} - \partial_{x_2} U_1^{\pm} \end{pmatrix} = \mathbf{0} \tag{4.0.2}$$

on $\mathbb{R}^2_{\pm} \times (0, T')$ with

$$\boldsymbol{U}^{\pm}(\cdot, 0) = \boldsymbol{U}^{(0),\pm} \qquad \text{on } \mathbb{R}^2_{\pm}, \tag{4.0.3}$$

and the interface conditions

$$[\![U_2]\!]_{2\mathrm{D}} = [\![U_3]\!]_{2\mathrm{D}} = 0 \qquad \text{on } \Gamma_2 \times [0, T'). \tag{4.0.4}$$

To get a solution of Maxwell's equations for a prescribed volume charge density ϱ_0 and a prescribed surface charge density ϱ_Γ, the initial condition $\boldsymbol{U}_E^{(0)}$ must also be chosen such that the divergence condition

$$\begin{aligned} \partial_{x_1} \mathcal{D}_1 \left(\boldsymbol{U}_E^{(0),\pm}\right) + \partial_{x_2} \mathcal{D}_2 \left(\boldsymbol{U}_E^{(0),\pm}\right) &= \partial_{x_1} \left(\epsilon_1^{\pm} U_1^{(0),\pm} + \epsilon_3^{\pm} \left(U_1^{(0),\pm^2} + U_2^{(0),\pm^2}\right) U_1^{(0),\pm}\right) \\ &\quad + \partial_{x_2} \left(\epsilon_1^{\pm} U_2^{(0),\pm} + \epsilon_3^{\pm} \left(U_1^{(0),\pm^2} + U_2^{(0),\pm^2}\right) U_2^{(0),\pm}\right) \\ &= \varrho_0 \qquad \text{on } \mathbb{R}^2_{\pm}, \end{aligned} \tag{4.0.5}$$

and the interface condition

$$\left[\!\!\left[\mathcal{D}_1 \left(\boldsymbol{U}_E^{(0)}\right)\right]\!\!\right]_{2\mathrm{D}} = \left[\!\!\left[\epsilon_1 U_1^{(0)} + \epsilon_3 \left(U_1^{(0)^2} + U_2^{(0)^2}\right) U_1^{(0)}\right]\!\!\right]_{2\mathrm{D}} = \varrho_\Gamma \qquad \text{on } \Gamma_2 \tag{4.0.6}$$

are satisfied. System (4.0.2), (4.0.3) and (4.0.4) is the problem treated by our approximation result and will be the main focus of our analysis in the following sections. The problem of finding suitable initial values for (4.0.5), (4.0.6) will be discussed separately in Chapter 7.

Remark 4.0.3
Note that the divergence equation $\nabla \cdot \boldsymbol{\mathcal{D}} = \varrho_0$ *is not contained in* (4.0.2). *We will nevertheless use this equation in Section 6.2 to estimate* $\partial_{x_1} \mathcal{D}_1$.

4.1. Construction of the Asymptotic Solution

Our main goal is the derivation and justification of an asymptotic solution of (4.0.2), (4.0.3) and (4.0.4). We will follow the same idea as presented in Example 2 in Section 2.3.

To treat the linear and nonlinear terms in a similar fashion, we remember Assumptions (A1), (A3) and assume analogously that there are constants $\epsilon_{3,m}^{\pm}$, $\epsilon_{3,M}^{\pm}$ such that

$$\epsilon_3^{\pm} \in C^3(\mathbb{R}_{\pm}) \cap W^{3,\infty}(\mathbb{R}_{\pm}), \quad \epsilon_{3,m}^{\pm} \leqslant \epsilon_3^{\pm}(x_1) \leqslant \epsilon_{3,M}^{\pm}, \qquad \forall\, x_1 \in \mathbb{R}\backslash\{0\} \tag{A6}$$

and that there are constants $\epsilon_3^{\pm\infty}$ such that

$$\epsilon_3^{\pm}(x_1) \to \epsilon_3^{\pm\infty} \quad \text{as} \quad x_1 \to \pm\infty. \tag{A7}$$

Furthermore, assume (A2), (A3) and (A4) to use all the results of Chapter 3. Then there exists a wave number $k_0 \in \mathbb{R}$, a wave frequency $\nu_0 = \omega(k_0)$, a group velocity $\nu_1 := \partial_k\omega(k_0)$, a constant $\nu_2 := \partial_k^2\omega(k_0)$ and the eigenfunction $\boldsymbol{m}(x_1) := \boldsymbol{w}(x_1, k_0)$.
Asymptotically, we consider a wave packet based on the carrier wave

$$\boldsymbol{m}(x_1)\mathrm{e}^{\mathrm{i}(k_0x_2-\nu_0t)}, \qquad (\boldsymbol{x}, t) \in \left(\mathbb{R}^2\backslash\Gamma_2\right) \times [0,\infty),$$

which solves the linear Maxwell problem, i.e. (4.0.2) for $\epsilon_3 = 0$ and with $\varrho_0 = 0$ and $\varrho_\Gamma = 0$.
Now we make the ansatz

$$\boldsymbol{U}_{\text{ans}}(\boldsymbol{x}, t) := \begin{pmatrix} \mathcal{E}_{\text{ans},1}(\boldsymbol{x}, t) \\ \mathcal{E}_{\text{ans},2}(\boldsymbol{x}, t) \\ \mathcal{H}_{\text{ans},3}(\boldsymbol{x}, t) \end{pmatrix} := \varepsilon A\left(\varepsilon(x_2 - \nu_1 t), \varepsilon^2 t\right) \boldsymbol{m}(x_1)\mathrm{e}^{\mathrm{i}(k_0x_2-\nu_0t)} + \text{c.c.} \tag{4.1.1}$$

with a complex envelope $A = A(X_2, T) : \mathbb{R} \times [0,\infty) \to \mathbb{C}$ and a small parameter $0 < \varepsilon \ll 1$. Since we are interested in TM-modes, we of course choose $\mathcal{E}_{\text{ans},3} = \mathcal{H}_{\text{ans},1} = \mathcal{H}_{\text{ans},2} = 0$.
The envelope depends on the slow variables $X_2 := \varepsilon(x_2 - \nu_1 t)$ and $T := \varepsilon^2 t$ and travels with the group velocity ν_1. If A is localized we get that $\boldsymbol{U}_{\text{ans}}$ is localized in x_1 and x_2 and is traveling in x_2-direction.
Our goal is to show that $\boldsymbol{U}_{\text{ans}}$ is a "good" approximation for a solution of (4.0.2) if A is a solution of an effective nonlinear Schrödinger equation. At the end, we want a theorem analogous to Theorem 2.3.1:

Goal 4.1.1 (Approximation Result)
Let $T_0 > 0$ and A be a solution of an effective nonlinear Schrödinger equation. Then there exists an $\varepsilon_0 > 0$ and a constant $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there is a solution $\boldsymbol{U}$ of (4.0.2), (4.0.3), (4.0.4) *in a suitable function space X with*

$$\|\boldsymbol{U}(t) - \boldsymbol{U}_{\text{ans}}(t)\|_X \leqslant C\varepsilon^{3/2}, \qquad \forall\, t \in \left[0, T_0\varepsilon^{-2}\right].$$

Remark 4.1.2

A result like Goal 4.1.1 would give us the existence of a solution $\boldsymbol{U}$ that is close to a wave packet over a long period of time, which is interesting from an analytical point of view.
From a numerical point of view we notice that it is in general much simpler to calculate $\boldsymbol{U}_{\text{ans}}$ than it is to calculate a solution of the full Maxwell problem with a standard numerical method, e.g. the finite element method. This has multiple reasons. First, we note that we no longer have to solve a problem on $\mathbb{R}^2 \times \left(0, T_0\varepsilon^{-2}\right)$, instead we have to solve the simpler problems for $\boldsymbol{m}$ and $\omega(k)$ on $\mathbb{R}$, see Section 3.4, and for A on $\mathbb{R} \times (0, T_0)$, see Section 2.4. Second, we see that due to the ε-dependence we can use a rougher discretization for the slow variables X_2 and T compared to the discretization of x_2 and t. Additionally, we have that the calculations for $\boldsymbol{m}$, $\omega(k)$ and A are independent of ε and that the calculated results can be reused for different ε. Numerical tests for the method of amplitude equations can be found in [84, 22, 26].

4.1.1. Derivation of the Nonlinear Schrödinger Equation

The first step to achieve Goal 4.1.1 is to find an equation for A and higher-order corrections of $\boldsymbol{U}_{\text{ans}}$ such that the residual

$$\mathbf{Res}(\boldsymbol{U}_{\text{ans}}) := \begin{pmatrix} \partial_t \mathcal{D}_1(\boldsymbol{U}_{\text{ans},E}) - \partial_{x_2} U_{\text{ans},3} \\ \partial_t \mathcal{D}_2(\boldsymbol{U}_{\text{ans},E}) + \partial_{x_1} U_{\text{ans},3} \\ -\partial_{x_2} U_{\text{ans},1} + \partial_{x_1} U_{\text{ans},2} + \mu_0 \partial_t U_{\text{ans},3} \end{pmatrix} \tag{4.1.2}$$

of (4.1.1) in the Maxwell problem (4.0.2) is small enough for the rigorous analysis of Chapter 6. Note that we use the abbreviation $\boldsymbol{U}_{\text{ans},E} := (\mathcal{E}_{\text{ans},1}, \mathcal{E}_{\text{ans},2}, 0)^\top$. This section will be focused on the analysis of (4.1.2), the discussion of the residual in the interface conditions will follow in Section 4.1.2.
In view of the linear eigenvalue problem it is reasonable to work in Fourier variables and transform between x_2 and k via

$$\mathcal{F}(f)(k) := \widehat{f}(k) := (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x_2) \mathrm{e}^{-\mathrm{i}kx_2} \, \mathrm{d}x_2.$$

The inverse transform is then given by

$$\mathcal{F}^{-1}(f)(x_2) := \check{f}(x_2) := (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(k) \mathrm{e}^{\mathrm{i}x_2 k} \, \mathrm{d}k$$

and we obtain with the convolution theorem, see e.g. [33],

$$\begin{aligned} \mathcal{F}(f * g) &= (2\pi)^{1/2} \mathcal{F}(f) \mathcal{F}(g), \\ \mathcal{F}(fg) &= (2\pi)^{1/2} \mathcal{F}(f) * \mathcal{F}(g). \end{aligned}$$

We compute

$$\begin{aligned}\int_{-\infty}^{\infty} \varepsilon A\left(\varepsilon(x_2-\nu_1 t),\varepsilon^2 t\right) \mathrm{e}^{\mathrm{i}(k_0x_2-\nu_0 t)}\mathrm{e}^{-\mathrm{i}kx_2}\,\mathrm{d}x_2 &= \int_{-\infty}^{\infty} \varepsilon A\left(\varepsilon(x_2-\nu_1 t),\varepsilon^2 t\right)\mathrm{e}^{-\mathrm{i}((k-k_0)x_2+\nu_0 t)}\,\mathrm{d}x_2\\ &= \int_{-\infty}^{\infty} A\left(X_2,\varepsilon^2 t\right)\mathrm{e}^{-\mathrm{i}\left(\frac{k-k_0}{\varepsilon}X_2+(k-k_0)\nu_1 t+\nu_0 t\right)}\,\mathrm{d}X_2\\ &= (2\pi)^{1/2}\widehat{A}\left(\frac{k-k_0}{\varepsilon},\varepsilon^2 t\right)\mathrm{e}^{-\mathrm{i}(\nu_0+(k-k_0)\nu_1)t},\end{aligned} \tag{4.1.3}$$

where we used the substitution $X_2=\varepsilon(x_2-\nu_1 t)$. Therefore, our transformed ansatz has the form

$$\widehat{\boldsymbol{U}}_{\text{ans}}(x_1,k,t) := \begin{pmatrix}\widehat{\mathcal{E}}_{\text{ans},1}(x_1,k,t)\\ \widehat{\mathcal{E}}_{\text{ans},2}(x_1,k,t)\\ \widehat{\mathcal{H}}_{\text{ans},3}(x_1,k,t)\end{pmatrix} := \widehat{A}\left(\frac{k-k_0}{\varepsilon},\varepsilon^2 t\right)\boldsymbol{m}(x_1)\mathrm{e}^{-\mathrm{i}(\nu_0+(k-k_0)\nu_1)t}+\widehat{\text{c.c.}}, \tag{4.1.4}$$

where $\widehat{\text{c.c.}}(\widehat{f})=\widehat{\text{c.c.}(f)}$ and we note that $\overline{\widehat{f}}(-k)=\widehat{\overline{f}}(k)$.
Maxwell's equations (4.0.2) transform to

$$\begin{cases}\partial_t\widehat{\mathcal{D}}_1-\mathrm{i}k\widehat{\mathcal{H}}_3=0,\\ \partial_t\widehat{\mathcal{D}}_2+\partial_{x_1}\widehat{\mathcal{H}}_3=0,\\ -\mathrm{i}k\widehat{\mathcal{E}}_1+\partial_{x_1}\widehat{\mathcal{E}}_2+\mu_0\partial_t\widehat{\mathcal{H}}_3=0,\end{cases} \tag{4.1.5}$$

with

$$\widehat{\boldsymbol{\mathcal{D}}}=\widehat{\boldsymbol{\mathcal{D}}}(\boldsymbol{\mathcal{E}})=\epsilon_1\widehat{\boldsymbol{\mathcal{E}}}+\epsilon_3((\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{E}})\boldsymbol{\mathcal{E}})^\wedge.$$

In what follows we use the notations $E_1:=\mathrm{e}^{-\mathrm{i}(\nu_0+(k-k_0)\nu_1)t}$, $F_1:=\mathrm{e}^{\mathrm{i}(k_0x_2-\nu_0 t)}$, $K:=\frac{k-k_0}{\varepsilon}$, $T:=\varepsilon^2 t$ and $X_2=\varepsilon(x_2-\nu_1 t)$. We will suppress the arguments of $\boldsymbol{m}$ and $\widehat{A}$ as well as their derivatives if they are obvious.
Looking at (4.1.5) we see that we need the following derivatives:

$$\begin{aligned}\partial_t\widehat{\boldsymbol{U}}_{\text{ans}} &= \boldsymbol{m}E_1\left(-i\nu_0\widehat{A}-\varepsilon\mathrm{i}K\nu_1\widehat{A}+\varepsilon^2\partial_T\widehat{A}\right)+\widehat{\text{c.c.}},\\ \partial_{x_1}\widehat{\boldsymbol{U}}_{\text{ans}} &= \partial_{x_1}\boldsymbol{m}E_1\widehat{A}+\widehat{\text{c.c.}}\end{aligned}$$

To analyze the Fourier transform of $\boldsymbol{\mathcal{D}}$ we split the displacement field in its linear and nonlinear parts $\boldsymbol{\mathcal{D}}=\boldsymbol{\mathcal{D}}_{\text{lin}}+\boldsymbol{\mathcal{D}}_{\text{nl}}$ with

$$\boldsymbol{\mathcal{D}}_{\text{lin}}:=\epsilon_1\boldsymbol{\mathcal{E}},\qquad \boldsymbol{\mathcal{D}}_{\text{nl}}:=\epsilon_3(\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{E}})\boldsymbol{\mathcal{E}}.$$

We first calculate the nonlinear term in physical variables and apply the Fourier transformation afterwards. For our chosen ansatz we get

$$\begin{aligned}\mathcal{D}_{\mathrm{nl},1}(\boldsymbol{U}_{\mathrm{ans},E}) = &\varepsilon^3\epsilon_3 F_1|A|^2A\left(3|m_1|^2m_1 + 2|m_2|^2m_1 + m_2^2\overline{m}_1\right)\\ &+ \varepsilon^3\epsilon_3F_1^3A^3\left(m_1^3 + m_1m_2^2\right) + \text{c.c.},\end{aligned}$$

where we used $|F_1| = 1$.
Now the temporal derivative is

$$\begin{aligned}\partial_t\mathcal{D}_{\mathrm{nl},1}(\boldsymbol{U}_{\mathrm{ans},E}) = &-\varepsilon^3 3\mathrm{i}\epsilon_3\nu_0F_1^3A^3\left(m_1^3 + m_1m_2^2\right)\\ &-\varepsilon^3\mathrm{i}\epsilon_3\nu_0F_1|A|^2A\left(3|m_1|^2m_1 + 2|m_2|^2m_1 + m_2^2\overline{m}_1\right) + \mathcal{O}(\varepsilon^4) + \text{c.c.}\end{aligned}$$

The Fourier transformation of $F_1|A|^2A$ can be calculated via

$$\begin{aligned}(2\pi)^{-1/2}&\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(k_0x_2-\nu_0t)}|A(X_2,T)|^2A(X_2,T)\mathrm{e}^{-\mathrm{i}kx_2}\,\mathrm{d}x_2\\ &= (2\pi)^{-1/2}\mathrm{e}^{-\mathrm{i}\nu_0t}\int_{-\infty}^{\infty}|A(X_2,T)|^2A(X_2,T)\mathrm{e}^{-\varepsilon\mathrm{i}Kx_2}\,\mathrm{d}x_2\\ &= (2\pi)^{-1/2}\varepsilon^{-1}E_1\int_{-\infty}^{\infty}|A(X_2,T)|^2A(X_2,T)\mathrm{e}^{-\mathrm{i}KX_2}\,\mathrm{d}X_2\\ &= 2\pi\varepsilon^{-1}E_1\left(\widehat{A}*_K\widehat{\overline{A}}*_K\widehat{A}\right)(K,T).\end{aligned}$$

We used $X_2 = \varepsilon(x_2 - \nu_1t)$, $k = k_0 + \varepsilon K$ and in the last step we applied the convolution theorem to the cubic term. The calculation for the so-called higher harmonics, i.e. the terms proportional to F_1^3, is similar but we have to introduce the new abbreviations $\widetilde{K} := \frac{k-3k_0}{\varepsilon}$ and $E_3 := \mathrm{e}^{-\mathrm{i}(3\nu_0+(k-3k_0)\nu_1)t}$:

$$\begin{aligned}(2\pi)^{-1/2}&\int_{-\infty}^{\infty}\mathrm{e}^{3\mathrm{i}(k_0x_2-\nu_0t)}A^3(X_2,T)\mathrm{e}^{-\mathrm{i}kx_2}\,\mathrm{d}x_2\\ &= (2\pi)^{-1/2}\mathrm{e}^{-3\mathrm{i}\nu_0t}\int_{-\infty}^{\infty}A^3(\varepsilon(x_2-\nu_1t),T)\mathrm{e}^{-\mathrm{i}(k-3k_0)x_2}\,\mathrm{d}x_2\\ &= (2\pi)^{-1/2}\mathrm{e}^{-\mathrm{i}(3\nu_0+(k-3k_0)\nu_1)t}\int_{-\infty}^{\infty}A^3(\varepsilon(x_2-\nu_1t),T)\mathrm{e}^{-\mathrm{i}\frac{k-3k_0}{\varepsilon}\varepsilon(x_2-\nu_1t)}\,\mathrm{d}x_2\\ &= (2\pi)^{-1/2}\varepsilon^{-1}E_3\int_{-\infty}^{\infty}A^3(X_2,T)\mathrm{e}^{-\mathrm{i}\widetilde{K}X_2}\,\mathrm{d}X_2\\ &= 2\pi\varepsilon^{-1}E_3\left(\widehat{A}*_K\widehat{A}*_K\widehat{A}\right)(\widetilde{K},T).\end{aligned}$$

All in all, we get

$$\begin{aligned}\widehat{\mathcal{D}}_{\mathrm{nl},1}(\boldsymbol{U}_{\mathrm{ans},E}) = &2\pi\varepsilon^2\epsilon_3E_1\left(3|m_1|^2m_1 + 2|m_2|^2m_1 + m_2^2\overline{m}_1\right)\left(\widehat{A}*_K\widehat{\overline{A}}*_K\widehat{A}\right)(K,T)\\ &+ 2\pi\varepsilon^2\epsilon_3E_3\left(m_1^3 + m_1m_2^2\right)\left(\widehat{A}*_K\widehat{A}*_K\widehat{A}\right)(\widetilde{K},T) + \widehat{\text{c.c.}}\end{aligned}$$

and

$$\begin{aligned}\partial_t \widehat{\mathcal{D}}_{\mathrm{nl},1}(\boldsymbol{U}_{\mathrm{ans},E}) = &-2\pi\mathrm{i}\varepsilon^2\epsilon_3\nu_0 E_1\left(3|m_1|^2 m_1 + 2|m_2|^2 m_1 + m_2^2\overline{m}_1\right)\left(\widehat{A} *_K \widehat{\overline{A}} *_K \widehat{A}\right)(K,T)\\ &-2\pi\mathrm{i}\varepsilon^2 3\epsilon_3\nu_0 E_3\left(m_1^3 + m_1 m_2^2\right)\left(\widehat{A} *_K \widehat{A} *_K \widehat{A}\right)(\widetilde{K},T) + \mathcal{O}(\varepsilon^3) + \widehat{\mathrm{c.c.}}\end{aligned}$$

The second components $\widehat{\mathcal{D}}_{\mathrm{nl},2}(\boldsymbol{U}_{\mathrm{ans},E})$ and $\partial_t\widehat{\mathcal{D}}_{\mathrm{nl},2}(\boldsymbol{U}_{\mathrm{ans},E})$ are obtained by simply switching the indices 1 and 2 and the third component is obviously always zero. In the following we will use that we selected the eigenfunction such that m_1 is real and m_2 is imaginary, see Remark 3.2.6, hence

$$\begin{aligned}3|m_1|^2 m_1 + 2|m_2|^2 m_1 + m_2^2\overline{m}_1 &= 3m_1^3 - m_2^2 m_1,\\ 3|m_2|^2 m_2 + 2|m_1|^2 m_2 + m_1^2\overline{m}_2 &= -3m_2^3 + m_1^2 m_2.\end{aligned}$$

Before we can start to put everything together and to compare powers of ε, we need to Taylor expand the eigenvalue problem (3.2.2) in k at k_0. By Corollary 3.3.9 the Taylor expansions of $\omega(k)$ and $\boldsymbol{w}(k)$ at k_0 exist and we get

$$\begin{aligned}\omega(k) &= \omega(k_0+\varepsilon K) = \nu_0 + \varepsilon K\nu_1 + \frac{1}{2}\varepsilon^2 K^2\nu_2 + \mathcal{O}(\varepsilon^3),\\ \boldsymbol{w}(k) &= \boldsymbol{w}(k_0+\varepsilon K) = \boldsymbol{m} + \varepsilon K\partial_k\boldsymbol{w}(k_0) + \frac{1}{2}\varepsilon^2 K^2\partial_k^2\boldsymbol{w}(k_0) + \mathcal{O}(\varepsilon^3).\end{aligned}$$

The Taylor expansion of the operator $L(k)$, see (3.1.11), is given by

$$L(k) = L(k_0+\varepsilon K) = L_0 + \varepsilon K L_1$$

with the operators L_0 and L_1 defined as

$$L_0\boldsymbol{m} := L(k_0)\boldsymbol{m} = \begin{pmatrix} k_0 m_3\\ \mathrm{i}\partial_{x_1} m_3\\ k_0 m_1 + \mathrm{i}\partial_{x_1} m_2\end{pmatrix},\qquad L_1\boldsymbol{m} := (\partial_k L(k_0))\boldsymbol{m} = \begin{pmatrix} m_3\\ 0\\ m_1\end{pmatrix}.$$

Note that all higher k-derivatives of L vanish.
Inserting the Taylor expansion into (3.2.2) and comparing the powers of ε, we get for ε^0

$$(L_0 + \nu_0\Lambda)\boldsymbol{m} = \boldsymbol{0}. \tag{4.1.6}$$

For ε^1 we get

$$K(L_1 + \nu_1\Lambda)\boldsymbol{m} + K(L_0 + \nu_0\Lambda)\partial_k\boldsymbol{w}(k_0) = \boldsymbol{0}. \tag{4.1.7}$$

Finally, for ε^2 we get the equation

$$\frac{1}{2}K^2\nu_2\Lambda\boldsymbol{m} + K^2(L_1 + \nu_1\Lambda)\partial_k\boldsymbol{w}(k_0) + \frac{1}{2}K^2(L_0 + \nu_0\Lambda)\partial_k^2\boldsymbol{w}(k_0) = \boldsymbol{0}. \tag{4.1.8}$$

Now we have all tools to determine the equation for A. Inserting ansatz (4.1.4) into the left-hand side of Maxwell's equations (4.1.5) gives us the following equation for the Fourier transformed residual:

$$\widehat{\mathbf{Res}}(\boldsymbol{U}_{\text{ans}}) := \begin{pmatrix} \partial_t \widehat{\mathcal{D}}_1(\boldsymbol{U}_{\text{ans},E}) - \mathrm{i}k\widehat{U}_{\text{ans},3} \\ \partial_t \widehat{\mathcal{D}}_2(\boldsymbol{U}_{\text{ans},E}) + \partial_{x_1}\widehat{U}_{\text{ans},3} \\ -\mathrm{i}k\widehat{U}_{\text{ans},1} + \partial_{x_1}\widehat{U}_{\text{ans},2} + \mu_0\partial_t \widehat{U}_{\text{ans},3} \end{pmatrix}. \tag{4.1.9}$$

We want that $\widehat{\mathbf{Res}}(\boldsymbol{U}_{\text{ans}})$ is formally of order ε^3. To order ε^0 the residual contains the term

$$-\mathrm{i}E_1\widehat{A}(L_0 + \nu_0\Lambda)\boldsymbol{m} + \widehat{\text{c.c.}}$$

We see that this term is zero due to (4.1.6).
To order ε^1 we get

$$-\mathrm{i}KE_1\widehat{A}(L_1 + \nu_1\Lambda)\boldsymbol{m} + \widehat{\text{c.c.}}$$

In comparison with (4.1.7) we are missing the term $K(L_0 + \nu_0\Lambda)\partial_k \boldsymbol{w}(k_0)$. In order to guarantee that the residual does not contain terms of order ε^1 we extend our ansatz to

$$\widehat{\boldsymbol{U}}_{\text{mod1}}(x_1, k, t) := \widehat{A}\left(\frac{k-k_0}{\varepsilon}, \varepsilon^2 t\right)(\boldsymbol{m}(x_1) + \varepsilon K\partial_k \boldsymbol{w}(x_1, k_0))\,\mathrm{e}^{-\mathrm{i}(\nu_0+(k-k_0)\nu_1)t} + \widehat{\text{c.c.}}$$

A simple computation shows that this new ansatz delivers the same terms of order ε^0 and has all required terms of order ε^1. Hence, our new modified ansatz satisfies Maxwell's equations up to a residual of order ε^2.
Looking at (4.1.8), it is reasonable to extend our ansatz even further. Inserting

$$\begin{aligned} &\widehat{\boldsymbol{U}}_{\text{mod2}}(x_1, k, t) \\ &\quad := \widehat{A}\left(\frac{k-k_0}{\varepsilon}, \varepsilon^2 t\right)\left(\boldsymbol{m}(x_1) + \varepsilon K\partial_k \boldsymbol{w}(x_1, k_0) + \frac{1}{2}\varepsilon^2 K^2 \partial_k^2 \boldsymbol{w}(x_1, k_0)\right)\mathrm{e}^{-\mathrm{i}(\nu_0+(k-k_0)\nu_1)t} + \widehat{\text{c.c.}} \end{aligned}$$

into (4.1.9) gives us the following terms of order ε^2:

$$\begin{aligned} &-\mathrm{i}E_1\Bigg(K^2\widehat{A}(L_1+\nu_1\Lambda)\partial_k\boldsymbol{w}(k_0) + \frac{1}{2}K^2\widehat{A}(L_0+\nu_0\Lambda)\partial_k^2\boldsymbol{w}(k_0) + \mathrm{i}\partial_T\widehat{A}\Lambda\boldsymbol{m} \\ &\qquad\qquad + 2\pi\epsilon_3\nu_0\left(\widehat{A} *_K \widehat{\overline{A}} *_K \widehat{A}\right)\begin{pmatrix} 3m_1^3 - m_1 m_2^2 \\ -3m_2^3 + m_1^2 m_2 \\ 0 \end{pmatrix}\Bigg) \\ &-6\pi\mathrm{i}\epsilon_3\nu_0 E_3\left(\widehat{A} *_K \widehat{A} *_K \widehat{A}\right)\begin{pmatrix} m_1^3 + m_1 m_2^2 \\ m_2^3 + m_1^2 m_2 \\ 0 \end{pmatrix} + \widehat{\text{c.c.}} \end{aligned} \tag{4.1.10}$$

Let us first remove the term proportional to E_3 by modifying the ansatz again:

$$\begin{aligned}
&\widehat{\boldsymbol{U}}_{\text{mod3}}(x_1,k,t)\\
&:= \widehat{A}\left(\frac{k-k_0}{\varepsilon},\varepsilon^2 t\right)\left(\boldsymbol{m}(x_1)+\varepsilon K\partial_k \boldsymbol{w}(x_1,k_0)+\frac{1}{2}\varepsilon^2 K^2\partial_k^2\boldsymbol{w}(x_1,k_0)\right)\mathrm{e}^{-\mathrm{i}(\nu_0+(k-k_0)\nu_1)t}\\
&\quad+2\pi\varepsilon^2\left(\widehat{A}*_K\widehat{A}*_K\widehat{A}\right)(\widetilde{K},T)\boldsymbol{h}(x_1)\mathrm{e}^{-\mathrm{i}(3\nu_0+(k-3k_0)\nu_1)t}+\widehat{\text{c.c.}}
\end{aligned}$$

With this correction the residual contains the additional terms

$$-2\pi\mathrm{i}\varepsilon^2 E_3\left(\widehat{A}*_K\widehat{A}*_K\widehat{A}\right)(\widetilde{K},T)(L(3k_0)+3\nu_0\Lambda)\boldsymbol{h}(x_1)+\mathcal{O}(\varepsilon^3)+\widehat{\text{c.c.}},$$

where $k=3k_0+\varepsilon\widetilde{K}$ was used. We therefore select the function $\boldsymbol{h}$ as the solution of

$$(L(3k_0)+3\nu_0\Lambda)\boldsymbol{h} = -3\nu_0\epsilon_3\begin{pmatrix} m_1^3+m_1m_2^2\\ m_2^3+m_2m_1^2\\ 0\end{pmatrix}. \tag{4.1.11}$$

All of the Assumptions (A1) – (A6) are now necessary to prove the existence of a solution $\boldsymbol{h}$, see Remark 3.3.8. First, we apply Lemma 3.3.11 to the linear eigenvalue problem to show that $\boldsymbol{m}\in\mathcal{H}^3(\mathbb{R})^3$, see also Lemma 4.2.6. By the Banach algebra property of $\mathcal{H}^3(\mathbb{R})$ it follows that the right-hand side in (4.1.11) is also in $\mathcal{H}^3(\mathbb{R})^3$. We can now apply case ii) of Lemma 3.3.7 for $(k,\omega)=(3k_0,3\omega)$ and get the existence of a solution $\boldsymbol{h}$.
The remaining terms of (4.1.10) can be further simplified with the help of equation (4.1.8) to obtain

$$K^2\widehat{A}(L_1+\nu_1\Lambda)\partial_k\boldsymbol{w}(k_0)+\frac{1}{2}K^2\widehat{A}(L_0+\nu_0\Lambda)\partial_k^2\boldsymbol{w}(k_0) = -\frac{1}{2}K^2\widehat{A}\nu_2\Lambda\boldsymbol{m}.$$

Therefore, the residual for our modified ansatz $\boldsymbol{U}_{\text{mod3}}$ to order ε^2 is given by $G\widehat{A}+\widehat{\text{c.c.}}$ with the operator G defined by

$$G\widehat{A} := -\mathrm{i}E_1\left(\left(\mathrm{i}\partial_T\widehat{A}-\frac{1}{2}K^2\widehat{A}\nu_2\right)\Lambda\boldsymbol{m}+2\pi\epsilon_3\nu_0\left(\widehat{A}*_K\widehat{\overline{A}}*_K\widehat{A}\right)\begin{pmatrix}3m_1^3-m_1m_2^2\\ -3m_2^3+m_1^2m_2\\ 0\end{pmatrix}\right).$$

By the right choice of $\widehat{A}$ we can guarantee that the L^2- projection $P_{\boldsymbol{m}}$ of $G\widehat{A}$ onto the kernel of $L_0+\nu_0\Lambda$ vanishes. Since ν_0 is by Assumption (A2) a simple eigenvalue, we have that $\mathrm{N}\,(L_0+\nu_0\Lambda)=\operatorname{span}\{\boldsymbol{m}\}$ and the projection is defined as usual by

$$(P_{\boldsymbol{m}}\boldsymbol{f})(x_1) := \frac{\langle \boldsymbol{f},\boldsymbol{m}\rangle_{L^2(\mathbb{R})^3}}{\langle \boldsymbol{m},\boldsymbol{m}\rangle_{L^2(\mathbb{R})^3}}\boldsymbol{m}(x_1) = \frac{\int_{-\infty}^{\infty}\boldsymbol{f}(\xi_1)\cdot\overline{\boldsymbol{m}}(\xi_1)\mathrm{d}\xi_1}{\int_{-\infty}^{\infty}\boldsymbol{m}(\xi_1)\cdot\overline{\boldsymbol{m}}(\xi_1)\mathrm{d}\xi_1}\boldsymbol{m}(x_1).$$

To guarantee that $G\widehat{A} = 0$ in the subspace $\mathrm{R}(P_{\boldsymbol{m}}) = \mathrm{N}(L_0 + \nu_0\Lambda)$ we have to choose $\widehat{A}$ such that

$$\begin{aligned} 0 =& \langle G\widehat{A}, \boldsymbol{m}\rangle_{L^2(\mathbb{R})^3} \\ =& \int_{-\infty}^{\infty} \left(\epsilon_1 \left(m_1^2 - m_2^2\right) + \mu_0 m_3^2\right)\, \mathrm{d}x_1 E_1 \left(\mathrm{i}\partial_T \widehat{A} - \frac{1}{2}K^2\nu_2\widehat{A}\right) \\ &+ 2\pi\nu_0 \int_{-\infty}^{\infty} \epsilon_3 \left(3m_1^4 - 2m_1^2m_2^2 + 3m_2^4\right) \mathrm{d}x_1 E_1 \left(\widehat{A} *_K \widehat{\overline{A}} *_K \widehat{A}\right), \end{aligned} \tag{4.1.12}$$

which is the Fourier transform of a nonlinear Schrödinger equation for A.
Conclusively, we fix A as the solution of

$$\mathrm{i}\partial_T A = -\frac{1}{2}\nu_2\partial_{X_2}^2 A + \kappa|A|^2 A \tag{4.1.13}$$

with

$$\kappa := -\nu_0 \int_{-\infty}^{\infty} \epsilon_3 \left(3m_1^4 - 2m_1^2m_2^2 + 3m_2^4\right) \mathrm{d}x_1,$$

where we used the normalization (3.2.4).
For our approximation result we will use that there exist smooth and localized solutions of (4.1.13), see Section 2.4.

Remark 4.1.3
The correction term

$$\frac{1}{2}\varepsilon^2 K^2 \partial_k^2 \boldsymbol{w}(x_1, k_0)\widehat{A}\left(\frac{k-k_0}{\varepsilon}, \varepsilon^2 t\right) \mathrm{e}^{-\mathrm{i}(\nu_0 + (k-k_0)\nu_1)t}$$

in $\widehat{\boldsymbol{U}}_{\mathrm{mod3}}$ *is not necessary to derive* (4.1.13). *Without this correction* (4.1.12) *would contain the additional term*

$$\frac{1}{2}\mathrm{i}K^2 E_1 \widehat{A}\left\langle (L_0 + \nu_0\Lambda)\partial_k^2 \boldsymbol{w}(k_0), \boldsymbol{m}\right\rangle_{L^2(\mathbb{R})^3}.$$

But this term vanishes since $(L_0 + \nu_0\Lambda)$ *is self-adjoint. We will nevertheless need the correction term to remove the residual to order* ε^2 *completely and not only on the subspace* $\mathrm{span}\{\boldsymbol{m}\}$.

We now study the residual in the range of the orthogonal projection $Q_{\boldsymbol{m}} := I - P_{\boldsymbol{m}}$. For one last time we have to extend our ansatz to

$$\begin{aligned}\widehat{\boldsymbol{U}}_{\text{ext}}(x_1,k,t) &:= \begin{pmatrix}\widehat{\mathcal{E}}_{\text{ext},1}(x_1,k,t)\\ \widehat{\mathcal{E}}_{\text{ext},2}(x_1,k,t)\\ \widehat{\mathcal{H}}_{\text{ext},3}(x_1,k,t)\end{pmatrix}\\ &:= \widehat{A}\left(\frac{k-k_0}{\varepsilon},\varepsilon^2 t\right)\Big(\boldsymbol{m}(x_1)+\varepsilon K\partial_k \boldsymbol{w}(x_1,k_0)+\frac{1}{2}\varepsilon^2 K^2\partial_k^2 \boldsymbol{w}(x_1,k_0)\Big)\mathrm{e}^{-\mathrm{i}(\nu_0+(k-k_0)\nu_1)t}\\ &\quad + 2\pi\varepsilon^2\left(\widehat{A} *_K \widehat{A} *_K \widehat{A}\right)\left(\frac{k-3k_0}{\varepsilon},\varepsilon^2 t\right)\boldsymbol{h}(x_1)\mathrm{e}^{-\mathrm{i}(3\nu_0+(k-3k_0)\nu_1)t}\\ &\quad + 2\pi\varepsilon^2\left(\widehat{A} *_K \widehat{\overline{A}} *_K \widehat{A}\right)\left(\frac{k-k_0}{\varepsilon},\varepsilon^2 t\right)\boldsymbol{p}(x_1)\mathrm{e}^{-\mathrm{i}(\nu_0+(k-k_0)\nu_1)t} + \widehat{\text{c.c.}},\end{aligned}$$

where the correction term in the last line is formally of order ε^2 and the existence of a suitable $\boldsymbol{p}$ has to be established next.
If we insert this extended ansatz into (4.1.9) and repeat the calculations we get that

$$\widehat{\mathbf{Res}}(\boldsymbol{U}_{\text{ext}}) = \varepsilon^2 G\widehat{A} - 2\pi\mathrm{i}\varepsilon^2 E_1(L_0+\nu_0\Lambda)\boldsymbol{p}\left(\widehat{A} * \widehat{\overline{A}} * \widehat{A}\right) + \widehat{\text{c.c.}} + \mathcal{O}(\varepsilon^3).$$

With (4.1.12) it follows that

$$G\widehat{A} = -2\pi\mathrm{i}E_1\left(\kappa\left(\widehat{A} *_K \widehat{\overline{A}} *_K \widehat{A}\right)\begin{pmatrix}\epsilon_1 m_1\\ \epsilon_1 m_2\\ \mu_0 m_3\end{pmatrix} + \epsilon_3\nu_0\left(\widehat{A} *_K \widehat{\overline{A}} *_K \widehat{A}\right)\begin{pmatrix}3m_1^3 - m_1m_2^2\\ -3m_2^3 + m_1^2 m_2\\ 0\end{pmatrix}\right).$$

Therefore, the terms of order ε^2 in $\widehat{\mathbf{Res}}(\boldsymbol{U}_{\text{ext}})$ vanish if $\boldsymbol{p}$ solves

$$(L_0+\nu_0\Lambda)\boldsymbol{p} = -\kappa\begin{pmatrix}\epsilon_1 m_1\\ \epsilon_1 m_2\\ \mu_0 m_3\end{pmatrix} - \epsilon_3\nu_0\begin{pmatrix}3m_1^3 - m_1m_2^2\\ -3m_2^3 + m_1^2 m_2\\ 0\end{pmatrix}. \tag{4.1.14}$$

To show that such a function $\boldsymbol{p}$ exists, we use case i) of Lemma 3.3.7 for $(k,\omega) = (k_0,\nu_0)$. Under Assumptions (A1) – (A6) we get that the right-hand side in (4.1.14) is in $L^2(\mathbb{R})^3$ and it only remains to show, that the right-hand side is orthogonal to $\boldsymbol{m}$:

$$\begin{aligned}&\left\langle -\kappa\begin{pmatrix}\epsilon_1 m_1\\ \epsilon_1 m_2\\ \mu_0 m_3\end{pmatrix} - \epsilon_3\nu_0\begin{pmatrix}3m_1^3 - m_1m_2^2\\ -3m_2^3 + m_1^2 m_2\\ 0\end{pmatrix}, \begin{pmatrix}m_1\\ m_2\\ m_3\end{pmatrix}\right\rangle_{L^2(\mathbb{R})^3}\\ &\quad = -\kappa\int_{-\infty}^{\infty}\left(\epsilon_1\left(m_1^2 - m_2^2\right) + \mu_0 m_3^2\right)\mathrm{d}x_1 - \nu_0\int_{-\infty}^{\infty}\epsilon_3\left(3m_1^4 - 2m_1^2 m_2^2 + 3m_2^4\right)\mathrm{d}x_1 = 0,\end{aligned}$$

where we used Remark 3.2.6, the normalization of $\boldsymbol{m}$ and the definition of κ.
Therefore, such a function $\boldsymbol{p}$ exists and $\widehat{\mathbf{Res}}(\boldsymbol{U}_{\text{ext}})$ is formally of order ε^3.

Remark 4.1.4
Note that it is possible to add additional correction terms to get an even smaller residual. We will nevertheless restrict ourselves to this order for the residual since it is already small enough to result in a meaningful approximation result.

4.1.2. Interface Properties of the Extended Ansatz

In this section we study the residual of the interface conditions

$$\widehat{\mathbf{Res}}_{\mathrm{IFC}}(\boldsymbol{U}_{\mathrm{ext}}) := \begin{pmatrix} \left[\!\left[\widehat{\mathcal{D}}_1\left(\boldsymbol{U}_{\mathrm{ext},E}\right)\right]\!\right]_{2\mathrm{D}} \\ \left[\!\left[\widehat{U}_{\mathrm{ext},2}\right]\!\right]_{2\mathrm{D}} \\ \left[\!\left[\widehat{U}_{\mathrm{ext},3}\right]\!\right]_{2\mathrm{D}} \end{pmatrix},$$

with $\boldsymbol{U}_{\mathrm{ext},E} := (\mathcal{E}_{\mathrm{ext},1}, \mathcal{E}_{\mathrm{ext},2}, 0)^\top$.
From Section 3.1 we know that $[\![\epsilon_1 w_1(k)]\!]_{1\mathrm{D}} = [\![w_2(k)]\!]_{1\mathrm{D}} = [\![w_3(k)]\!]_{1\mathrm{D}} = 0$ for all $k \in \mathbb{R}$. We can therefore differentiate the jump-conditions in k and get for $j \in \{0, 1, 2\}$

$$\left[\!\left[\epsilon_1 \partial_k^j w_1(k)\right]\!\right]_{1\mathrm{D}} = \left[\!\left[\partial_k^j w_2(k)\right]\!\right]_{1\mathrm{D}} = \left[\!\left[\partial_k^j w_3(k)\right]\!\right]_{1\mathrm{D}} = 0$$

and hence

$$\begin{aligned} [\![\epsilon_1 m_1]\!]_{1\mathrm{D}} + \varepsilon K [\![\epsilon_1 \partial_k w_1(k_0)]\!]_{1\mathrm{D}} + \frac{1}{2}\varepsilon^2 K^2 \left[\!\left[\epsilon_1 \partial_k^2 w_1(k_0)\right]\!\right]_{1\mathrm{D}} &= 0, \\ [\![m_2]\!]_{1\mathrm{D}} + \varepsilon K [\![\partial_k w_2(k_0)]\!]_{1\mathrm{D}} + \frac{1}{2}\varepsilon^2 K^2 \left[\!\left[\partial_k^2 w_2(k_0)\right]\!\right]_{1\mathrm{D}} &= 0, \\ [\![m_3]\!]_{1\mathrm{D}} + \varepsilon K [\![\partial_k w_3(k_0)]\!]_{1\mathrm{D}} + \frac{1}{2}\varepsilon^2 K^2 \left[\!\left[\partial_k^2 w_3(k_0)\right]\!\right]_{1\mathrm{D}} &= 0. \end{aligned}$$

With this it is easy to see that $\widehat{\mathbf{Res}}_{\mathrm{IFC}}$ has no terms of order ε^0 and ε^1. To order ε^2 we get for the first component of $\widehat{\mathbf{Res}}_{\mathrm{IFC}}$

$$\begin{aligned} &-2\pi \mathrm{i} \nu_0 E_1 \left(\widehat{A} *_K \widehat{\overline{A}} *_K \widehat{A}\right)(K, T) \left([\![\epsilon_1 p_1]\!]_{1\mathrm{D}} + \left[\!\left[\epsilon_3 \left(3 m_1^3 - m_1 m_2^2\right)\right]\!\right]_{1\mathrm{D}}\right) \\ &-2\pi \mathrm{i} \nu_0 E_3 \left(\widehat{A} *_K \widehat{A} *_K \widehat{A}\right)(\widetilde{K}, T) \left([\![\epsilon_1 h_1]\!]_{1\mathrm{D}} + \left[\!\left[\epsilon_3 \left(m_1^3 + m_1 m_2^2\right)\right]\!\right]_{1\mathrm{D}}\right) + \widehat{\mathrm{c.c.}} \end{aligned}$$

To see that these terms vanish, we use that $\boldsymbol{h}$ and $\boldsymbol{p}$ solve (4.1.11) and (4.1.14), respectively, and that we can use Lemma 3.3.11 to improve the regularity of h_1, p_1 on both sides of the interface, see Lemma 4.2.6. For $\boldsymbol{m}, \boldsymbol{h}, \boldsymbol{p} \in \mathcal{H}^3(\mathbb{R})^3$ we can then use the Sobolev embedding $H^1(\mathbb{R}_\pm) \hookrightarrow C(\mathbb{R}_\pm)$, which gives us that

$$\begin{aligned} [\![\epsilon_1 p_1]\!]_{1\mathrm{D}} &= -\left[\!\left[\epsilon_3 \left(3 m_1^3 - m_1 m_2^2\right)\right]\!\right]_{1\mathrm{D}}, \\ [\![\epsilon_1 h_1]\!]_{1\mathrm{D}} &= -\left[\!\left[\epsilon_3 \left(m_1^3 + m_1 m_2^2\right)\right]\!\right]_{1\mathrm{D}}. \end{aligned}$$

For the other two components we get

$$\begin{aligned}\widehat{\mathrm{Res}}_{\mathrm{IFC},2} =& 2\pi\varepsilon^2 E_1 \left(\widehat{A} *_K \widehat{\overline{A}} *_K \widehat{A}\right)(K,T)[\![p_2]\!]_{1\mathrm{D}} \\ &+ 2\pi\varepsilon^2 E_3 \left(\widehat{A} *_K \widehat{A} *_K \widehat{A}\right)(\widetilde{K},T)[\![h_2]\!]_{1\mathrm{D}} + \widehat{\mathrm{c.c.}}, \\ \widehat{\mathrm{Res}}_{\mathrm{IFC},3} =& 2\pi\varepsilon^2 E_1 \left(\widehat{A} *_K \widehat{\overline{A}} *_K \widehat{A}\right)(K,T)[\![p_3]\!]_{1\mathrm{D}} \\ &+ 2\pi\varepsilon^2 E_3 \left(\widehat{A} *_K \widehat{A} *_K \widehat{A}\right)(\widetilde{K},T)[\![h_3]\!]_{1\mathrm{D}} + \widehat{\mathrm{c.c.}}\end{aligned}$$

and $\widehat{\mathrm{Res}}_{\mathrm{IFC},2}, \widehat{\mathrm{Res}}_{\mathrm{IFC},3}$ vanish since $\boldsymbol{h} \in \mathrm{D}\left(L(3k_0) + 3\nu_0\Lambda\right)$ and $\boldsymbol{p} \in \mathrm{D}\left(L(k_0) + \nu_0\Lambda\right)$ imply that

$$[\![p_2]\!]_{1\mathrm{D}} = [\![p_3]\!]_{1\mathrm{D}} = [\![h_2]\!]_{1\mathrm{D}} = [\![h_3]\!]_{1\mathrm{D}} = 0.$$

Remark 4.1.5
Note that for our extended ansatz the Fourier transform of $\mathrm{Res}_{\mathrm{div}}(\boldsymbol{U}_{\mathrm{ext},E}) := \nabla \cdot \mathcal{D}(\boldsymbol{U}_{\mathrm{ext},E})$ *is given by*

$$\begin{aligned}\partial_{x_1}\widehat{\mathcal{D}}_1(\boldsymbol{U}_{\mathrm{ext},E}) + \mathrm{i}k\widehat{\mathcal{D}}_2(\boldsymbol{U}_{\mathrm{ext},E}) =& (\partial_{x_1}(\epsilon_1 m_1) + \mathrm{i}k_0\epsilon_1 m_2)\widehat{A}E_1 \\ &+ \varepsilon K(\partial_{x_1}(\epsilon_1\partial_k \boldsymbol{w}(k_0)) + \mathrm{i}k_0\epsilon_1\partial_k\boldsymbol{w}(k_0) + \mathrm{i}\epsilon_1 m_2)\widehat{A}E_1 + \widehat{\mathrm{c.c.}} + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Similar to (3.1.9) *we get that* $\partial_{x_1}(\epsilon_1 w_1(k)) + \mathrm{i}k\epsilon_1 w_2 = 0$*, for solutions* $\boldsymbol{w}$ *of* (3.2.2). *A Taylor expansion of this expression in* k *at* k_0 *gives us*

$$\partial_{x_1}(\epsilon_1 m_1) + \mathrm{i}k_0\epsilon_1 m_2 + \varepsilon K(\partial_{x_1}(\epsilon_1\partial_k\boldsymbol{w}(k_0)) + \mathrm{i}k_0\epsilon_1\partial_k\boldsymbol{w}(k_0) + \mathrm{i}\epsilon_1 m_2) + \mathcal{O}(\varepsilon^2) = 0.$$

A comparison of powers of ε *shows us that* $\widehat{\mathrm{Res}}_{\mathrm{div}}(\boldsymbol{U}_{\mathrm{ext},E})$ *is formally of order* ε^2*.*
Note that $\mathrm{Res}_{\mathrm{div}}(\boldsymbol{U}_{\mathrm{ext},E})$ *and* $\mathbf{Res}_{\mathrm{IFC}}(\boldsymbol{U}_{\mathrm{ext}})$ *are not part of our error estimates in Chapter 6 and don't have to be small for our approximation result.*

All in all, we have seen that $\widehat{\boldsymbol{U}}_{\mathrm{ext}}(x_1,k,t)$ is formally a "good" approximation for a solution of (4.1.5), i.e. formally only terms of order ε^3 and higher remain in $\widehat{\mathbf{Res}}(\boldsymbol{U}_{\mathrm{ext}})$. Now we have to rigorously estimate the residual in a suitable norm.

4.2. Estimation of the Residual and its Derivatives

In this section we estimate the residual $\mathbf{Res}(\boldsymbol{U}_{\mathrm{ext}})$ and its time-derivatives in the $\mathcal{H}^3(\mathbb{R}^2)^3$-norm rigorously under the assumption that A solves (4.1.13) and is regular enough.

We go back to space-variables and by applying the inverse Fourier transformation we obtain

$$\mathbf{Res} := \mathbf{Res}(\boldsymbol{U}_{\mathrm{ext}}) = \begin{pmatrix} \partial_t\mathcal{D}_1(\boldsymbol{U}_{\mathrm{ext},E}) - \partial_{x_2}U_{\mathrm{ext},3} \\ \partial_t\mathcal{D}_2(\boldsymbol{U}_{\mathrm{ext},E}) + \partial_{x_1}U_{\mathrm{ext},3} \\ -\partial_{x_2}U_{\mathrm{ext},1} + \partial_{x_1}U_{\mathrm{ext},2} + \mu_0\partial_t U_{\mathrm{ext},3} \end{pmatrix}$$

and

$$\begin{aligned}\boldsymbol{U}_{\text{ext}}(x_1,x_2,t) =& \varepsilon A(X_2,T)\boldsymbol{m}(x_1)\mathrm{e}^{\mathrm{i}(k_0x_2-\nu_0 t)}\\ &-\varepsilon^2\mathrm{i}\partial_{X_2}A(X_2,T)\partial_k\boldsymbol{w}(x_1,k_0)\mathrm{e}^{\mathrm{i}(k_0x_2-\nu_0 t)}\\ &-\varepsilon^3\frac{1}{2}\partial_{X_2}^2A(X_2,T)\partial_k^2\boldsymbol{w}(x_1,k_0)\mathrm{e}^{\mathrm{i}(k_0x_2-\nu_0 t)}\\ &+\varepsilon^3|A(X_2,T)|^2A(X_2,T)\boldsymbol{p}(x_1)\mathrm{e}^{\mathrm{i}(k_0x_2-\nu_0 t)}\\ &+\varepsilon^3A^3(X_2,T)\boldsymbol{h}(x_1)\mathrm{e}^{3\mathrm{i}(k_0x_2-\nu_0 t)}+\text{c.c.},\end{aligned}\tag{4.2.1}$$

recalling that $X_2=\varepsilon(x_2-\nu_1 t)$ and $T=\varepsilon^2 t$.
For our error analysis in Chapter 6 we need that **Res** and its temporal derivatives up to order 3 are bounded in the $\mathcal{H}^3(\mathbb{R}^2)^3$-norm by $C\varepsilon^{7/2}$ for a time interval of length $\mathcal{O}(\varepsilon^{-2})$, i.e. on the interval $\left[0,T_0\varepsilon^{-2}\right]$ for some $T_0>0$.

Remark 4.2.1
Before we start the estimates, let us note some changes in the powers of ε due to transformations. First, note that due to the ε-dependency of X_2 and K the inverse Fourier transformation comes with an additional factor ε, compare (4.1.3). *In Section 4.1 we showed that $\widehat{\mathbf{Res}}(\boldsymbol{U}_{\text{ext}})$ is formally of order ε^3 after Fourier transformation we now have that $\mathbf{Res}(\boldsymbol{U}_{\text{ext}})$ is formally of order ε^4.*
Second, we lose half an order of ε when we take the L^2-norm of a function that depends on $\varepsilon(x_2-\nu_1 t)$, e.g.

$$\begin{aligned}\|A(\varepsilon(\cdot-\nu_1 t),T)\|_{L^2(\mathbb{R})} &= \left(\int_{\mathbb{R}}|A(\varepsilon(x_2-\nu_1 t),T)|^2\,\mathrm{d}x_2\right)^{1/2}\\ &=\left(\varepsilon^{-1}\int_{\mathbb{R}}|A(X_2,T)|^2\,\mathrm{d}X_2\right)^{1/2}\\ &=\varepsilon^{-1/2}\|A(\cdot,T)\|_{L^2(\mathbb{R})}.\end{aligned}\tag{4.2.2}$$

4.2.1. Estimation of the Residual

Our goal is now to write down **Res** explicitly and to estimate all the occurring terms. To shorten the notation, we take as before $F_1=\mathrm{e}^{\mathrm{i}(k_0x_2-\nu_0 t)}$ and suppress the dependency on x_1,x_2,t. The derivatives of $\boldsymbol{U}_{\text{ext}}$ are then given by

$$\begin{aligned}\partial_{x_1}\boldsymbol{U}_{\text{ext}} =& F_1\left(\varepsilon A\partial_{x_1}\boldsymbol{m}-\varepsilon^2\mathrm{i}\partial_{X_2}A\partial_{x_1}\partial_k\boldsymbol{w}(k_0)-\varepsilon^3\frac{1}{2}\partial_{X_2}^2A\partial_{x_1}\partial_k^2\boldsymbol{w}(k_0)\right)\\ &+F_1\left(\varepsilon^3|A|^2A\partial_{x_1}\boldsymbol{p}+F_1^2\varepsilon^3A^3\partial_{x_1}\boldsymbol{h}\right)+\text{c.c.},\\ \partial_{x_2}\boldsymbol{U}_{\text{ext}} =& \mathrm{i}k_0F_1\left(\varepsilon A\boldsymbol{m}-\varepsilon^2\mathrm{i}\partial_{X_2}A\partial_k\boldsymbol{w}(k_0)-\varepsilon^3\frac{1}{2}\partial_{X_2}^2A\partial_k^2\boldsymbol{w}(k_0)+\varepsilon^3|A|^2A\boldsymbol{p}+3F_1^2\varepsilon^3A^3\partial_{x_1}\boldsymbol{h}\right)\\ &+F_1\left(\varepsilon^2\partial_{X_2}A\boldsymbol{m}-\varepsilon^3\mathrm{i}\partial_{X_2}^2A\partial_k\boldsymbol{w}(k_0)-\varepsilon^4\frac{1}{2}\partial_{X_2}^3A\partial_k^2\boldsymbol{w}(k_0)\right)\\ &+F_1\left(\varepsilon^4\left(2|A|^2\partial_{X_2}A+A^2\partial_{X_2}\overline{A}\right)\boldsymbol{p}+3F_1^2\varepsilon^4A^2\partial_{X_2}A\boldsymbol{h}\right)+\text{c.c.}\end{aligned}$$

and

$$\begin{aligned}
\partial_t \boldsymbol{U}_{\text{ext}} = &-\mathrm{i}\nu_0 F_1 \left(\varepsilon A\boldsymbol{m} - \varepsilon^2 \mathrm{i}\partial_{X_2} A\partial_k \boldsymbol{w}(k_0) - \varepsilon^3 \frac{1}{2}\partial_{X_2}^2 A\partial_k^2 \boldsymbol{w}(k_0) + \varepsilon^3 |A|^2 A\boldsymbol{p} + 3F_1^2\varepsilon^3 A^3 \partial_{x_1}\boldsymbol{h}\right) \\
&+ F_1 \left(-\varepsilon^2 \nu_1 \partial_{X_2} A\boldsymbol{m} + \varepsilon^3 \mathrm{i}\nu_1 \partial_{X_2}^2 A\partial_k \boldsymbol{w}(k_0) + \varepsilon^4 \frac{\nu_1}{2}\partial_{X_2}^3 A\partial_k^2 \boldsymbol{w}(k_0)\right) \\
&+ F_1 \left(-\varepsilon^4 \nu_1 \left(2|A|^2\partial_{X_2} A + A^2 \partial_{X_2}\overline{A}\right)\boldsymbol{p} - 3F_1^2\varepsilon^4\nu_1 A^2\partial_{X_2} A\boldsymbol{h}\right) \\
&+ F_1 \left(\varepsilon^3 \partial_T A\boldsymbol{m} - \varepsilon^4 \mathrm{i}\partial_T\partial_{X_2} A\partial_k \boldsymbol{w}(k_0) - \varepsilon^5 \frac{1}{2}\partial_T\partial_{X_2}^2 A\partial_k^2 \boldsymbol{w}(k_0)\right) \\
&+ F_1 \left(\varepsilon^5 \left(2|A|^2\partial_T A + A^2\partial_T\overline{A}\right)\boldsymbol{p} + 3F_1^2\varepsilon^5 A^2\partial_T A\boldsymbol{h}\right) + \text{c.c.}
\end{aligned}$$

Remark 4.2.2

Note that derivatives in t and x_2 generate higher-order terms since $T = \varepsilon^2 t$ and $X_2 = \varepsilon(x_2 - \nu_1 t)$ depend on ε. A derivative in x_1 does not change the order in ε. For higher-order derivatives of $\boldsymbol{U}_{\text{ext}}$ the analogous statement holds true.

With the calculations of Chapter 4.1 we know that **Res** only contains terms of order ε^4 and higher. In particular, we note that therefore the derivatives $\partial_{x_1} U_{\text{ext},2}$ and $\partial_{x_1} U_{\text{ext},3}$ no longer appear in the residual.

The parts of Res_1 and Res_2 that are linear in $\boldsymbol{U}_{\text{ext}}$ are given by

$$\begin{aligned}
\text{Res}_{\text{lin},1} = &F_1\varepsilon^4 \left(\frac{1}{2}\partial_{X_2}^3 A\partial_k^2 w_3(k_0) - \left(2|A|^2\partial_{X_2} A + A^2\partial_{X_2}\overline{A}\right) p_3\right) - 3F_1^3\varepsilon^4 A^2 \partial_{X_2} A h_3 \\
&+ F_1\varepsilon^4\epsilon_1 \left(\frac{\nu_1}{2}\partial_{X_2}^3 A\partial_k^2 w_1(k_0) - \mathrm{i}\partial_T\partial_{X_2} A\partial_k w_1(k_0) - \nu_1\left(2|A|^2\partial_{X_2} A + A^2\partial_{X_2}\overline{A}\right) p_1\right) \\
&- 3F_1^3\varepsilon^4\epsilon_1\nu_1 A^2\partial_{X_2} A h_1 \\
&+ F_1\varepsilon^5\epsilon_1 \left(-\frac{1}{2}\partial_T\partial_{X_2}^2 A\partial_k^2 w_1(k_0) + \left(2|A|^2\partial_T A + A^2\partial_T\overline{A}\right) p_1\right) + 3F_1^3\varepsilon^5\epsilon_1 A^2\partial_T A h_1 + \text{c.c.}, \\
\text{Res}_{\text{lin},2} = &F_1\varepsilon^4\epsilon_1 \left(\frac{\nu_1}{2}\partial_{X_2}^3 A\partial_k^2 w_2(k_0) - \mathrm{i}\partial_T\partial_{X_2} A\partial_k w_2(k_0) - \nu_1\left(2|A|^2\partial_{X_2} A + A^2\partial_{X_2}\overline{A}\right) p_2\right) \\
&- 3F_1^3\varepsilon^4\epsilon_1\nu_1 A^2\partial_{X_2} A h_2 \\
&+ F_1\varepsilon^5\epsilon_1 \left(-\frac{1}{2}\partial_T\partial_{X_2}^2 A\partial_k^2 w_2(k_0) + \left(2|A|^2\partial_T A + A^2\partial_T\overline{A}\right) p_2\right) + 3F_1^3\varepsilon^5\epsilon_1 A^2\partial_T A h_2 + \text{c.c.}
\end{aligned}$$

For the part of Res_1 that is nonlinear in $\boldsymbol{U}_{\text{ext}}$ we get

$$\begin{aligned}
\text{Res}_{\text{nl},1} = &-\varepsilon^4\epsilon_3\Big[3\nu_1F_1^3A^2\partial_{X_2}A\left(m_1^3+m_1m_2^2\right)\\
&+\nu_0F_1^3A^2\partial_{X_2}A\left(3m_1^2\partial_k w_1(k_0)+m_2^2\partial_k w_1(k_0)+2m_1m_2\partial_k w_2(k_0)\right)\\
&+\nu_1F_1A^2\partial_{X_2}\overline{A}\left(3|m_1|^2m_1+2m_1|m_2|^2+\overline{m}_1m_2^2\right)\\
&+\nu_0F_1A^2\partial_{X_2}\overline{A}\left(3m_1^2\partial_k\overline{w}_1(k_0)+m_2^2\partial_k\overline{w}_1(k_0)+2m_1m_2\partial_k\overline{w}_2(k_0)\right)\\
&+2\nu_1F_1|A|^2\partial_{X_2}A\left(3|m_1|^2m_1+2m_1|m_2|^2+\overline{m}_1m_2^2\right)\\
&+2\nu_0F_1|A|^2\partial_{X_2}A\left(3|m_1|^2\partial_k w_1(k_0)+|m_2|^2\partial_k w_1(k_0)+\overline{m}_1m_2\partial_k w_2(k_0)+m_1m_2\partial_k w_2(k_0)\right)\Big]\\
&+\text{c.c.}+\mathcal{O}(\varepsilon^5),
\end{aligned}$$

and for $\text{Res}_{\text{nl},2}$ we have to change the indices of the components of $\boldsymbol{m}$ and $\partial_k\boldsymbol{w}(k_0)$. The third component of the residual is given by

$$\begin{aligned}
\text{Res}_3 = &F_1\varepsilon^4\left(\frac{1}{2}\partial_{X_2}^3A\partial_k^2w_1(k_0)-\left(2|A|^2\partial_{X_2}A+A^2\partial_{X_2}\overline{A}\right)p_1\right)-3F_1^3\varepsilon^4\partial_{X_2}AA^2h_1\\
&+F_1\varepsilon^4\mu_0\left(\frac{\nu_1}{2}\partial_{X_2}^3A\partial_k^2w_3(k_0)-\mathrm{i}\partial_T\partial_{X_2}A\partial_k w_3(k_0)-\nu_1\left(2|A|^2\partial_{X_2}A+A^2\partial_{X_2}\overline{A}\right)p_3\right)\\
&-3F_1^3\varepsilon^4\mu_0\nu_1A^2\partial_{X_2}Ah_3\\
&+F_1\varepsilon^5\mu_0\left(-\frac{1}{2}\partial_T\partial_{X_2}^2A\partial_k^2w_3(k_0)+\left(2|A|^2\partial_TA+A^2\partial_T\overline{A}\right)p_3\right)+3F_1^3\varepsilon^5\mu_0A^2\partial_TAh_3+\text{c.c.}
\end{aligned}$$

For the estimate of $\|\mathbf{Res}(\cdot,t)\|_{L^2(\mathbb{R}^2)^3}$ we will discuss the terms of order ε^4 in detail and will sketch the idea for higher-order terms.
At first, we list the different types of terms that appear and how to handle them:

- Constants like μ_0, ν_1, ν_2 are trivial.
- The exponential function F_1 satisfies $|F_1| = 1$.
- The material functions ϵ_1, ϵ_3 are bounded since $\epsilon_1^\pm, \epsilon_3^\pm \in C^3(\mathbb{R}_\pm)\cap W^{3,\infty}(\mathbb{R}_\pm)$ by Assumptions (A1) and (A6).
- The envelope has to be smooth enough such that $\left\|\partial_{X_2}^{j_1}\partial_T^{j_2}A(\cdot,T)\right\|_{L^2(\mathbb{R})} \leqslant C$ for all appearing $j_1, j_2 \in \mathbb{N}_0$. To be precise, we will need

$$A \in \bigcap_{k=0}^{1} C^{1-k}\left([0,T_0], H^{2+k}(\mathbb{R})\right), \tag{4.2.3}$$

  for some $T_0 > 0$ since the highest order derivatives that can appear are $\partial_{X_2}^3A$ and $\partial_{X_2}^2\partial_TA$. For the nonlinear terms in A we notice that one factor with three derivatives can at most appear once and all other factors contain at most two derivatives, we can therefore always use the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ to bound the lower order terms. Some examples for estimates of such products are (suppressing the time dependence)

$$\begin{aligned}\left\||A|^2\partial_{X_2}A\right\|_{L^2(\mathbb{R})} &\leqslant \|A\|^2_{L^\infty(\mathbb{R})}\|\partial_{X_2}A\|_{L^2(\mathbb{R})} \leqslant C\|A\|^3_{H^1(\mathbb{R})},\\ \left\||\partial_{X_2}A|^2\partial^2_{X_2}A\right\|_{L^2(\mathbb{R})} &\leqslant \|\partial_{X_2}A\|^2_{L^\infty(\mathbb{R})}\left\|\partial^2_{X_2}A\right\|_{L^2(\mathbb{R})} \leqslant C\|A\|^3_{H^2(\mathbb{R})},\\ \left\||\partial^2_{X_2}A|^2\partial_T\partial^2_{X_2}A\right\|_{L^2(\mathbb{R})} &\leqslant \left\|\partial^2_{X_2}A\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_T\partial^2_{X_2}A\right\|_{L^2(\mathbb{R})} \leqslant C\|A\|^2_{H^3(\mathbb{R})}\|\partial_T A\|_{H^2(\mathbb{R})}.\end{aligned}$$

- The x_1-dependent functions have to satisfy

$$\boldsymbol{m}, \partial_k\boldsymbol{w}(\cdot,k_0), \partial_k^2\boldsymbol{w}(\cdot,k_0), \boldsymbol{h}, \boldsymbol{p} \in L^2(\mathbb{R})^3 \cap L^\infty(\mathbb{R})^3. \tag{4.2.4}$$

  As stated before, there are no x_1-derivatives left in **Res**, therefore $(L^2(\mathbb{R})^3 \cap L^\infty(\mathbb{R})^3)$-functions are enough to estimate all appearing linear and nonlinear terms, e.g. the first term in $\mathrm{Res}_{\mathrm{nl},1}$ can be estimated by

$$\left\|m_1^3 + m_1m_2^2\right\|_{L^2(\mathbb{R})} \leqslant \left(\|m_1\|^2_{L^\infty(\mathbb{R})} + \|m_2\|^2_{L^\infty(\mathbb{R})}\right)\|m_1\|_{L^2(\mathbb{R})} \leqslant C. \tag{4.2.5}$$

  Note that due to the form of the nonlinearity, the third components of these vector functions never appear in a nonlinear term and it would be enough when these components are only in $L^2(\mathbb{R})$.

- We will use that the terms in **Res** allow for a separation of variables. Terms of the form $g(\boldsymbol{x}) := \varepsilon^b f_1(x_1)f_2(\varepsilon x_2)f_3(x_2)$ with $b \geqslant 4$, $f_1, f_2 \in L^2(\mathbb{R})$ and $f_3 \in L^\infty(\mathbb{R})$ can then be estimated by

$$\|g\|_{L^2(\mathbb{R}^2)} \leqslant \varepsilon^{b-\frac{1}{2}}\|f_1\|_{L^2(\mathbb{R})}\|f_2\|_{L^2(\mathbb{R})}\|f_3\|_{L^\infty(\mathbb{R})}.$$

Now let us estimate the terms of order ε^4 of Res_3:

$$\begin{aligned}&\|\mathrm{Res}_3(\cdot,t)\|_{L^2(\mathbb{R}^2)}\\ &\leqslant C\varepsilon^{7/2}\Big[\left\|\partial^3_{X_2}A(\cdot,T)\right\|_{L^2(\mathbb{R})}\left(\left\|\partial_k^2 w_1(\cdot,k_0)\right\|_{L^2(\mathbb{R})} + \left\|\partial_k^2 w_3(\cdot,k_0)\right\|_{L^2(\mathbb{R})}\right)\\ &+ \|\partial_T\partial_{X_2}A(\cdot,T)\|_{L^2(\mathbb{R})}\|\partial_k w_3(\cdot,k_0)\|_{L^2(\mathbb{R})}\\ &+ \left(\left\||A(\cdot,T)|^2\partial_{X_2}A(\cdot,T)\right\|_{L^2(\mathbb{R})} + \left\|A^2(\cdot,T)\partial_{X_2}\overline{A}(\cdot,T)\right\|_{L^2(\mathbb{R})}\right)\left(\|p_1\|_{L^2(\mathbb{R})} + \|p_3\|_{L^2(\mathbb{R})}\right)\\ &+ \left\|A^2(\cdot,T)\partial_{X_2}A(\cdot,T)\right\|_{L^2(\mathbb{R})}\left(\|h_1\|_{L^2(\mathbb{R})} + \|h_3\|_{L^2(\mathbb{R})}\right)\Big]\\ &\leqslant C\varepsilon^{7/2}\Big[\left\|\partial^3_{X_2}A(\cdot,T)\right\|_{L^2(\mathbb{R})} + \|\partial_T\partial_{X_2}A(\cdot,T)\|_{L^2(\mathbb{R})} + \left\||A(\cdot,T)|^2\partial_{X_2}A(\cdot,T)\right\|_{L^2(\mathbb{R})}\\ &+ \left\|A^2(\cdot,T)\partial_{X_2}\overline{A}(\cdot,T)\right\|_{L^2(\mathbb{R})} + \left\|A^2(\cdot,T)\partial_{X_2}A(\cdot,T)\right\|_{L^2(\mathbb{R})}\Big]\\ &\leqslant C\varepsilon^{7/2}.\end{aligned}$$

Note that we used (4.2.2) and that the loss of half a derivative is due to the transformation of variables and does not change for products of X_2-dependent functions. To estimate the other

components of **Res** we additionally use estimates like (4.2.5) and proceed analogously. Terms of at least order ε^5 can also be estimated with the same techniques. The regularity assumptions in (4.2.3) and (4.2.4) are enough to cover all derivatives and nonlinearities that can appear. For instance the appearing quartic terms in A can be estimated like the cubic terms above by using $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.
All in all, we conclude that

$$\|\mathbf{Res}(\boldsymbol{U}_{\text{ext}})(\cdot,\cdot,t)\|_{L^2(\mathbb{R}^2)^3} \leqslant C\varepsilon^{\frac{7}{2}}, \qquad t \in \left[0, T_0\varepsilon^{-2}\right], \tag{4.2.6}$$

under condition (4.2.3) and (4.2.4), where the constant C depends on the norms of A, $\boldsymbol{m}$, $\partial_k \boldsymbol{w}(\cdot, k_0)$, $\partial_k^2 \boldsymbol{w}(\cdot, k_0)$, $\boldsymbol{h}$ and $\boldsymbol{p}$.

4.2.2. Estimation of the Derivatives of the Residual

Let us now estimate all the derivatives of **Res** up to order 3. This can be done similarly to the estimates of Section 4.2.1.

Let $T_0 > 0$. We have to estimate $\left\|\partial^{\boldsymbol{\beta}}\mathbf{Res}(\cdot, t)\right\|_{L^2(\mathbb{R}^2)^3}$ for $t \in \left[0, T_0\varepsilon^{-2}\right]$ with $\boldsymbol{\beta} \in \mathbb{N}_0^3$, $|\boldsymbol{\beta}| \leqslant 3$.

Remark 4.2.3
We use the multi-index notation for derivatives. For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_t)^\top \in \mathbb{N}_0^3$ *we simply write* $\partial^{\boldsymbol{\alpha}}$ *instead of* $\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_t^{\alpha_t}$. *We also use the typical conventions* $|\boldsymbol{\alpha}| := \sum_j \alpha_j$ *and for* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^3$ *we say* $\boldsymbol{\alpha} \leqslant \boldsymbol{\beta}$ *if* $\alpha_j \leqslant \beta_j$ *for all* j.

For the next step we have to discuss the necessary regularity. To estimate the derivatives up to order three, we have to increase the regularity assumptions of (4.2.3) and (4.2.4) by 3 orders:

- **Res** contains the derivatives $\partial_T^{\alpha_1}\partial_{X_2}^{\alpha_2} A$ with $\alpha_1 \leqslant 1$ and $\alpha_1 + \alpha_2 \leqslant 3$. It follows that $\partial^{\boldsymbol{\beta}}\mathbf{Res}$ contains $\partial_T^{\gamma_1}\partial_{X_2}^{\gamma_2} A$ with $\gamma_1 \leqslant 4$ and $\gamma_1 + \gamma_2 \leqslant 6$. Hence, we need the stricter condition

$$A \in \bigcap_{k=0}^{4} C^{4-k}\left([0, T_0], H^{2+k}(\mathbb{R})\right). \tag{4.2.7}$$

- **Res** contains no derivatives of $\boldsymbol{m}, \partial_k \boldsymbol{w}(k_0), \partial_k^2 \boldsymbol{w}(k_0), \boldsymbol{p}, \boldsymbol{h}$.
Therefore, we now have the condition

$$\boldsymbol{m}, \partial_k \boldsymbol{w}(k_0), \partial_k^2 \boldsymbol{w}(k_0), \boldsymbol{h}, \boldsymbol{p} \in \mathcal{H}^3(\mathbb{R})^3. \tag{4.2.8}$$

Note that we needed $L^\infty(\mathbb{R})$-regularity in (4.2.4) to estimate the nonlinear terms. But since $H^3(\mathbb{R}_\pm)$ are Banach algebras, no additional condition is necessary in (4.2.8).

- We also need to estimate up to three derivatives of the material functions ϵ_1, ϵ_3. Here we can again use that $\epsilon_1^\pm, \epsilon_3^\pm \in C^3(\mathbb{R}_\pm) \cap W^{3,\infty}(\mathbb{R}_\pm)$ by Assumptions (A1) and (A6).

Now we can proceed as in Section 4.2.1 to infer the following lemma.

Lemma 4.2.4 (Estimation of the Residual)
Let $T_0 > 0$, $\boldsymbol{m}, \partial_k \boldsymbol{w}(k_0), \partial_k^2 \boldsymbol{w}(k_0), \boldsymbol{h}, \boldsymbol{p} \in \mathcal{H}^3(\mathbb{R})^3$ *and* $A \in \bigcap_{k=0}^4 C^{4-k}\left([0,T_0], H^{2+k}(\mathbb{R})\right)$. *Then*

$$\left\|\partial^{\beta}\mathbf{Res}(\cdot,t)\right\|_{L^2(\mathbb{R}^2)^3} \leqslant C\varepsilon^{7/2} \tag{4.2.9}$$

for all $t \in \left[0, T_0\varepsilon^{-2}\right]$ *and* $\boldsymbol{\beta} \in \mathbb{N}_0^3$ *with* $|\boldsymbol{\beta}| \leqslant 3$. *The constant C depends on the norms of* A, $\boldsymbol{m}$, $\partial_k \boldsymbol{w}(\cdot, k_0)$, $\partial_k^2 \boldsymbol{w}(\cdot, k_0)$, $\boldsymbol{h}$ *and* $\boldsymbol{p}$.

Remark 4.2.5
As mentioned before, by extending the ansatz even further to remove terms of order ε^4 *and higher and under stronger regularity assumptions, it is possible to make the residual even smaller. It should therefore be possible to show* $\left\|\partial^{\beta}\mathbf{Res}(\cdot,t)\right\|_{L^2(\mathbb{R}^2)^3} \leqslant C\varepsilon^{\tau}$ *with an improved exponent* $\tau > \frac{7}{2}$.

The regularity assumptions of Lemma 4.2.4 can easily be satisfied. For the regularity of the envelope A we only need sufficiently smooth initial data to apply Theorem 2.4.1 with $m = 10$.
For the regularity of the x_1-dependent functions we will apply Lemma 3.3.11 for different right-hand sides.

Lemma 4.2.6 (Higher Regularity of $\boldsymbol{U}_{\text{ext}}$ in the x_1-Variable)
Let $\boldsymbol{m}, \partial_k \boldsymbol{w}(k_0), \partial_k^2 \boldsymbol{w}(k_0), \boldsymbol{h}, \boldsymbol{p} \in L^2(\mathbb{R})^3$ *be defined as before. Assume that* $\epsilon_1, \epsilon_3 \in \mathcal{W}^{3,\infty}(\mathbb{R})$.
Then $\boldsymbol{m}, \partial_k \boldsymbol{w}(k_0), \partial_k^2 \boldsymbol{w}(k_0), \boldsymbol{p}, \boldsymbol{h} \in \mathcal{H}^3(\mathbb{R})^3$.

PROOF: We start the proof with the analysis of $\boldsymbol{m}$. Since $(L(k_0) + \nu_0\Lambda)\boldsymbol{m} = \boldsymbol{0}$, we can apply Lemma 3.3.11 with $\boldsymbol{f} = \boldsymbol{0}$ to see that $\boldsymbol{m} \in \mathcal{H}^3(\mathbb{R})^3$.
Next, by differentiating $(L(k) + \omega\Lambda)\boldsymbol{w} = \boldsymbol{0}$ in k, we see that $\partial_k \boldsymbol{w}(k_0)$ and $\partial_k^2 \boldsymbol{w}(k_0)$ solve

$$\begin{aligned}
(L(k_0) + \nu_0\Lambda)\partial_k \boldsymbol{w}(k_0) &= -(\partial_k L(k_0) + \partial_k\omega(k_0)\Lambda)\boldsymbol{m}, \\
(L(k_0) + \nu_0\Lambda)\partial_k^2 \boldsymbol{w}(k_0) &= -2(\partial_k L(k_0) + \partial_k\omega(k_0)\Lambda)\partial_k \boldsymbol{w}(k_0) - \left(\partial_k^2 L(k_0) + \partial_k^2\omega(k_0)\Lambda\right)\boldsymbol{m}.
\end{aligned}$$

Since $\boldsymbol{m} \in \mathcal{H}^3(\mathbb{R})^3$, the functions

$$(\partial_k L(k_0) + \partial_k\omega(k_0)\Lambda)\boldsymbol{m} = \begin{pmatrix} \epsilon_1\nu_1 m_1 + m_3 \\ \epsilon_1\nu_1 m_2 \\ m_1 + \mu_0\nu_1 m_3 \end{pmatrix}, \qquad \left(\partial_k^2 L(k_0) + \partial_k^2\omega(k_0)\Lambda\right)\boldsymbol{m} = \begin{pmatrix} \epsilon_1\nu_2 m_1 \\ \epsilon_1\nu_2 m_2 \\ \mu_0\nu_2 m_3 \end{pmatrix}$$

belong to $\mathcal{H}^3(\mathbb{R})^3$. Therefore, the assumptions of Lemma 3.3.11 are satisfied and we infer that $\partial_k \boldsymbol{w}(k_0) \in \mathcal{H}^3(\mathbb{R})^3$. This fact implies that

$$(\partial_k L(k_0) + \partial_k\omega(k_0)\Lambda)\partial_k \boldsymbol{w}(k_0) \in \mathcal{H}^3(\mathbb{R})^3$$

and consequently $\partial_k^2 \boldsymbol{w}(k_0) \in \mathcal{H}^3(\mathbb{R})^3$ by Lemma 3.3.11.

To treat $\boldsymbol{p}$ and $\boldsymbol{h}$, we note that the right-hand sides in

$$(L(k_0)+\nu_0\Lambda)\boldsymbol{p} = -\kappa\begin{pmatrix}\epsilon_1 m_1\\ \epsilon_1 m_2\\ \mu_0 m_3\end{pmatrix} - \epsilon_3\nu_0\begin{pmatrix}3m_1^3 - m_1 m_2^2\\ -3m_2^3 + m_1^2 m_2\\ 0\end{pmatrix}$$

and

$$(L(3k_0)+3\nu_0\Lambda)\boldsymbol{h} = -3\nu_0\epsilon_3\begin{pmatrix}m_1^3 + m_1 m_2^2\\ m_2^3 + m_2 m_1^2\\ 0\end{pmatrix}$$

are also contained in $\mathcal{H}^3(\mathbb{R})^3$ since $\boldsymbol{m} \in \mathcal{H}^3(\mathbb{R})^3$. Hence, the statement follows as before.

□

Let us finish this section by studying the regularity of $\boldsymbol{U}_{\text{ext}}$ under the regularity assumptions of Lemma 4.2.4 and state some useful estimates.

Remark 4.2.7

By the structure of $\boldsymbol{U}_{\text{ext}}$ it follows under the Assumptions (4.2.7) *and* (4.2.8) *that*

$$\boldsymbol{U}_{\text{ext}} \in \bigcap_{k=0}^{4} C^{4-k}\left([0, T_0\varepsilon^{-2}], \mathcal{H}^{\min\{3,k\}}(\mathbb{R}^2)\right)^3 .$$

Indeed, the space regularity is given by $\min\{3,k\}$ *since $\boldsymbol{U}_{\text{ext}}$ contains $\partial_{X_2}^2 A$ and is therefore the sum of products of functions in (X_2, t) that are at least in $\bigcap_{k=0}^4 C^{4-k}\left([0,T_0], H^k(\mathbb{R})\right)$ and functions in x_1 that are in $\mathcal{H}^3(\mathbb{R})$. Due to the simpler structure, $\boldsymbol{U}_{\text{ans}}$ even satisfies*

$$\boldsymbol{U}_{\text{ans}} \in \bigcap_{k=0}^{4} C^{4-k}\left(\left[0, T_0\varepsilon^{-2}\right], \mathcal{H}^{\min\{3,k+2\}}(\mathbb{R}^2)\right)^3 . \tag{4.2.10}$$

In Section 6.2 it will be necessary to estimate $\partial^{\boldsymbol{\alpha}}\boldsymbol{U}_{\text{ext}}$ and $\partial_t\partial^{\boldsymbol{\alpha}}\boldsymbol{U}_{\text{ext}}$ for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_t)^\top \in \mathbb{N}_0^3$ with $|\boldsymbol{\alpha}| \leqslant 3$.

To get such estimates we use (4.2.7) *and* (4.2.8)*. In addition, we note that the structure of $\boldsymbol{U}_{\text{ext}}$ allows us to use the Sobolev embedding $\mathcal{H}^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ in both space dimensions separately, thus avoiding the less favorable embedding $\mathcal{H}^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$.*

Let us now start with estimates for $\partial^{\boldsymbol{\alpha}}\boldsymbol{U}_{\text{ext}}$. For $|\boldsymbol{\alpha}| \leqslant 3$ and $\alpha_1 \leqslant 2$ we get with the just mentioned Sobolev embedding

$$\|\partial^{\boldsymbol{\alpha}}\boldsymbol{U}_{\text{ext}}\|_{L^\infty(\mathbb{R}^2\times(0,T_0\varepsilon^{-2}))^3} \leqslant C\varepsilon. \tag{4.2.11}$$

Note the factor ε since all terms in $\boldsymbol{U}_{\text{ext}}$ are at lest of order ε.

For $\alpha_1 = 3$ we can use the Sobolev embedding only in the x_2-dimension and get

$$\left\|\partial_{x_1}^3\boldsymbol{U}_{\text{ext}}(\cdot,t)\right\|_{L^2(\mathbb{R}^2)^3}^2 \leqslant \int_{\mathbb{R}} \sup_{x_2\in\mathbb{R}} |\partial_{x_1}^3\boldsymbol{U}_{\text{ext}}(x_1,x_2,t)|^2 \,\mathrm{d}x_1 \leqslant C\varepsilon^2, \quad \forall\, t \in \left(0, T_0\varepsilon^{-2}\right). \tag{4.2.12}$$

To estimate $\partial^{\boldsymbol{\alpha}}\partial_t \boldsymbol{U}_{\text{ext}}$*, we start with* $|\boldsymbol{\alpha}| \leqslant 3$*,* $\alpha_1 \in \{1,2\}$*, where we can apply the Sobolev embedding in both space dimensions and get*

$$\|\partial^{\boldsymbol{\alpha}}\partial_t \boldsymbol{U}_{\text{ext}}\|_{L^\infty(\mathbb{R}^2\times(0,T_0\varepsilon^{-2}))^3} \leqslant C\varepsilon. \tag{4.2.13}$$

For $\alpha_1 = 3$ *we get*

$$\left\|\partial_{x_1}^3\partial_t \boldsymbol{U}_{\text{ext}}(\cdot,t)\right\|_{L^2(\mathbb{R}^2)^3}^2 \leqslant \int_{\mathbb{R}} \sup_{x_2\in\mathbb{R}} |\partial_{x_1}^3\partial_t \boldsymbol{U}_{\text{ext}}(x_1,x_2,t)|^2 \,\mathrm{d}x_1 \leqslant C\varepsilon^2. \tag{4.2.14}$$

Finally, for $|\boldsymbol{\alpha}| = 3$*,* $\alpha_1 = 0$ *we cannot simply use an estimate analogous to* (4.2.12) *by switching the roles of* x_1 *and* x_2*, since integrals in* x_2 *would lose half an order of* ε*, compare* (4.1.3)*, which is not enough for our estimates in Section 6.2. But we can use the structure of* $\boldsymbol{U}_{\text{ext}}$ *and write* $\partial^{\boldsymbol{\alpha}}\partial_t \boldsymbol{U}_{\text{ext}} = \boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{B}}$ *with*

$$\begin{aligned}
\boldsymbol{\mathcal{A}}(x_1,x_2,t) &:= \varepsilon A(X_2,T)\boldsymbol{m}(x_1)\partial^{\boldsymbol{\alpha}}\partial_t\left(\mathrm{e}^{\mathrm{i}(k_0x_2-\nu_0 t)}\right),\\
\boldsymbol{\mathcal{B}}(x_1,x_2,t) &:= \partial^{\boldsymbol{\alpha}}\partial_t \boldsymbol{U}_{\text{ext}}(x_1,x_2,t) - \boldsymbol{\mathcal{A}}(x_1,x_2,t).
\end{aligned}$$

Now the terms in $\boldsymbol{\mathcal{B}}$ *are at least of order* ε^2 *and we can compensate the loss of half an order of* ε*. We get*

$$\|\boldsymbol{\mathcal{A}}\|_{L^\infty(\mathbb{R}^2\times(0,T_0\varepsilon^{-2}))^3} \leqslant C\varepsilon, \qquad \int_{\mathbb{R}} \sup_{x_1\in\mathbb{R}} |\boldsymbol{\mathcal{B}}(x_1,x_2,t)|^2 \,\mathrm{d}x_2 \leqslant C\varepsilon^2. \tag{4.2.15}$$

Note that it is possible to improve the regularity of $\boldsymbol{U}_{\text{ext}}$ *by selecting more regular functions* $A^{(0)}$*,* ϵ_1*,* ϵ_3 *in our construction of* $\boldsymbol{U}_{\text{ext}}$*. With a sufficiently regular* $\boldsymbol{U}_{\text{ext}}$ *we could always use the Sobolev embedding and estimate*

$$\left\|\partial^{\boldsymbol{\alpha}}\partial_t^k \boldsymbol{U}_{\text{ext}}\right\|_{L^\infty(\mathbb{R}^2\times(0,T_0\varepsilon^{-2}))^3} \leqslant C\varepsilon,$$

for some $k \in \mathbb{N}_0$ *and* $\boldsymbol{\alpha} \in \mathbb{N}_0^3$*.*

To summarize the chapter, we have constructed a formal approximative solution $\boldsymbol{U}_{\text{ext}}$ of Maxwell's equations such that estimate (4.2.9) holds true for the residual $\mathbf{Res}(\boldsymbol{U}_{\text{ext}})$. This estimate and the estimates for $\boldsymbol{U}_{\text{ext}}$ itself will play an essential role in our rigorous analysis of the approximation properties in Chapter 6.

5. Local Existence for Hyperbolic Systems

In this chapter we will rewrite the Maxwell problem as a hyperbolic system with the goal of applying the local existence results and a priori estimates for linear and nonlinear hyperbolic systems from [67]. Only some slight adaptions for our setting are necessary, we will therefore mostly refer to [67] and the accompanying results in [75, 76, 77] for the proofs. These results will be the central tools for the error analysis in Chapter 6.

Remark 5.0.1
There are the following differences in [67] compared to our setting.
First, slightly more regular coefficients are used in [67] to shorten some of the computations. But the adaption to our setting is straightforward, see Remark 5.1.3.
Second, since we are working with TM-modes and x_3*-independent functions, our hyperbolic system contains only three equations dependent on* x_1, x_2, t*. This is in some sense easier than the case covered in [67] where the full Maxwell problem with six equations is studied. Note that due to the reduction from a problem on* $\mathbb{R}^3 \times [0, T']$ *to a problem on* $\mathbb{R}^2 \times [0, T']$ *one has to check the arguments that are dependent on the dimension. Studying the proofs in [67] reveals that one has to check if the Sobolev embeddings and Banach algebra property still hold for* $W^{m,p}(\Omega)$*, with* $\Omega \subset \mathbb{R}^2$ *instead of* $\Omega \subset \mathbb{R}^3$*. This is obviously the case since* $mp > 3$ *implies* $mp > 2$*.*
Third, in [67] more general domains are studied and a lengthy localization argument is used to transform the problem to a half-space setting. For our problem a simple reflection is enough to transform the interface problem with two half-spaces to a boundary problem on one half-space, see Remark 5.1.5.

Throughout this chapter let $T' > 0$ and $J := (0, T')$. In contrast to [67], we will work in the space domain $\mathbb{R}^2$, which will be separated by the interface $\Gamma_2 = \{x \in \mathbb{R}^2 \mid x_1 = 0\}$ into the two half-spaces $\mathbb{R}^2_+ = \mathbb{R}_+ \times \mathbb{R}$ and $\mathbb{R}^2_- = \mathbb{R}_- \times \mathbb{R}$ and for a function $f : \mathbb{R}^2 \to \mathbb{R}$ we will denote the restrictions to $\mathbb{R}^2_+$ and $\mathbb{R}^2_-$ by f^+ and f^-, respectively.

5.1. Linear Hyperbolic Systems

We start with a linear symmetric hyperbolic system for $\boldsymbol{u} : \mathbb{R}^2 \times J \to \mathbb{R}^3$,

$$\begin{cases} A_{\mathrm{t}}^{\pm}(\boldsymbol{x},t)\partial_t \boldsymbol{u}^{\pm} + \sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{u}^{\pm} + M^{\pm}(\boldsymbol{x},t)\boldsymbol{u}^{\pm} = \boldsymbol{f}^{\pm}, & \boldsymbol{x} \in \mathbb{R}^2_{\pm}, \quad t \in J, \\ B_\Gamma \begin{pmatrix} \boldsymbol{u}^+ \\ \boldsymbol{u}^- \end{pmatrix} = \mathbf{0}, & \boldsymbol{x} \in \Gamma_2, \quad t \in J, \\ \boldsymbol{u}^{\pm}(\cdot,0) = \boldsymbol{u}^{(0),\pm}, & \boldsymbol{x} \in \mathbb{R}^2_{\pm}, \end{cases} \tag{5.1.1}$$

where $A_{\mathrm{t}} : \mathbb{R}^2 \times J \to \mathbb{R}^{3\times 3}$, A_{t} is symmetric for all $(\boldsymbol{x},t)$,

$$A_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$M : \mathbb{R}^2 \times J \to \mathbb{R}^{3\times 3}$ and

$$B_\Gamma := \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

Clearly, $B_\Gamma(\boldsymbol{u}^+,\boldsymbol{u}^-)^\top$ encodes the interface conditions $[\![u_2]\!]_{2\mathrm{D}} = [\![u_3]\!]_{2\mathrm{D}} = 0$ and $\sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{u}^{\pm}$ contains the spatial derivatives we have seen in (4.0.2). Note that we are not going to use (5.1.1) in order to study the linear part of (4.0.2), but rather to study a fixed-point problem in the bootstrapping argument for the nonlinear system in Section 6.2. Hence, we need the inhomogeneous term $\boldsymbol{f}$, the linear term $M\boldsymbol{U}$ and a matrix A_{t} different from Λ in (5.1.1). The concrete A_{t}, M and $\boldsymbol{f}$ that connect (5.1.1) with Maxwell's equations (4.0.2) will be discussed in Section 6.2.

Note that the coefficients and functions in (5.1.1) are in general discontinuous in $x_1 = 0$.

Definition 5.1.1 (Weak Solution of the Linear Hyperbolic Problem)
Under a weak solution of (5.1.1) *we understand a function* $\boldsymbol{u} \in C\left(\overline{J}, L^2(\mathbb{R}^2)\right)^3$ *that satisfies*

$$\int_J \int_{\mathbb{R}^2} \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = -\int_J \int_{\mathbb{R}^2} \Big(\boldsymbol{u} \cdot \partial_t (A_{\mathrm{t}} \boldsymbol{\varphi}) + \boldsymbol{u} \cdot \partial_{x_1}(A_1 \boldsymbol{\varphi}) + \boldsymbol{u} \cdot \partial_{x_2}(A_2 \boldsymbol{\varphi}) - \boldsymbol{u} \cdot M^\top \boldsymbol{\varphi} \Big) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t$$

for all test functions

$$\boldsymbol{\varphi} \in \left\{ \boldsymbol{\psi} \,\middle|\, \boldsymbol{\psi}^+ \in H_0^1\left(\mathbb{R}^2_+ \times J\right)^3, \, \boldsymbol{\psi}^- \in H_0^1\left(\mathbb{R}^2_- \times J\right)^3 \right\},$$

$\mathrm{Tr}_\Gamma\left(B_\Gamma(\boldsymbol{u}^+,\boldsymbol{u}^-)^\top\right) = \mathbf{0}$ *and* $\boldsymbol{u}(\cdot,0) = \boldsymbol{u}^{(0)}$.

For $\boldsymbol{u} \in C\left(\overline{J}, L^2(\mathbb{R}^2)\right)^3$ it is not immediately clear how and if $\mathrm{Tr}_\Gamma\left(B_\Gamma(\boldsymbol{u}^+,\boldsymbol{u}^-)^\top\right)$ is well-defined, but the special structure of the matrices A_1 and B_Γ allow us to define this trace.

We will present an in-depth discussion in Section 5.4. Note that we will often omit writing the trace operator and implicitly assume all jump conditions in the sense of traces.
More regularity will be necessary for the existence results of this chapter. We therefore define the space

$$\mathcal{G}^m(\mathbb{R}^2 \times J)^3 := \bigcap_{j=0}^{m} C^j\left(\bar{J}, \mathcal{H}^{m-j}(\mathbb{R}^2)\right)^3$$

with $m \in \mathbb{N}_0$ and the norm

$$\|\boldsymbol{u}\|_{\mathcal{G}^m(\mathbb{R}^2\times J)^3} := \max_{0\leqslant j\leqslant m} \left\|\partial_t^j \boldsymbol{u}\right\|_{L^\infty\left(J,\mathcal{H}^{m-j}(\mathbb{R}^2)\right)^3}.$$

Remark 5.1.2
The estimates of our final approximation result will be done in the $\mathcal{G}^3$-norm, hence we have to control the norm of all space and temporal derivatives up to order 3.

The main goal of this section is to show that there exists a solution $\boldsymbol{u} \in \mathcal{G}^m(\mathbb{R}^2 \times J)^3$ of (5.1.1) and that $\boldsymbol{u}$ satisfies a certain a priori estimate, see Theorem 5.1.9.
But before we can prove Theorem 5.1.9, we first define some additional function spaces. For any open $\Omega \subset \mathbb{R}^2$, $m, n \in \mathbb{N}_0$ we will use

$$\begin{aligned}
F^{m,n}(\Omega \times J) &:= \Big\{A \in W^{1,\infty}(\Omega \times J)^{n\times n} \,\Big|\, \partial^{\boldsymbol{\alpha}} A \in L^\infty(J, H^{m-|\boldsymbol{\alpha}|}(\Omega))^{n\times n} + W^{m-|\boldsymbol{\alpha}|,\infty}(\Omega \times J)^{n\times n} \\
&\qquad \text{for all } \boldsymbol{\alpha} \in \mathbb{N}_0^3 \text{ with } 1 \leqslant |\boldsymbol{\alpha}| \leqslant m\Big\}, \\
\|A\|_{F^{m,n}(\Omega\times J)} &:= \max\left\{\|A\|_{W^{1,\infty}(\Omega\times J)^{n\times n}}, \max_{1\leqslant|\boldsymbol{\alpha}|\leqslant m} \|\partial^{\boldsymbol{\alpha}} A\|_{L^\infty(J,H^{m-|\boldsymbol{\alpha}|}(\Omega))^{n\times n}+W^{m-|\boldsymbol{\alpha}|,\infty}(\Omega\times J)^{n\times n}}\right\}, \\
\mathcal{F}^{m,n}(\mathbb{R}^2 \times J) &:= \Big\{A \in \mathcal{W}^{1,\infty}(\mathbb{R}^2 \times J)^{n\times n} \,\Big|\, A^- \in F^{m,n}\left(\mathbb{R}^2_- \times J\right), A^+ \in F^{m,n}\left(\mathbb{R}^2_+ \times J\right)\Big\}, \\
\|A\|_{\mathcal{F}^{m,n}(\mathbb{R}^2\times J)} &:= \max\left\{\|A^-\|_{F^{m,n}\left(\mathbb{R}^2_-\times J\right)}, \|A^+\|_{F^{m,n}\left(\mathbb{R}^2_+\times J\right)}\right\},
\end{aligned}$$

with the usual definition for the sum of two vector spaces

$$\begin{aligned}
&L^\infty(J, H^{m-|\boldsymbol{\alpha}|}(\Omega)) + W^{m-|\boldsymbol{\alpha}|,\infty}(\Omega \times J) \\
&\qquad := \Big\{A : \Omega \times J \to \mathbb{R} \,\Big|\, A = \widetilde{B} + \widetilde{C}, \widetilde{B} \in L^\infty(J, H^{m-|\boldsymbol{\alpha}|}(\Omega)), \widetilde{C} \in W^{m-|\boldsymbol{\alpha}|,\infty}(\Omega \times J)\Big\}, \\
&\|A\|_{L^\infty(J,H^{m-|\boldsymbol{\alpha}|}(\Omega))+W^{m-|\boldsymbol{\alpha}|,\infty}(\Omega\times J)} \\
&\qquad := \inf\Big\{\|\widetilde{B}\|_{L^\infty(J,H^{m-|\boldsymbol{\alpha}|}(\Omega))} + \|\widetilde{C}\|_{W^{m-|\boldsymbol{\alpha}|,\infty}(\Omega\times J)} \,\Big|\, A = \widetilde{B} + \widetilde{C}, \\
&\qquad\qquad \widetilde{B} \in L^\infty(J, H^{m-|\boldsymbol{\alpha}|}(\Omega)), \widetilde{C} \in W^{m-|\boldsymbol{\alpha}|,\infty}(\Omega \times J)\Big\}.
\end{aligned}$$

For a fixed time instant we use the spaces

$$\begin{aligned}
F_0^{m,n}(\Omega) &:= \Big\{A \in L^\infty(\Omega)^{n\times n} \,\Big|\, \partial^{\boldsymbol{\alpha}} A \in H^{m-|\boldsymbol{\alpha}|}(\Omega)^{n\times n} + W^{m-|\boldsymbol{\alpha}|,\infty}(\Omega)^{n\times n} \\
&\qquad \text{for all } \boldsymbol{\alpha} \in \mathbb{N}_0^2 \text{ with } 1 \leqslant |\boldsymbol{\alpha}| \leqslant m\Big\}, \\
\|A\|_{F_0^{m,n}(\Omega)} &:= \max\Big\{\|A\|_{L^\infty(\Omega)^{n\times n}}, \max_{1\leqslant|\boldsymbol{\alpha}|\leqslant m} \|\partial^{\boldsymbol{\alpha}} A\|_{H^{m-|\boldsymbol{\alpha}|}(\Omega)^{n\times n}+W^{m-|\boldsymbol{\alpha}|,\infty}(\Omega)^{n\times n}}\Big\}, \\
\mathcal{F}_0^{m,n}(\mathbb{R}^2) &:= \{A \in L^\infty(\mathbb{R}^2)^{n\times n} \,|\, A^- \in F_0^{m,n}(\mathbb{R}^2_-),\, A^+ \in F_0^{m,n}(\mathbb{R}^2_+)\}, \\
\|A\|_{\mathcal{F}_0^{m,n}(\mathbb{R}^2)} &:= \max\Big\{\|A^-\|_{F_0^{m,n}(\mathbb{R}^2_-)}, \|A^+\|_{F_0^{m,n}(\mathbb{R}^2_+)}\Big\}.
\end{aligned}$$

Finally, we will define some subspaces of $\mathcal{F}^{m,n}$ and indicate them by certain subscripts. The subscript $\eta > 0$ indicates that additionally A is symmetric positive definite with constant η, the subscript "cp" means that A is constant outside of a compact set, and the subscript "cv" means that A is convergent for $|(\boldsymbol{x}, t)| \to \infty$, i.e. for $\eta > 0$ we define

$$\begin{aligned}
\mathcal{F}_\eta^{m,n}(\mathbb{R}^2 \times J) &:= \Big\{A \in \mathcal{F}^{m,n}(\mathbb{R}^2 \times J) \,\Big|\, A = A^\top, \boldsymbol{v}^\top A \boldsymbol{v} \geqslant \eta|\boldsymbol{v}|^2 \text{ for all } \boldsymbol{v} \in \mathbb{R}^n\Big\}, \\
\mathcal{F}_{\mathrm{cp}}^{m,n}(\mathbb{R}^2 \times J) &:= \Big\{A \in \mathcal{F}^{m,n}(\mathbb{R}^2 \times J) \,\Big|\, \text{for all } A \text{ there exists a matrix } \widetilde{A} \in \mathbb{R}^{n\times n} \\
&\qquad \text{and a compact set } M \subset \mathbb{R}^2 \times J \text{ with } A(\boldsymbol{x}, t) = \widetilde{A} \text{ for } (\boldsymbol{x}, t) \notin M\Big\}, \\
\mathcal{F}_{\mathrm{cv}}^{m,n}(\mathbb{R}^2 \times J) &:= \Big\{A \in \mathcal{F}^{m,n}(\mathbb{R}^2 \times J) \,\Big|\, \text{for all } A \text{ there exists a matrix } \widetilde{A} \in \mathbb{R}^{n\times n} \\
&\qquad \text{with } A(\boldsymbol{x}, t) \to \widetilde{A} \text{ for } |(\boldsymbol{x}, t)| \to \infty\Big\}.
\end{aligned}$$

Subspaces with multiple subscripts are possible, e.g.

$$\mathcal{F}_{\eta,\mathrm{cv}}^{m,n}(\mathbb{R}^2 \times J) := \mathcal{F}_\eta^{m,n}(\mathbb{R}^2 \times J) \cap \mathcal{F}_{\mathrm{cv}}^{m,n}(\mathbb{R}^2 \times J).$$

We are mainly interested in the case $\Omega = \mathbb{R}^2_\pm$ and $m = n = 3$.

Remark 5.1.3
Let $A \in F^{m,n}(\Omega \times J)$. Note that the part of $\partial^{\boldsymbol{\alpha}} A$ in $W^{m-|\boldsymbol{\alpha}|,\infty}(\Omega \times J)$ can often be estimated more easily than the part in $L^\infty(J, H^{m-|\boldsymbol{\alpha}|}(\Omega))$, e.g. the product of an L^∞-function and an L^2-function is easier to estimate than the product of two L^2-functions. Therefore, the $W^{m-|\boldsymbol{\alpha}|,\infty}(\Omega \times J)$-part is mostly omitted in the proofs of [67] and a different definition for the spaces $F^{m,n}$ and $\mathcal{F}^{m,n}$ is used, see Remark 6.1 in [67]. Since such terms appear in our setting, we will not omit them.

The next lemma discusses products between $\mathcal{H}^m(\mathbb{R}^2)$- and $\mathcal{F}_0^{m,1}(\mathbb{R}^2)$-functions.

Lemma 5.1.4 (Product Estimates)
Let $m_1, m_2 \in \mathbb{N}_0$ with $m_1 \geqslant m_2$ and $m_1 \geqslant 2$.

i) Let $j \in \{0, \ldots, m_1\}$, $f \in \mathcal{H}^{m_1-j}(\mathbb{R}^2)$ and $g \in \mathcal{H}^j(\mathbb{R}^2)$. Then $fg \in L^2(\mathbb{R}^2)$ and

$$\|fg\|_{L^2(\mathbb{R}^2)} \leqslant C \|f\|_{\mathcal{H}^{m_1-j}(\mathbb{R}^2)} \|g\|_{\mathcal{H}^j(\mathbb{R}^2)}.$$

ii) Let $f \in \mathcal{H}^{m_1}(\mathbb{R}^2)$ and $g \in \mathcal{H}^{m_2}(\mathbb{R}^2)$. Then $fg \in \mathcal{H}^{m_2}(\mathbb{R}^2)$ and

$$\|fg\|_{\mathcal{H}^{m_2}(\mathbb{R}^2)} \leqslant C \|f\|_{\mathcal{H}^{m_1}(\mathbb{R}^2)} \|g\|_{\mathcal{H}^{m_2}(\mathbb{R}^2)}.$$

iii) Let $f \in \mathcal{F}_0^{m_1,1}(\mathbb{R}^2)$ and $g \in \mathcal{H}^{m_2}(\mathbb{R}^2)$. Then $fg \in \mathcal{H}^{m_2}(\mathbb{R}^2)$ and

$$\|fg\|_{\mathcal{H}^{m_2}(\mathbb{R}^2)} \leqslant C \|f\|_{\mathcal{F}_0^{m_1,1}(\mathbb{R}^2)} \|g\|_{\mathcal{H}^{m_2}(\mathbb{R}^2)}.$$

iv) Let $f \in \mathcal{F}^{m_1,1}(\mathbb{R}^2 \times J)$ and $g \in \mathcal{G}^{m_2}(\mathbb{R}^2 \times J)$. Then $fg \in \mathcal{G}^{m_2}(\mathbb{R}^2 \times J)$ and

$$\|fg\|_{\mathcal{G}^{m_2}(\mathbb{R}^2 \times J)} \leqslant C \|f\|_{\mathcal{F}^{m_1,1}(\mathbb{R}^2 \times J)} \|g\|_{\mathcal{G}^{m_2}(\mathbb{R}^2 \times J)}.$$

v) Let $f \in \mathcal{F}^{m_2,1}(\mathbb{R}^2 \times J)$ and $g \in \mathcal{G}^{m_1}(\mathbb{R}^2 \times J)$. Then $fg \in \mathcal{G}^{m_2}(\mathbb{R}^2 \times J)$ and

$$\|fg\|_{\mathcal{G}^{m_2}(\mathbb{R}^2 \times J)} \leqslant C \|f\|_{\mathcal{F}^{m_2,1}(\mathbb{R}^2 \times J)} \|g\|_{\mathcal{G}^{m_1}(\mathbb{R}^2 \times J)}.$$

vi) Let $f \in \mathcal{F}_0^{m_1,1}(\mathbb{R}^2)$ and $g \in \mathcal{F}_0^{m_2,1}(\mathbb{R}^2)$. Then $fg \in \mathcal{F}_0^{m_2,1}(\mathbb{R}^2)$ and

$$\|fg\|_{\mathcal{F}_0^{m_2,1}(\mathbb{R}^2)} \leqslant C \|f\|_{\mathcal{F}_0^{m_1,1}(\mathbb{R}^2)} \|g\|_{\mathcal{F}_0^{m_2,1}(\mathbb{R}^2)}.$$

PROOF: The proof is based on the Hölder inequality and Sobolev embeddings. Details for the three-dimensional case can be found in the proof of Lemma 2.22 in [75].
To prove the assertions above it is sufficient to show the analogous results for each half-space. W.l.o.g. we will focus on the half-space $\mathbb{R}^2_+$. The main tool will be the Sobolev embeddings $H^2\left(\mathbb{R}^2_+\right) \hookrightarrow L^q\left(\mathbb{R}^2_+\right)$ for $2 \leqslant q \leqslant \infty$ and $H^1\left(\mathbb{R}^2_+\right) \hookrightarrow L^q\left(\mathbb{R}^2_+\right)$ for $2 \leqslant q < \infty$.
i): Let w.l.o.g. $m_1 - j \geqslant j$. For $j = 0$ we have $f \in H^2\left(\mathbb{R}^2_+\right)$ and hence

$$\|fg\|_{L^2(\mathbb{R}^2_+)} \leqslant \|f\|_{L^\infty(\mathbb{R}^2_+)} \|g\|_{L^2(\mathbb{R}^2_+)} \leqslant C \|f\|_{H^{m_1}(\mathbb{R}^2_+)} \|g\|_{H^0(\mathbb{R}^2_+)}.$$

For the remaining cases of j we know that f, g are at least $H^1\left(\mathbb{R}^2_+\right)$-functions and therefore we can use the generalized Hölder inequality to show

$$\|fg\|_{L^2(\mathbb{R}^2_+)} \leqslant \|f\|_{L^3(\mathbb{R}^2_+)} \|g\|_{L^6(\mathbb{R}^2_+)} \leqslant C \|f\|_{H^{m_1-j}(\mathbb{R}^2_+)} \|g\|_{H^{m_2}(\mathbb{R}^2_+)}.$$

ii): For $\boldsymbol{\alpha} \in \mathbb{N}_0^2$ with $|\boldsymbol{\alpha}| \leqslant m_2$ we have

$$\partial^{\boldsymbol{\alpha}}(fg) = \sum_{0 \leqslant \boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \partial^{\boldsymbol{\beta}} f \partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}} g. \tag{5.1.2}$$

Here we used the multidimensional Leibniz rule with $\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} := \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!(\boldsymbol{\alpha}-\boldsymbol{\beta})!}$ and $\boldsymbol{\alpha}! := \prod_{j=1}^{2} \alpha_j!$ for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^2$.
Now for every fixed $\boldsymbol{\beta}$ we have for $j := |\boldsymbol{\beta}|$ that $\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}} g \in H^{m_2-|\boldsymbol{\alpha}|+j}\left(\mathbb{R}_+^2\right) \subset H^j\left(\mathbb{R}_+^2\right)$ and $\partial^{\boldsymbol{\beta}} f \in H^{m_1-j}\left(\mathbb{R}_+^2\right)$. Now the statement follows from the repeated application of i).
iii): We again use (5.1.2) for $|\boldsymbol{\alpha}| \leqslant m_2$ and set $j := |\boldsymbol{\beta}|$. For $j = 0$ we have

$$\left\|\partial^{\boldsymbol{\beta}} f \partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}} g\right\|_{L^2\left(\mathbb{R}_+^2\right)} = \|f \partial^{\boldsymbol{\alpha}} g\|_{L^2\left(\mathbb{R}_+^2\right)} \leqslant \|f\|_{L^\infty\left(\mathbb{R}_+^2\right)} \|\partial^{\boldsymbol{\alpha}} g\|_{L^2\left(\mathbb{R}_+^2\right)} \leqslant C \|f\|_{\mathcal{F}_0^{m_1,1}(\mathbb{R}^2)} \|g\|_{\mathcal{H}^{m_2}(\mathbb{R}^2)}.$$

For $j \geqslant 1$ we have $\partial^{\boldsymbol{\beta}} f = \widetilde{B} + \widetilde{C}$ with $\widetilde{B} \in H^{m_1-j}\left(\mathbb{R}_+^2\right)$, $\widetilde{C} \in W^{m_1-j,\infty}\left(\mathbb{R}_+^2\right)$ and $\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}} g \in H^j\left(\mathbb{R}_+^2\right)$. The statement follows from the triangle inequality,

$$\left\|\widetilde{C} \partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}} g\right\|_{L^2\left(\mathbb{R}_+^2\right)} \leqslant \|\widetilde{C}\|_{L^\infty\left(\mathbb{R}_+^2\right)} \left\|\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}} g\right\|_{L^2\left(\mathbb{R}_+^2\right)} \leqslant C \|f\|_{F_0^{m_1,1}\left(\mathbb{R}_+^2\right)} \|g\|_{H^{m_2}\left(\mathbb{R}_+^2\right)}$$

and the repeated application of i) to estimate $\left\|B \partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}} g\right\|_{L^2\left(\mathbb{R}_+^2\right)}$.
iv) By definition of $F^{m_1,1}\left(\mathbb{R}_+^2 \times J\right)$ and $\mathcal{G}^{m_2}(\mathbb{R}^2 \times J)$ it follows that f, g and all their derivatives are L^∞-functions in time. The statement then follows from an application of iii) for all fixed time points.
v): This follows analogously to iv) since i) is symmetric in f, g.
vi): We again use (5.1.2) and set $j := |\boldsymbol{\beta}|$. First, we have

$$\|fg\|_{L^\infty\left(\mathbb{R}_+^2\right)} \leqslant \|f\|_{L^\infty\left(\mathbb{R}_+^2\right)} \|g\|_{L^\infty\left(\mathbb{R}_+^2\right)} \leqslant \|f\|_{F_0^{m_1,1}\left(\mathbb{R}_+^2\right)} \|g\|_{F_0^{m_2,1}\left(\mathbb{R}_+^2\right)}.$$

To estimate $\|\partial^{\boldsymbol{\alpha}}(fg)\|_{H^{m-|\boldsymbol{\alpha}|}(\Omega)^{n\times n} + W^{m-|\boldsymbol{\alpha}|,\infty}(\Omega)^{n\times n}}$ we proceed as before and use that for $j \geqslant 1$ we have $\partial^{\boldsymbol{\beta}} f \in H^{m_1-j}\left(\mathbb{R}_+^2\right) + W^{m_1-j,\infty}\left(\mathbb{R}_+^2\right)$ and $\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}} g \in H^{m_2-|\boldsymbol{\alpha}|+j}\left(\mathbb{R}_+^2\right) + W^{m_2-|\boldsymbol{\alpha}|+j,\infty}(\mathbb{R}_+^2)$. The statement follows again from the repeated application of i) and obvious estimates for the $W^{m,\infty}(\mathbb{R}_+^2)$-terms.

□

Let us now describe the transformation of the interface problem (5.1.1) to a boundary value half-space problem.

Remark 5.1.5
One key step to prove the existence of a solution of (5.1.1) *is the transformation of the interface problem to a boundary value problem on the half-space* $\mathbb{R}_+^2$. *To achieve this, the left part of the problem will be reflected to the right. This reflection can easily be done in our setting.*

We define the new matrices

$$\mathcal{A}_\mathrm{t}(\boldsymbol{x},\boldsymbol{t}) := \begin{pmatrix} A_\mathrm{t}(x_1,x_2,t) & 0 \\ 0 & A_\mathrm{t}(-x_1,x_2,t) \end{pmatrix}, \quad \mathcal{M}(\boldsymbol{x},\boldsymbol{t}) := \begin{pmatrix} M(x_1,x_2,t) & 0 \\ 0 & M(-x_1,x_2,t) \end{pmatrix},$$
$$\mathcal{A}_1(\boldsymbol{x},\boldsymbol{t}) := \begin{pmatrix} A_1 & 0 \\ 0 & -A_1 \end{pmatrix}, \qquad \mathcal{A}_2(\boldsymbol{x},\boldsymbol{t}) := \begin{pmatrix} A_2 & 0 \\ 0 & A_2 \end{pmatrix}$$

and new vector functions

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{x},t) &:= \left(\boldsymbol{u}^+(x_1,x_2,t), \boldsymbol{u}^-(-x_1,x_2,t)\right)^\top, \\ \boldsymbol{v}^{(0)}(\boldsymbol{x}) &:= \left(\boldsymbol{u}^{(0),+}(x_1,x_2), \boldsymbol{u}^{(0),-}(-x_1,x_2)\right)^\top, \\ \boldsymbol{g}(\boldsymbol{x},t) &:= \left(\boldsymbol{f}^+(x_1,x_2,t), \boldsymbol{f}^-(-x_1,x_2,t)\right)^\top \end{aligned}$$

and get an equivalent system to (5.1.1) *on* $\mathbb{R}^2_+$*:*

$$\begin{cases} \mathcal{A}_\mathrm{t}\partial_t \boldsymbol{v} + \sum_{j=1}^{2} \mathcal{A}_j \partial_{x_j} \boldsymbol{v} + \mathcal{M}\boldsymbol{v} = \boldsymbol{g}, & \boldsymbol{x} \in \mathbb{R}^2_+, \quad t \in J, \\ B_\Gamma \boldsymbol{v} = \boldsymbol{0}, & \boldsymbol{x} \in \Gamma_2, \quad t \in J, \\ \boldsymbol{v}(\cdot,0) = \boldsymbol{v}^{(0)}, & \boldsymbol{x} \in \mathbb{R}^2_+. \end{cases} \tag{5.1.3}$$

Analogously to Definition 5.1.1 we define the weak solution of (5.1.3) *as a function* $\boldsymbol{v} \in C\left(\bar{J}, L^2\left(\mathbb{R}^2_+\right)\right)^6$ *that satisfies*

$$\int_J \int_{\mathbb{R}^2_+} \boldsymbol{g} \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = -\int_J \int_{\mathbb{R}^2_+} \left(\boldsymbol{v} \cdot \partial_t(\mathcal{A}_\mathrm{t}\boldsymbol{\varphi}) + \boldsymbol{v} \cdot \partial_{x_1}(\mathcal{A}_1\boldsymbol{\varphi}) + \boldsymbol{v} \cdot \partial_{x_2}(\mathcal{A}_2\boldsymbol{\varphi}) - \boldsymbol{v} \cdot \mathcal{M}^\top \boldsymbol{\varphi}\right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

for all test functions $\boldsymbol{\varphi} \in H^1_0\left(\mathbb{R}^2_+ \times J\right)^6$, $\mathrm{Tr}_\Gamma\left(B_\Gamma \boldsymbol{v}\right) = \boldsymbol{0}$ *and* $\boldsymbol{v}(\cdot,0) = \boldsymbol{v}^{(0)}$*, see Definition 3.1 in [75]. An in-depth discussion of the transformation of a more general interface problem to a half-space problem can be found in [67].*

The existence of a solution of (5.1.3) was proven in [29]. We will state this result for our setting:

Theorem 5.1.6 (Existence Result in the Half-Space Setting)
Let $T' > 0$ *and* $J = (0, T')$*. Take coefficients* $\mathcal{A}_\mathrm{t} \in W^{1,\infty}\left(\mathbb{R}^2_+ \times J\right)^{6\times 6}$ *and* $\mathcal{M} \in L^\infty\left(\mathbb{R}^2_+ \times J\right)^{6\times 6}$*. Assume additionally that* $\mathcal{A}_\mathrm{t}$ *is uniformly positive definite,* $\mathcal{A}_1$ *has the same number of positive and negative eigenvalues and that there is a matrix* C_Γ *such that*

$$\mathcal{A}_1 = \frac{1}{2}\left(C_\Gamma^\top B_\Gamma + B_\Gamma^\top C_\Gamma\right). \tag{5.1.4}$$

Choose $\boldsymbol{g} \in L^2\left(\mathbb{R}^2_+ \times J\right)^6$, $\boldsymbol{v}^{(0)} \in L^2\left(\mathbb{R}^2_+\right)^6$*.*
Then there is a unique weak solution $\boldsymbol{v} \in C\left(J, L^2\left(\mathbb{R}^2_+\right)\right)^6$ *of* (5.1.3)*.*

PROOF: This result and its proof can be found as Proposition 5.1 in [29] for the general n-dimensional problem.

□

We will now check the algebraic conditions on $\mathcal{A}_1$ from Theorem 5.1.6 for our setting.

Remark 5.1.7

Take $\mathcal{A}_1$ and B_Γ as defined above in this chapter. We can easily check that for

$$C_\Gamma := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

condition (5.1.4) *is satisfied. Furthermore, we note that $\mathcal{A}_1$ has the eigenvalues $\lambda_{1,2} = 0$, $\lambda_{3,4} = 1$ and $\lambda_{5,6} = -1$.*

Since we are interested in Maxwell's equation with a nonlinear displacement field, we have to improve on Theorem 5.1.6 and need solutions of higher regularity.

Remark 5.1.8

The existence result for linear symmetric hyperbolic initial boundary value problems (Theorem 5.1.6) requires coefficients in $W^{1,\infty}\left(\mathbb{R}^2_+ \times J\right)^{6\times 6}$ and yields solutions in $C\left(J, L^2\left(\mathbb{R}^2_+\right)\right)^6$. To apply a fixed-point argument, which delivers an existence result for the nonlinear case, it is necessary that the solution space can be embedded into $W^{1,\infty}\left(\mathbb{R}^2_+ \times J\right)^6$.
Hence, we have to find $m \in \mathbb{N}$ for which

$$C\left(J, H^m\left(\mathbb{R}^2_+\right)\right)^6 \cap C^1\left(J, H^{m-1}\left(\mathbb{R}^2_+\right)\right)^6 \hookrightarrow W^{1,\infty}\left(\mathbb{R}^2_+ \times J\right)^6$$

holds true. Here we will need the Sobolev embedding $H^s(\Omega) \hookrightarrow L^\infty(\Omega)$ for $s > 1$ and sufficiently well-behaved domains $\Omega \subset \mathbb{R}^2$. Now we can show that

$$\begin{aligned} \|u\|_{W^{1,\infty}\left(\mathbb{R}^2_+ \times J\right)} &= \sup_{t\in J} \|u(\cdot,t)\|_{L^\infty\left(\mathbb{R}^2_+\right)} + \sup_{t\in J} \|\partial_t u(\cdot,t)\|_{L^\infty\left(\mathbb{R}^2_+\right)} + \sum_{j=1}^{2} \sup_{t\in J} \left\|\partial_{x_j} u(\cdot,t)\right\|_{L^\infty\left(\mathbb{R}^2_+\right)} \\ &\leqslant C\left(\sup_{t\in J} \|u(\cdot,t)\|_{H^{s+1}\left(\mathbb{R}^2_+\right)} + \sup_{t\in J} \|\partial_t u(\cdot,t)\|_{H^s\left(\mathbb{R}^2_+\right)}\right) \\ &\leqslant C\left(\|u\|_{C\left(J,H^m\left(\mathbb{R}^2_+\right)\right)} + \|u\|_{C^1\left(J,H^{m-1}\left(\mathbb{R}^2_+\right)\right)}\right), \end{aligned}$$

where the first inequality holds for $s > 1$ due to the Sobolev embedding and where we set $m := s + 1$ in the second inequality. We therefore have to work with $m > 2$, more details can be found in [75].

Now an adaptation of Theorem 3.1 of [67] gives us the needed higher regularity.

Theorem 5.1.9 (Linear Existence Result)
Let $\eta, T', r > 0$, $m \in \{0,1,2,3\}$ and $J = (0,T')$. Take coefficients $A_{\mathrm{t}} \in \mathcal{F}^{3,3}_{\eta,\mathrm{cv}}(\mathbb{R}^2 \times J)$ and $M \in \mathcal{F}^{3,3}_{\mathrm{cv}}(\mathbb{R}^2 \times J)$ with

$$\|A_{\mathrm{t}}\|_{\mathcal{F}^{3,3}(\mathbb{R}^2\times J)}, \|A_{\mathrm{t}}(\cdot,0)\|_{\mathcal{F}^{2,3}_0(\mathbb{R}^2)}, \|\partial_t A_{\mathrm{t}}(\cdot,0)\|_{\mathcal{H}^1(\mathbb{R}^2)^{3\times3}}, \left\|\partial_t^2 A_{\mathrm{t}}(\cdot,0)\right\|_{L^2(\mathbb{R}^2)^{3\times3}} \leqslant r,$$
$$\|M\|_{\mathcal{F}^{3,3}(\mathbb{R}^2\times J)}, \|M(\cdot,0)\|_{\mathcal{F}^{2,3}_0(\mathbb{R}^2)}, \|\partial_t M(\cdot,0)\|_{\mathcal{H}^1(\mathbb{R}^2)^{3\times3}}, \left\|\partial_t^2 M(\cdot,0)\right\|_{L^2(\mathbb{R}^2)^{3\times3}} \leqslant r.$$

Choose $\boldsymbol{f} \in \mathcal{H}^m(\mathbb{R}^2 \times J)^3$ and $\boldsymbol{u}^{(0)} \in \mathcal{H}^m(\mathbb{R}^2)^3$ such that the linear compatibility conditions of order m are satisfied, see Section 5.3.
Then there is a unique weak solution $\boldsymbol{u}$ of (5.1.1) in $\mathcal{G}^m(\mathbb{R}^2 \times J)^3$ and a constant $C_m = C_m(r,T') \geqslant 1$ such that

$$\|\boldsymbol{u}\|^2_{\mathcal{G}^m(\mathbb{R}^2\times J)^3} \leqslant C_m \left(\left\|\boldsymbol{u}^{(0)}\right\|^2_{\mathcal{H}^m(\mathbb{R}^2)^3} + \|\boldsymbol{f}\|^2_{\mathcal{H}^m(\mathbb{R}^2\times J)^3} + \sum_{j=0}^{m-1} \left\|\partial_t^j \boldsymbol{f}(\cdot,0)\right\|^2_{\mathcal{H}^{m-1-j}(\mathbb{R}^2)^3} \right), \tag{5.1.5}$$

where the sum is empty if $m = 0$.

PROOF: As mentioned in Remark 5.0.1 we note that Theorem 3.1 of [67] deals with spatial domains in $\mathbb{R}^3$ instead of $\mathbb{R}^2$ and the solution vector takes values in $\mathbb{R}^6$ instead of $\mathbb{R}^3$, but the proof can be repeated for our setting in an analogous way. Thus, we will only give a sketch of the proof, where we describe the essential steps.
The proof is divided into six steps. First, the coefficients for the localized half-space problem are determined. This is much simpler for our setting and is already done in Remark 5.1.5. In Step II and Step III the compatibility conditions for the localized problem are studied and the connection to the compatibility conditions for the interface problem is established. This is again much simpler to do for our setting, see Remark 5.3.3. In Step IV an a priori estimate and regularity for a solution of (5.1.1) is established. The proof is based on a version of Theorem 5.1.6 in combination with further results from [75, 67, 77]. Here techniques similar to our estimates in Section 6.2 are used and the structure of Maxwell's equations is utilized to estimate the normal and tangential derivatives in two different ways. Finally, in the last two steps a fixed-point argument is used to show the existence of a solution $\boldsymbol{u}$ of the interface problem. Here one goes back from the half-space problem to the interface problem, which is much simpler in our setting.
Note that the a priori estimates in [67] work in spaces $\mathcal{G}^m$ with time-weighted norms. For bounded time intervals this is obviously equivalent to the norm for $\mathcal{G}^m$ we introduced above.
□

With this existence result we are well-equipped to study the nonlinear problem. But before we do this in the next section, we will use Theorem 5.1.9 to formulate an approximation argument that will be used later on in Section 6.2.

Lemma 5.1.10 (Approximation Result in Linear Hyperbolic Systems)
Let $T' > 0$, $J = (0, T')$, $\boldsymbol{u}^{(0)} \in L^2(\mathbb{R}^2)^3$, $A_t \in \mathcal{F}^{3,3}_{\eta,\mathrm{cv}}(\mathbb{R}^2 \times J)$, $M \in \mathcal{F}^{3,3}_{\mathrm{cv}}(\mathbb{R}^2 \times J)$ and $\boldsymbol{f} \in \mathcal{G}^0(\mathbb{R}^2 \times J)^3$. Take a weak solution $\boldsymbol{u} \in \mathcal{G}^0(\mathbb{R}^2 \times J)^3$ of (5.1.1) for the data $\left(\boldsymbol{f}, \boldsymbol{u}^{(0)}\right)$. Then the following statements are true:

i) There are sequences

$$\left(\boldsymbol{u}_n^{(0)}\right)_n \subset \mathcal{D}_\Gamma(\mathbb{R}^2)^3 := \left\{\boldsymbol{\varphi} \,\middle|\, \boldsymbol{\varphi}^+ \in C^\infty\left(\mathbb{R}^2_+\right)^3, \boldsymbol{\varphi}^- \in C^\infty\left(\mathbb{R}^2_-\right)^3, \operatorname{supp} \boldsymbol{\varphi} \subset \mathbb{R}^2 \text{ compact}\right\}$$

and $\left(\boldsymbol{f}_n\right)_n \subset \mathcal{H}^1(\mathbb{R}^2 \times J)^3$ with $\boldsymbol{u}_n^{(0)} \to \boldsymbol{u}^{(0)}$ in $L^2(\mathbb{R}^2)^3$ and $\boldsymbol{f}_n \to \boldsymbol{f}$ in $L^2(\mathbb{R}^2 \times J)^3$ for $n \to \infty$ and $B_\Gamma \left(\boldsymbol{u}_n^{(0),+}, \boldsymbol{u}_n^{(0),-}\right)^\top = 0$.

ii) There exists a sequence $(\boldsymbol{u}_n)_n \subset \mathcal{G}^1(\mathbb{R}^2 \times J)^3$ such that for all $n \in \mathbb{N}$ the function $\boldsymbol{u}_n$ solves (5.1.1) for the data $\left(\boldsymbol{f}_n, \boldsymbol{u}_n^{(0)}\right)$ and $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\mathcal{G}^0(\mathbb{R}^2 \times J)^3$ for $n \to \infty$.

PROOF: For i) we use the fact that $C_c^\infty(\Omega)$ and $H^1(\Omega)$ are dense in $L^2(\Omega)$ for any domain Ω. Therefore, we can choose sequences $(\boldsymbol{w}_n)_n \subset \mathcal{D}_\Gamma(\mathbb{R}^2)^3$ and $\left(\boldsymbol{f}_n\right)_n \subset \mathcal{H}^1(\mathbb{R}^2 \times J)^3$ with $\boldsymbol{w}_n \to \boldsymbol{u}^{(0)}$ in $L^2(\mathbb{R}^2)^3$ and $\boldsymbol{f}_n \to \boldsymbol{f}$ in $L^2(\mathbb{R}^2 \times J)^3$ for $n \to \infty$.
To guarantee the interface condition we introduce the characteristic function χ_{M_n} with

$$M_n := \mathbb{R}^2 \setminus \left\{\boldsymbol{x} \in \mathbb{R}^2 \,\middle|\, x_1 \in \left[-\frac{1}{n}, \frac{1}{n}\right]\right\}.$$

Now we show that $\boldsymbol{w}_n \chi_{M_n} \to \boldsymbol{u}^{(0)}$ in $L^2(\mathbb{R}^2)^3$. With the definition of χ_{M_n} and the Minkowski inequality we get

$$\begin{aligned} \left\|\boldsymbol{w}_n \chi_{M_n} - \boldsymbol{u}^{(0)}\right\|_{L^2(\mathbb{R}^2)^3} &\leqslant \left\|\left(\boldsymbol{w}_n - \boldsymbol{u}^{(0)}\right)\chi_{M_n}\right\|_{L^2(\mathbb{R}^2)^3} + \left\|\boldsymbol{u}^{(0)}\chi_{M_n} - \boldsymbol{u}^{(0)}\right\|_{L^2(\mathbb{R}^2)^3} \\ &\leqslant \left\|\boldsymbol{w}_n - \boldsymbol{u}^{(0)}\right\|_{L^2(\mathbb{R}^2)^3} + \left\|\boldsymbol{u}^{(0)}(\chi_{M_n} - 1)\right\|_{L^2(\mathbb{R}^2)^3}. \end{aligned} \tag{5.1.6}$$

The first term on the right-hand side in (5.1.6) vanishes for $n \to \infty$ since $\boldsymbol{w}_n \to \boldsymbol{u}^{(0)}$ in $L^2(\mathbb{R}^2)^3$. For the second term we use $\left|\boldsymbol{u}^{(0)}(\chi_{M_n} - 1)\right| \leqslant |\boldsymbol{u}^{(0)}|$ and $\boldsymbol{u}^{(0)}(\chi_{M_n} - 1) \to 0$ in $L^2(\mathbb{R}^2)^3$. Hence, Lebesgue's dominated convergence theorem can be applied and we see that the second term in (5.1.6) also vanishes for $n \to \infty$.
Since $\Gamma_2 \cap M_n = \varnothing$, we also get that $\mathrm{Tr}_\Gamma(B_\Gamma \boldsymbol{w}_n \chi_{M_n}) = \boldsymbol{0}$. Now we mollify $\boldsymbol{w}_n \chi_{M_n}$ to produce functions $\boldsymbol{u}_n^{(0)} \in \mathcal{D}_\Gamma(\mathbb{R}^2)^3$ with the stated properties.
The existence of $(\boldsymbol{u}_n)_n$ in assertion ii) is a direct consequence of Theorem 5.1.9. To show the convergence we use that (5.1.1) is a linear problem, consequently $\boldsymbol{u}_n - \boldsymbol{u}$ is a weak solution of (5.1.1) for the data $\left(\boldsymbol{f}_n - \boldsymbol{f}, \boldsymbol{u}_n^{(0)} - \boldsymbol{u}^{(0)}\right)$. Estimate (5.1.5) thus yields

$$\|\boldsymbol{u}_n - \boldsymbol{u}\|_{\mathcal{G}^0(\mathbb{R}^2 \times J)^3} \leqslant C\left(\left\|\boldsymbol{u}_n^{(0)} - \boldsymbol{u}^{(0)}\right\|_{L^2(\mathbb{R}^2)^3} + \|\boldsymbol{f}_n - \boldsymbol{f}\|_{L^2(\mathbb{R}^2 \times J)^3}\right).$$

The convergence properties of $(f_n)_n$ and $\left(u_n^{(0)}\right)_n$ complete the proof. □

5.2. Nonlinear Hyperbolic Systems

The reduced nonlinear Maxwell system (4.0.2), (4.0.3), (4.0.4) is a special case of a nonlinear hyperbolic problem and can be written as

$$\begin{cases} \widetilde{S}(\boldsymbol{x}, \boldsymbol{U}^{\pm})\partial_t \boldsymbol{u}^{\pm} + \sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{U}^{\pm} = \mathbf{0}, & \boldsymbol{x} \in \mathbb{R}^2_{\pm}, \quad t \in J, \\ B_\Gamma \begin{pmatrix} \boldsymbol{U}^+ \\ \boldsymbol{U}^- \end{pmatrix} = \mathbf{0}, & \boldsymbol{x} \in \Gamma_2, \quad t \in J, \\ \boldsymbol{U}^{\pm}(\cdot, 0) = \boldsymbol{U}^{(0),\pm}, & \boldsymbol{x} \in \mathbb{R}^2_{\pm}, \end{cases} \tag{5.2.1}$$

where for $\boldsymbol{v} \in \mathbb{R}^3$ we set

$$\widetilde{S}(\boldsymbol{x}, \boldsymbol{v}) := \Lambda(x_1) + \epsilon_3(x_1)\theta(\boldsymbol{v}),$$
$$\Lambda(x_1) := \begin{pmatrix} \epsilon_1(x_1) & 0 & 0 \\ 0 & \epsilon_1(x_1) & 0 \\ 0 & 0 & \mu_0 \end{pmatrix}, \quad \theta(\boldsymbol{v}) := \begin{pmatrix} 3v_1^2 + v_2^2 & 2v_1v_2 & 0 \\ 2v_1v_2 & v_1^2 + 3v_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

With this matrix function $\widetilde{S}$ we now have a quasilinear hyperbolic problem and we need new function spaces for the coefficients, namely

$$\mathcal{ML}^{m,k}\left(\mathbb{R}^2, \Omega_{\pm}\right) := \Big\{ S : \left(\mathbb{R}^2_+ \times \Omega_+\right) \cup \left(\mathbb{R}^2_- \times \Omega_-\right) \to \mathbb{R}^{k\times k} \,\Big|\, S^{\pm} \in C^m\left(\mathbb{R}^2_{\pm} \times \Omega_{\pm}, \mathbb{R}^{k\times k}\right),$$
$$\sup_{(\boldsymbol{x},\boldsymbol{u}) \in \mathbb{R}^2_{\pm} \times \mathcal{U}_{\pm}} |\partial^{\boldsymbol{\alpha}} S(\boldsymbol{x}, \boldsymbol{u})| < \infty \text{ for all compact sets } \mathcal{U}_{\pm} \subset \Omega_{\pm} \text{ and } \boldsymbol{\alpha} \in \mathbb{N}_0^5 \text{ with } |\boldsymbol{\alpha}| \leqslant m \Big\},$$

where $\Omega_{\pm} \subset \mathbb{R}^3$ are open and S^+, S^- are the restrictions of S to $\mathbb{R}^2_+ \times \Omega_+$ and $\mathbb{R}^2_- \times \Omega_-$, respectively. We will again use the subscripts η and "cv" to denote the additional conditions that the matrix is symmetric positive definite and convergent, i.e. for $\eta > 0$ we define

$$\mathcal{ML}^{m,k}_{\eta}(\mathbb{R}^2, \Omega_{\pm}) := \Big\{ S \in \mathcal{ML}^{m,k}\left(\mathbb{R}^2, \Omega_{\pm}\right) \,\Big|\, S = S^{\top}, \boldsymbol{v}^{\top} S \boldsymbol{v} \geqslant \eta |\boldsymbol{v}| \text{ on } \mathbb{R}^2_{\pm} \times \Omega_{\pm} \text{ for all } \boldsymbol{v} \in \mathbb{R}^k \Big\},$$
$$\mathcal{ML}^{m,k}_{\mathrm{cv}}(\mathbb{R}^2, \Omega_{\pm}) := \Big\{ S \in \mathcal{ML}^{m,k}\left(\mathbb{R}^2, \Omega_{\pm}\right) \,\Big|\, \text{there exists } A \in \mathbb{R}^{k\times k} \text{ such that for all}$$
$$(\boldsymbol{x}_n, \boldsymbol{u}_n)_n \subset \mathbb{R}_{\pm} \times \Omega_{\pm} \text{ with } |\boldsymbol{x}_n| \to \infty, \boldsymbol{u}_n \to \mathbf{0} : \lim_{n\to\infty} S(\boldsymbol{x}_n, \boldsymbol{u}_n) = A \Big\},$$

and

$$\mathcal{ML}^{m,k}_{\eta,\mathrm{cv}}\left(\mathbb{R}^2, \Omega_{\pm}\right) := \mathcal{ML}^{m,k}_{\eta}\left(\mathbb{R}^2, \Omega_{\pm}\right) \cap \mathcal{ML}^{m,k}_{\mathrm{cv}}\left(\mathbb{R}^2, \Omega_{\pm}\right).$$

Definition 5.2.1 (Solution of the Nonlinear Hyperbolic Problem)
A solution of (5.2.1) *is a function* $\boldsymbol{U} \in \mathcal{G}^1(\mathbb{R}^2 \times J)^3 \cap L^\infty(\mathbb{R}^2 \times J)^3$ *with* $\overline{\operatorname{im} \boldsymbol{U}^\pm} \subset \Omega_\pm$ *that satisfies*

$$\widetilde{S}(\boldsymbol{U})\partial_t \boldsymbol{U} + \sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{U} = \mathbf{0}$$

for almost all $\boldsymbol{x} \in \mathbb{R}^2 \backslash \Gamma_2$ *and for all* $t \in J$, $\operatorname{Tr}_\Gamma \left(B_\Gamma(\boldsymbol{U}^+, \boldsymbol{U}^-)^\top\right) = \mathbf{0}$ *and* $\boldsymbol{U}(\cdot, 0) = \boldsymbol{U}^{(0)}$. *Here* $\operatorname{im} \boldsymbol{U}^\pm$ *denotes the image of* $\boldsymbol{U}^+$ *and* $\boldsymbol{U}^-$*, respectively. Note that for* $\boldsymbol{U} \in \mathcal{G}^1(\mathbb{R}^2 \times J)$ *we can use the usual trace for* H^1*-functions, see Section 5.4.*

Remark 5.2.2
Note that a solution $\boldsymbol{U}$ *of* (5.2.1) *in* $\mathcal{G}^3(\mathbb{R}^2 \times J)^3$ *is a classical solution of* (5.2.1) *because of the Sobolev embeddings* $H^3\left(\mathbb{R}^2_\pm\right) \hookrightarrow C^1\left(\mathbb{R}^2_\pm\right)$.

The following local existence result for (5.2.1) follows from Proposition 6.1 and Theorem 6.1 of [67].

Theorem 5.2.3 (Nonlinear Existence Result)
Let $\eta > 0$, $\Omega_\pm \subset \mathbb{R}^3$ *and* $\widetilde{S} \in \mathcal{ML}^{3,3}_{\eta,\mathrm{cv}}\left(\mathbb{R}^2, \Omega_\pm\right)$. *Assume that* $\boldsymbol{U}^{(0)} \in \mathcal{H}^3(\mathbb{R}^2)^3$ *satisfies the nonlinear compatibility conditions of order* 3*, see Section 5.3, and* $\overline{\operatorname{im} \boldsymbol{U}^{(0),\pm}} \subset \Omega_\pm$ *with*

$$\operatorname{dist}\left(\overline{\operatorname{im} \boldsymbol{U}^{(0),\pm}}, \partial\Omega_\pm\right) > \kappa \tag{5.2.2}$$

for some $\kappa > 0$.
Then the following statements are true.

i) There exists a unique solution $\boldsymbol{U} \in \mathcal{G}^3\left(\mathbb{R}^2 \times (0, t_M)\right)^3$ *of* (5.2.1)*, where* $t_M > 0$ *is the maximal existence time.*

ii) If $t_M < \infty$*, then* $\lim_{t \nearrow t_M} \|\boldsymbol{U}(\cdot, t)\|_{\mathcal{H}^3(\mathbb{R}^2)^3} = \infty$ *or* $\liminf_{t \nearrow t_M} \operatorname{dist}\left(\overline{\operatorname{im} \boldsymbol{U}^\pm(t)}, \partial\Omega_\pm\right) = 0$.

Proof: As explained in Remark 5.0.1 and the proof of Theorem 5.1.9, the results of [67] treat a somewhat different but more difficult situation. So Theorem 5.2.3 follows from Theorem 5.1.9 by the same arguments as in Theorem 6.1 of [67] and Theorem 3.3 of [76]. We will therefore only sketch the idea of the proof.
To show the existence of a solution of (5.2.1) a fixed-point argument is used.
First, we fix $\boldsymbol{U} = \widetilde{\boldsymbol{U}}$ in $\widetilde{S}(\boldsymbol{x}, \boldsymbol{U})$, which results in a linear hyperbolic system

$$\begin{cases} S(\widetilde{\boldsymbol{U}})\partial_t \boldsymbol{U} + \sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{U} = \mathbf{0}, & \boldsymbol{x} \in \mathbb{R}^2_\pm, \quad t \in J, \\ B_\Gamma \begin{pmatrix} \boldsymbol{U}^+ \\ \boldsymbol{U}^- \end{pmatrix} = \mathbf{0}, & \boldsymbol{x} \in \Gamma_2, \quad t \in J, \\ \boldsymbol{U}(\cdot, 0) = \boldsymbol{U}^{(0)}, & \boldsymbol{x} \in \mathbb{R}^2_\pm. \end{cases} \tag{5.2.3}$$

Now we apply Theorem 5.1.9 to show the existence of a solution $\boldsymbol{U}$ of (5.2.3). To get a solution of the original nonlinear problem we apply the Banach fixed-point theorem.
The exact proofs of Theorem 6.1 in [67] and Theorem 3.3 in [76] are then based on careful estimates to show that the solution operator $\Phi : \widetilde{\boldsymbol{U}} \mapsto \boldsymbol{U}$ for the right choice of $\mathrm{D}(\Phi)$ is a self-mapping and a contraction.
Standard techniques are then used to construct the maximal existence interval, see Proposition 6.1 in [67] and Lemma 4.1 in [76].

□

5.3. Compatibility Conditions

One can show that for solutions of higher regularity some conditions involving the coefficients and the data are necessary, these conditions are called compatibility conditions.

To derive the compatibility conditions we start with a smooth solution $\boldsymbol{U} \in \mathcal{G}^3(\mathbb{R}^2 \times J)^3$ of (5.2.1) with $J := (0, t_M)$. We can differentiate (5.2.1) twice in time and get new equations that are still satisfied for all $t \in J$. By continuity these new equations have to be satisfied at $t = 0$ as well. This gives us necessary conditions on the initial values for $\boldsymbol{U} \in \mathcal{G}^3(\mathbb{R}^2 \times J)^3$.
If $\widetilde{S}(\boldsymbol{U})$ is positive definite, then $\widetilde{S}(\boldsymbol{U})$ is invertible and (5.2.1) implies

$$\begin{aligned} \partial_t \boldsymbol{U} &= -\widetilde{S}(\boldsymbol{U})^{-1} \left(\sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{U} \right) =: \widetilde{\boldsymbol{V}}^{(1)}(\boldsymbol{U}), \qquad (5.3.1) \\ [\![U_2]\!]_{\mathrm{2D}} &= [\![U_3]\!]_{\mathrm{2D}} = 0. \end{aligned}$$

Differentiation in time gives us the following new equations:

$$\begin{aligned} \partial_t^2 \boldsymbol{U} &= -\widetilde{S}(\boldsymbol{U})^{-1} \left(\sum_{j=1}^{2} A_j \partial_{x_j} \partial_t \boldsymbol{U} + \partial_t \left(\widetilde{S}(\boldsymbol{U}) \right) \partial_t \boldsymbol{U} \right) =: \widetilde{\boldsymbol{V}}^{(2)}(\boldsymbol{U}, \partial_t \boldsymbol{U}), \qquad (5.3.2) \\ [\![\partial_t U_2]\!]_{\mathrm{2D}} &= [\![\partial_t U_3]\!]_{\mathrm{2D}} = 0, \\ \partial_t^3 \boldsymbol{U} &= -\widetilde{S}(\boldsymbol{U})^{-1} \left(\sum_{j=1}^{2} A_j \partial_{x_j} \partial_t^2 \boldsymbol{U} + 2\partial_t \left(\widetilde{S}(\boldsymbol{U}) \right) \partial_t^2 \boldsymbol{U} + \partial_t^2 \left(\widetilde{S}(\boldsymbol{U}) \right) \partial_t \boldsymbol{U} \right) \\ &=: \widetilde{\boldsymbol{V}}^{(3)}(\boldsymbol{U}, \partial_t \boldsymbol{U}, \partial_t^2 \boldsymbol{U}), \qquad (5.3.3) \\ [\![\partial_t^2 U_2]\!]_{\mathrm{2D}} &= [\![\partial_t^2 U_3]\!]_{\mathrm{2D}} = 0. \end{aligned}$$

We can now iteratively define

$$\begin{aligned} \boldsymbol{V}^{(0)}(\boldsymbol{U}) &:= \boldsymbol{U}, & \boldsymbol{V}^{(1)}(\boldsymbol{U}) &:= \widetilde{\boldsymbol{V}}^{(1)}(\boldsymbol{V}^{(0)}(\boldsymbol{U})), \\ \boldsymbol{V}^{(2)}(\boldsymbol{U}) &:= \widetilde{\boldsymbol{V}}^{(2)}\left(\boldsymbol{V}^{(0)}(\boldsymbol{U}), \boldsymbol{V}^{(1)}(\boldsymbol{U})\right), & \boldsymbol{V}^{(3)}(\boldsymbol{U}) &:= \widetilde{\boldsymbol{V}}^{(3)}\left(\boldsymbol{V}^{(0)}(\boldsymbol{U}), \boldsymbol{V}^{(1)}(\boldsymbol{U}), \boldsymbol{V}^{(2)}(\boldsymbol{U})\right) \end{aligned}$$

to get operators $\boldsymbol{V}^{(j)}$ that only contain spatial derivatives and no temporal derivatives of $\boldsymbol{U}$. The equations above imply that

$$\partial_t^j \boldsymbol{U}(\cdot,0) = \boldsymbol{V}^{(j)}(\boldsymbol{U}(\cdot,0)),$$
$$\left[\!\left[V_2^{(j-1)}(\boldsymbol{U}(\cdot,0)) \right]\!\right]_{2\mathrm{D}} = \left[\!\left[V_3^{(j-1)}(\boldsymbol{U}(\cdot,0)) \right]\!\right]_{2\mathrm{D}} = 0$$

for $j \in \{1,2,3\}$. Hence, the initial values $\boldsymbol{U}^{(0)}$ have to satisfy the necessary conditions

$$\left[\!\left[V_2^{(j)}\left(\boldsymbol{U}^{(0)}\right) \right]\!\right]_{2\mathrm{D}} = \left[\!\left[V_3^{(j)}\left(\boldsymbol{U}^{(0)}\right) \right]\!\right]_{2\mathrm{D}} = 0 \tag{5.3.4}$$

for $j \in \{0,1,2\}$. Note that for higher regularity additional compatibility conditions are necessary, but we will focus our analysis on solutions in $\mathcal{G}^3(\mathbb{R}^2 \times J)^3$. A general formula for the compatibility conditions can be found in [67].

Definition 5.3.1 (Nonlinear Compatibility Conditions)
Let $m \in \{1,2,3\}$. We say that an initial value $\boldsymbol{U}^{(0)} \in \mathcal{H}^m(\mathbb{R}^2)^3$ satisfies the nonlinear compatibility conditions of order m for (5.2.1) *if and only if* (5.3.4) *is true for $j \in \{0,\dots,m-1\}$.*

Remark 5.3.2
The compatibility conditions for the linear problem (5.1.1) *can be derived analogously. In comparison to* (5.3.1), (5.3.2) *and* (5.3.3) *we have to replace $\widetilde{S}(\boldsymbol{U})$ by A_t and include the additional terms $M(\boldsymbol{x},t)\boldsymbol{u}$, $\boldsymbol{f}$ and their temporal derivatives:*

$$\widetilde{\boldsymbol{V}}_{\mathrm{lin}}^{(1)}(\boldsymbol{u}) = -A_\mathrm{t}^{-1}\left(\sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{u} + M\boldsymbol{u} - \boldsymbol{f} \right),$$
$$\widetilde{\boldsymbol{V}}_{\mathrm{lin}}^{(2)}(\boldsymbol{u}, \partial_t \boldsymbol{u}) = -A_\mathrm{t}^{-1}\left(\sum_{j=1}^{2} A_j \partial_{x_j} \partial_t \boldsymbol{u} + \partial_t A_\mathrm{t} \partial_t \boldsymbol{u} + \partial_t (M\boldsymbol{u}) - \partial_t \boldsymbol{f} \right),$$
$$\widetilde{\boldsymbol{V}}_{\mathrm{lin}}^{(3)}(\boldsymbol{u}, \partial_t \boldsymbol{u}, \partial_t^2 \boldsymbol{u}) = -A_\mathrm{t}^{-1}\left(\sum_{j=1}^{2} A_j \partial_{x_j} \partial_t^2 \boldsymbol{u} + \partial_t^2 A_\mathrm{t} \partial_t \boldsymbol{u} + 2\partial_t A_\mathrm{t} \partial_t^2 \boldsymbol{u} + \partial_t^2 (M\boldsymbol{u}) - \partial_t^2 \boldsymbol{f} \right).$$

Remark 5.3.3
The compatibility conditions for the half-space problem (5.1.3) *follow in the same way by replacing A_t, A_1, A_2, M, $\boldsymbol{u}$, $\boldsymbol{f}$ by their counterparts $\mathcal{A}_\mathrm{t}$, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{M}$, $\boldsymbol{v}$, $\boldsymbol{g}$ as defined in Remark 5.1.5.*

5.4. Trace Operator

In this section, we want to discuss the trace operator for solutions of the linear problem (5.1.1) and the nonlinear problem (5.2.1) and we will also analyze some differentiability properties of the trace operator. See Chapter 2.1 in [75] for more details.

We will first define a trace for problems on the right half-space $\mathbb{R}^2_+$ and then use a reflection as in Remark 5.1.5 to define a trace for the interface problem on $\mathbb{R}^2$. The trace for functions in the left half-space can be defined analogously. The trace operator will be based on properties of the space-time divergence operator, we therefore define the following special Sobolev spaces.

Definition 5.4.1 (Special Sobolev Spaces)
Let $T' > 0$ and $J := (0, T')$. Then we define

$$H(\operatorname{div}_t, \mathbb{R}^2_+ \times J) := \left\{ (v_1, v_2, v_t)^\top \in L^2(\mathbb{R}^2_+ \times J)^3 \,\middle|\, \operatorname{div}_t \boldsymbol{v} := \partial_{x_1} v_1 + \partial_{x_2} v_2 + \partial_t v_t \in L^2(\mathbb{R}^2_+ \times J) \right\},$$
$$H(\operatorname{div}_t, \mathbb{R}^2_+ \times J)_1 := \left\{ \tilde{v} \in L^2(\mathbb{R}^2_+ \times J) \,\middle|\, \text{there exists a function } \boldsymbol{v} \in H(\operatorname{div}_t, \mathbb{R}^2_+ \times J) \text{ with } v_1 = \tilde{v} \right\},$$

with the norms

$$\|\boldsymbol{v}\|_{H(\operatorname{div}_t, \mathbb{R}^2_+ \times J)} := \left(\|\boldsymbol{v}\|^2_{L^2(\mathbb{R}^2_+ \times J)^3} + \|\operatorname{div}_t \boldsymbol{v}\|^2_{L^2(\mathbb{R}^2_+ \times J)} \right)^{\frac{1}{2}},$$
$$\|\tilde{v}\|_{H(\operatorname{div}_t, \mathbb{R}^2_+ \times J)_1} := \inf_{q \in V_{\tilde{v}}} \|q\|_{H(\operatorname{div}_t, \mathbb{R}^2_+ \times J)},$$

where $V_{\tilde{v}} := \left\{ \boldsymbol{v} \in H(\operatorname{div}_t, \mathbb{R}^2_+ \times J) \,\middle|\, v_1 = \tilde{v} \right\}$.

The following lemma gives us the existence of a trace operator on $H(\operatorname{div}_t, \mathbb{R}^2_+ \times J)_1$, see Lemma 2.5. in [75] for the proof.

Lemma 5.4.2 (Trace Operator on $H(\operatorname{div}_t, \mathbb{R}^2_+ \times J)_1$)
There exists a unique linear and continuous trace operator

$$\operatorname{Tr}_\Gamma : H(\operatorname{div}_t, \mathbb{R}^2_+ \times J)_1 \longrightarrow H^{-1/2}(\Gamma_2 \times J),$$

which extends the restriction

$$C_c^\infty(\overline{\mathbb{R}^2_+} \times \overline{J}) \longrightarrow C_c^\infty(\Gamma_2 \times J), \qquad \phi \mapsto \phi|_{\Gamma_2}.$$

We will now apply Lemma 5.4.2 to solutions of linear hyperbolic problems, see Remark 2.14 in [75].

Remark 5.4.3
Let $\mathcal{A}_t \in W^{1,\infty}\left(\mathbb{R}^2_+ \times J\right)^{n\times n}$, $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{R}^{n\times n}$ *be symmetric,* $\mathcal{M} \in L^\infty\left(\mathbb{R}^2_+ \times J\right)^{n\times n}$ *and* $\boldsymbol{g} \in L^2\left(\mathbb{R}^2_+ \times J\right)^n$. *Let* $\boldsymbol{v} \in L^2\left(\mathbb{R}^2_+ \times J\right)^n$ *be a weak solution of*

$$L_h\boldsymbol{v} := \mathcal{A}_t\partial_t\boldsymbol{v} + \mathcal{A}_1\partial_{x_1}\boldsymbol{v} + \mathcal{A}_2\partial_{x_2}\boldsymbol{v} + \mathcal{M}\boldsymbol{v} = \boldsymbol{g}, \tag{5.4.1}$$

cf. (5.1.3). *At first one only has* $L_h\boldsymbol{v} \in H^{-1}\left(\mathbb{R}^2_+ \times J\right)^n$, *but from*

$$\langle L_h\boldsymbol{v}, \boldsymbol{\varphi}\rangle_{H^{-1}\times H^1_0} = \langle \boldsymbol{g}, \boldsymbol{\varphi}\rangle_{H^{-1}\times H^1_0} = \langle \boldsymbol{g}, \boldsymbol{\varphi}\rangle_{L^2\times L^2}$$

for all $\boldsymbol{\varphi} \in C^\infty_c\left(\mathbb{R}^2_+ \times J\right)^n$, *it follows that* $L_h\boldsymbol{v} = \boldsymbol{g}$ *in* $L^2\left(\mathbb{R}^2_+ \times J\right)^n$, *which implies that*

$$\sum_j \partial_j(\mathcal{A}_j\boldsymbol{v}) = \boldsymbol{g} + \sum_j \partial_j\mathcal{A}_j\boldsymbol{v} - \mathcal{M}\boldsymbol{v} \in L^2\left(\mathbb{R}^2_+ \times J\right).$$

Therefore, $(\mathcal{A}_1\boldsymbol{v})_k \in H(\mathrm{div}_t, \mathbb{R}^2_+ \times J)_1$ *for all* $k \in \{1, \dots, n\}$.
We can now define the trace of $\mathcal{A}_1\boldsymbol{v}$ *on* $\Gamma_2 \times J$ *with Lemma 5.4.2 as*

$$\mathrm{Tr}_\Gamma(\mathcal{A}_1\boldsymbol{v}) := (\mathrm{Tr}_\Gamma(\mathcal{A}_1\boldsymbol{v})_1, \dots, \mathrm{Tr}_\Gamma(\mathcal{A}_1\boldsymbol{v})_n)^\top.$$

We will now connect the boundary conditions with the Matrix $\mathcal{A}_1$ to define a trace operator, see Definition 2.16 in [75].

Definition 5.4.4 (Trace Operator)
Take the same assumptions as in Remark 5.4.3. Additionally, assume that there are matrices $B, T \in \mathbb{R}^{k\times n}$ *such that* $B = T\mathcal{A}_1$ *is satisfied. Then we define the trace of* $B\boldsymbol{v}$ *on* $\Gamma_2 \times J$ *via:*

$$\mathrm{Tr}_\Gamma(B\boldsymbol{v}) := T\,\mathrm{Tr}_\Gamma(\mathcal{A}_1\boldsymbol{v}).$$

Remark 5.4.5
To define traces for the interface conditions $B_\Gamma(\boldsymbol{u}^+, \boldsymbol{u}^-)^\top = 0$ *in* (5.1.1) *we will go back to the situation of Remark 5.1.5, where we transformed the problem to a half-space problem with* $\boldsymbol{v} := (\boldsymbol{u}^+, \boldsymbol{u}^-)^\top$ *and matrices* $\mathcal{A}_j$, $\mathcal{M}$. *Since* $B_\Gamma = T\mathcal{A}_1$ *with*

$$T := \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix},$$

we can use Definition 5.4.4 to define the trace of $B_\Gamma\boldsymbol{v}$ *for* $\boldsymbol{v} \in C\left(\overline{J}, L^2\left(\mathbb{R}^2_+\right)\right)^6$. *With this the trace of* $B_\Gamma(\boldsymbol{u}^+, \boldsymbol{u}^-)^\top$ *is well-defined for* $\boldsymbol{u} \in \mathcal{G}^0(\mathbb{R}^2 \times J)^3$.

For more regular functions $\boldsymbol{u} \in \mathcal{G}^1(\mathbb{R}^2 \times J)$, e.g. a solution of the nonlinear hyperbolic problem (5.2.1), there is a second way to define a trace operator. We can use the transformation to a half-space problem and the usual trace operator for H^1-functions

$$\operatorname{tr} : H^1\left(\mathbb{R}^2_\pm\right) \to H^{1/2}(\Gamma_2),$$

to define the trace operator

$$\widetilde{\operatorname{Tr}}_\Gamma\left(B_\Gamma(\boldsymbol{u}^+,\boldsymbol{u}^-)^\top\right) := B_\Gamma(\operatorname{tr}(\boldsymbol{u}^+),\operatorname{tr}(\boldsymbol{u}^-))^\top$$

for all $t \in J$, see e.g. [9, Chapter 9] for more on the standard trace operator for Sobolev functions. It is shown in Remark 2.17 of [75] that the two trace operators Tr_Γ and $\widetilde{\operatorname{Tr}}_\Gamma$ coincide on $\mathcal{G}^1(\mathbb{R}^2 \times J)$.
We will end this section with two lemmata that will allow us to differentiate the trace operator and to apply partial integration.

Lemma 5.4.6 (Differentiation of the Trace Operator)
Let $\mathcal{A}_\mathrm{t} \in W^{1,\infty}\left(\mathbb{R}^2_+ \times J\right)^{n\times n}$, $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{R}^{n\times n}$ be symmetric, $\mathcal{M} \in L^\infty\left(\mathbb{R}^2_+ \times J\right)^{n\times n}$ and define L_h as in (5.4.1). *Assume that there are matrices $B, T \in \mathbb{R}^{k\times n}$ with $B = T\mathcal{A}_1$. Let $\boldsymbol{v} \in L^2\left(\mathbb{R}^2_+ \times J\right)^n$ with $L_\mathrm{h}\boldsymbol{v} \in L^2\left(\mathbb{R}^2_+ \times J\right)^n$.*
If additionally $L_\mathrm{h}\partial_t\boldsymbol{v}, L_\mathrm{h}\partial_2\boldsymbol{v} \in L^2(\mathbb{R}^2_+ \times J)^n$, then the distributional derivative $\partial_j \operatorname{Tr}_\Gamma(B\boldsymbol{v})$ exists in $H^{-1/2}(\Gamma_2)^k$ and

$$\partial_j \operatorname{Tr}_\Gamma(B\boldsymbol{v}) = \operatorname{Tr}_\Gamma(B\partial_j\boldsymbol{v}) + \operatorname{Tr}_\Gamma(\partial_j B\boldsymbol{v})$$

for $j \in \{t, 2\}$.

PROOF: See Corollary 2.6 and Corollary 2.18. in [75].

□

Lemma 5.4.7 (Partial Integration in $H^1\left(\mathbb{R}^2_+\right)$)
Let $f, g \in H^1\left(\mathbb{R}^2_+\right)$, then

$$\begin{aligned} \int_{\mathbb{R}^2_+} \partial_{x_1} f g \,\mathrm{d}\boldsymbol{x} &= -\int_{\mathbb{R}^2_+} f\partial_{x_1} g \,\mathrm{d}\boldsymbol{x} - \int_{\partial\mathbb{R}^2_+} \operatorname{Tr}_\Gamma(f)\operatorname{Tr}_\Gamma(g)\,\mathrm{d}\boldsymbol{x},\\ \int_{\mathbb{R}^2_+} \partial_{x_2} f g \,\mathrm{d}\boldsymbol{x} &= -\int_{\mathbb{R}^2_+} f\partial_{x_2} g \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{5.4.2}$$

PROOF: Take sequences $(f_n)_n, (g_n)_n \subset C_c^\infty\left(\mathbb{R}^2_+\right)$ with $f_n \to f$ and $g_n \to g$ in $H^1\left(\mathbb{R}^2_+\right)$ for $n \to \infty$. Using the compact support and the classical partial integration formula we get:

$$\begin{aligned} \int_{\mathbb{R}^2_+} \partial_{x_1} f_n g_n \,\mathrm{d}\boldsymbol{x} &= -\int_{\mathbb{R}^2_+} f_n\partial_{x_1} g_n \,\mathrm{d}\boldsymbol{x} - \int_{\partial\mathbb{R}^2_+} \operatorname{Tr}_\Gamma(f_n)\operatorname{Tr}_\Gamma(g_n)\,\mathrm{d}\boldsymbol{x},\\ \int_{\mathbb{R}^2_+} \partial_{x_2} f_n g_n \,\mathrm{d}\boldsymbol{x} &= -\int_{\mathbb{R}^2_+} f_n\partial_{x_2} g_n \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{5.4.3}$$

Now all the integrals in (5.4.3) converge for $n \to \infty$ to their counterparts in (5.4.2), e.g.

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2_+} \partial_{x_1} f g - \partial_{x_1} f_n g_n \,\mathrm{d}\boldsymbol{x} \right| \\
&\qquad \leqslant \|\partial_{x_1} f - \partial_{x_1} f_n\|_{L^2(\mathbb{R}^2_+)} \|g\|_{L^2(\mathbb{R}^2_+)} + \|\partial_{x_1} f_n\|_{L^2(\mathbb{R}^2_+)} \|g - g_n\|_{L^2(\mathbb{R}^2_+)} \to 0, \\
&\left| \int_{\mathbb{R}^2_+} \mathrm{Tr}_\Gamma(f)\,\mathrm{Tr}_\Gamma(g) - \partial_{x_1} \mathrm{Tr}_\Gamma(f_n)\,\mathrm{Tr}_\Gamma(g_n) \,\mathrm{d}\boldsymbol{x} \right| \\
&\qquad \leqslant \|\mathrm{Tr}_\Gamma(f - f_n)\|_{L^2(\mathbb{R}^2_+)} \|\mathrm{Tr}_\Gamma(g)\|_{L^2(\mathbb{R}^2_+)} + \|\mathrm{Tr}_\Gamma(f_n)\|_{L^2(\mathbb{R}^2_+)} \|\mathrm{Tr}_\Gamma(g - g_n)\|_{L^2(\mathbb{R}^2_+)} \to 0,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality and that the trace operator is linear and bounded, see e.g. [1, Theorem 5.36].

□

6. Rigorous Analysis of the Asymptotic Solution

In this chapter we will rigorously prove the approximation properties of the asymptotic solution constructed in Chapter 4. The main tools of the proof will be the local existence result of Chapter 5 and a bootstrapping argument to extend the local existence to an asymptotically long time interval for initial data close to the small asymptotic ansatz.

6.1. Error Equations

Following the procedure presented in Section 2.3 we start the rigorous analysis by analyzing the error

$$\boldsymbol{R}(\boldsymbol{x},t) := \varepsilon^{-a}(\boldsymbol{U}(\boldsymbol{x},t) - \boldsymbol{U}_{\text{ext}}(\boldsymbol{x},t)),$$

where $\boldsymbol{U}$ is a solution of the reduced Maxwell's equations (4.0.2), (4.0.3), (4.0.4), $\boldsymbol{U}_{\text{ext}}$ is the extended asymptotic ansatz defined in (4.2.1) and a is a positive number. Note that for our approximation result we want a to be as large as possible.
Recall that in the hyperbolic form $\boldsymbol{U}$ solves

$$\begin{cases} \widetilde{S}(\boldsymbol{x},\boldsymbol{U}^{\pm})\partial_t\boldsymbol{U}^{\pm} + \sum_{j=1}^{2} A_j\partial_{x_j}\boldsymbol{U}^{\pm} = \boldsymbol{0}, & \boldsymbol{x}\in\mathbb{R}^2_{\pm}, \quad t\in J, \\ B_\Gamma\begin{pmatrix}\boldsymbol{U}^+\\ \boldsymbol{U}^-\end{pmatrix} = \boldsymbol{0}, & \boldsymbol{x}\in\Gamma_2, \quad t\in J, \\ \boldsymbol{U}^{\pm}(\cdot,0) = \boldsymbol{U}^{(0),\pm}, & \boldsymbol{x}\in\mathbb{R}^2_{\pm}, \end{cases} \tag{6.1.1}$$

with the matrix functions

$$\widetilde{S}(\boldsymbol{x},\boldsymbol{U}) := \Lambda(x_1) + \epsilon_3(x_1)\theta(\boldsymbol{U}),$$

$$\Lambda(x_1) = \begin{pmatrix}\epsilon_1(x_1) & 0 & 0\\ 0 & \epsilon_1(x_1) & 0\\ 0 & 0 & \mu_0\end{pmatrix}, \quad \theta(\boldsymbol{U}) := \begin{pmatrix}3U_1^2+U_2^2 & 2U_1U_2 & 0\\ 2U_1U_2 & U_1^2+3U_2^2 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$A_1 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}, \quad A_2 = \begin{pmatrix}0&0&-1\\0&0&0\\-1&0&0\end{pmatrix}.$$

We can now substitute $\boldsymbol{U} = \boldsymbol{U}_{\text{ext}} + \varepsilon^a \boldsymbol{R}$ and obtain

$$\begin{cases} S(\boldsymbol{x},t,\boldsymbol{R}^{\pm})\partial_t \boldsymbol{R}^{\pm} + \sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{R}^{\pm} + W(\boldsymbol{x},t,\boldsymbol{R}^{\pm})\boldsymbol{R}^{\pm} = -\varepsilon^{-a}\mathbf{Res}, & \boldsymbol{x} \in \mathbb{R}^2_{\pm}, \quad t \in J, \\ B_\Gamma \begin{pmatrix} \boldsymbol{R}^+ \\ \boldsymbol{R}^- \end{pmatrix} = \mathbf{0}, & \boldsymbol{x} \in \Gamma_2, \quad t \in J, \\ \boldsymbol{R}^{\pm}(\cdot,0) = \boldsymbol{R}^{(0),\pm}, & \boldsymbol{x} \in \mathbb{R}^2_{\pm}, \end{cases} \tag{6.1.2}$$

with

$$\begin{aligned}
\boldsymbol{R}^{(0)}(\boldsymbol{x}) &:= \varepsilon^{-a}\left(\boldsymbol{U}^{(0)}(\boldsymbol{x}) - \boldsymbol{U}^{(0)}_{\text{ext}}(\boldsymbol{x})\right), \\
\mathbf{Res}(\boldsymbol{x},t) &:= \mathbf{Res}(\boldsymbol{U}_{\text{ext}}) = \widetilde{S}(\boldsymbol{x},\boldsymbol{U}_{\text{ext}})\partial_t \boldsymbol{U}_{\text{ext}} + \sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{U}_{\text{ext}}, \\
S(\boldsymbol{x},t,\boldsymbol{R}) &:= \widetilde{S}(\boldsymbol{x},\boldsymbol{U}_{\text{ext}} + \varepsilon^a \boldsymbol{R}) = \Lambda(\boldsymbol{x}) + \epsilon_3(x_1)\varepsilon^{2a}\theta(\boldsymbol{R}) + \varphi(\boldsymbol{x},t,\boldsymbol{R}), \\
\varphi(\boldsymbol{x},t,\boldsymbol{R}) &:= \epsilon_3(x_1)\varepsilon^a \begin{pmatrix} 6U_{\text{ext},1}R_1 + 2U_{\text{ext},2}R_2 & 2U_{\text{ext},1}R_2 + 2U_{\text{ext},2}R_1 & 0 \\ 2U_{\text{ext},1}R_2 + 2U_{\text{ext},2}R_1 & 2U_{\text{ext},1}R_1 + 6U_{\text{ext},2}R_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \epsilon_3(x_1)\theta(\boldsymbol{U}_{\text{ext}}), \\
W(\boldsymbol{x},t,\boldsymbol{R})\boldsymbol{R} &:= \varepsilon^{-a}\big(S(\boldsymbol{x},t,\boldsymbol{R}) - \widetilde{S}(\boldsymbol{x},\boldsymbol{U}_{\text{ext}})\big)\partial_t \boldsymbol{U}_{\text{ext}} \\
&= \epsilon_3(x_1) \begin{pmatrix} 6U_{\text{ext},1}\partial_t U_{\text{ext},1} + 2U_{\text{ext},2}\partial_t U_{\text{ext},2} & 2\partial_t U_{\text{ext},1}U_{\text{ext},2} + 2U_{\text{ext},1}\partial_t U_{\text{ext},2} & 0 \\ 2\partial_t U_{\text{ext},1}U_{\text{ext},2} + 2U_{\text{ext},1}\partial_t U_{\text{ext},2} & 6U_{\text{ext},2}\partial_t U_{\text{ext},2} + 2U_{\text{ext},1}\partial_t U_{\text{ext},1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad + \varepsilon^a \epsilon_3(x_1) \begin{pmatrix} 3\partial_t U_{\text{ext},1}R_1 + 2\partial_t U_{\text{ext},2}R_2 & \partial_t U_{\text{ext},1}R_2 & 0 \\ \partial_t U_{\text{ext},2}R_1 & 3\partial_t U_{\text{ext},2}R_2 + 2\partial_t U_{\text{ext},1}R_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Note that the matrix $\varphi(\boldsymbol{R})$ is symmetric and that $\varphi(\boldsymbol{R})$ and $W(\boldsymbol{R})$ are independent of R_3.
The interface condition of (6.1.2) is a consequence of $B_\Gamma\left(\boldsymbol{U}^+_{\text{ext}}, \boldsymbol{U}^-_{\text{ext}}\right)^\top = \mathbf{0}$, as explained in Section 4.1.2.
For a fixed $\boldsymbol{U}_{\text{ext}}$, systems (6.1.1) and (6.1.2) are equivalent. Our rough strategy is to use the local existence Theorem 5.2.3 for (6.1.1) in order to get the existence of $\boldsymbol{R}$ on the time interval $(0, t_M)$ and then apply a bootstrapping argument on (6.1.2) to show that $t_M \geqslant T_0\varepsilon^{-2}$ for some $T_0 > 0$ and that

$$\|\varepsilon^a \boldsymbol{R}\|_{\mathcal{G}^3(\mathbb{R}^2\times(0,T_0\varepsilon^{-2}))^3} = \|\boldsymbol{U} - \boldsymbol{U}_{\text{ext}}\|_{\mathcal{G}^3(\mathbb{R}^2\times(0,T_0\varepsilon^{-2}))^3} \leqslant C\varepsilon^a \tag{6.1.3}$$

holds for all small enough $\varepsilon > 0$ and $\boldsymbol{R}^{(0)}$. Note that we do the estimates in the $\mathcal{G}^3\left(\mathbb{R}^2 \times \left(0, T_0\varepsilon^{-2}\right)\right)^3$-norm since this is the natural space for our solution $\boldsymbol{U}$ given by Theorem 5.2.3.

Let us now check under which conditions on the material functions ϵ_1, ϵ_3 Theorem 5.2.3 is applicable.

Remark 6.1.1
To apply Theorem 5.2.3 we need to find $\Omega_\pm$ such that $\widetilde{S} \in \mathcal{ML}^{3,3}_{\eta,\mathrm{cv}}\left(\mathbb{R}^2, \Omega_\pm\right)$ for some $\eta > 0$.
First, to satisfy $\widetilde{S} \in \mathcal{ML}^{3,3}_{\mathrm{cv}}\left(\mathbb{R}^2, \Omega_\pm\right)$ we need $\epsilon_3^\pm \in C^3(\mathbb{R}_\pm) \cap W^{3,\infty}(\mathbb{R}_\pm)$ and ϵ_1, ϵ_3 have to converge for $|x_1| \to \infty$, as we have assumed in (A1), (A3), (A6) *and* (A7).
Second, $\widetilde{S}(\boldsymbol{x}, \boldsymbol{v})$ has to be symmetric positive definite for all $\boldsymbol{x} \in \mathbb{R}^2_\pm$ and $\boldsymbol{v} \in \Omega_\pm$. By the definition of $\widetilde{S}$ it is clear that $\widetilde{S}$ is symmetric and a short computation gives us the three eigenvalues $\lambda_1 = \mu_0$, $\lambda_2 = \epsilon_1 + \epsilon_3\left(v_1^2 + v_2^2\right)$ and $\lambda_3 = \epsilon_1 + 3\epsilon_3\left(v_1^2 + v_2^2\right)$ dependent from $\boldsymbol{v}$. We now have to show that there exists an $\eta > 0$ and domains $\Omega_\pm$ such that $\lambda_1, \lambda_2, \lambda_3 \geqslant \eta > 0$ for all $\boldsymbol{v} \in \Omega_\pm$.
Recall the bounds on ϵ_1 and ϵ_3 in (A1) *and* (A6)*. If $\epsilon^\pm_{3,m} \geqslant 0$, then clearly $\lambda_2, \lambda_3 > \epsilon^\pm_{1,m}$ and the choice $\eta := \min\{\mu_0, \epsilon^+_{1,m}, \epsilon^-_{1,m}\}$ and $\Omega_\pm := \mathbb{R}^3$ is possible. If $\epsilon^\pm_{3,m} < 0$, we impose*

$$\epsilon^\pm_{1,m} + 3\epsilon^\pm_{3,m}\left(v_1^2 + v_2^2\right) > \eta > 0 \;\; \textit{for all } \boldsymbol{v} \in \Omega_\pm.$$

Choosing $0 < \eta < \min\left\{\mu_0, \epsilon^+_{1,m}, \epsilon^-_{1,m}\right\}$ and

$$\Omega_\pm := \begin{cases} \left\{ \boldsymbol{v} \in \mathbb{R}^3 \,\middle|\, v_1^2 + v_2^2 < \frac{\eta - \epsilon^\pm_{1,m}}{3\epsilon^\pm_{3,m}} \right\}, & \epsilon^\pm_{3,m} < 0, \\ \mathbb{R}^3, & \epsilon^\pm_{3,m} \geqslant 0, \end{cases}$$

we infer $\widetilde{S} \in \mathcal{ML}^{3,3}_{\eta,\mathrm{cv}}\left(\mathbb{R}^2, \Omega_\pm\right)$. Since $S(\boldsymbol{x}, t, \boldsymbol{R}) = \widetilde{S}(\boldsymbol{x}, \boldsymbol{U}_{\mathrm{ext}}(\boldsymbol{x}, t) + \varepsilon^a \boldsymbol{R})$, we also conclude that $S(\boldsymbol{R})$ is uniformly positive definite and symmetric. Note that for $\epsilon^\pm_{3,m} \geqslant 0$ the conditions $\overline{\operatorname{im} \boldsymbol{U}^{(0),\pm}} \subset \Omega_\pm$ and (5.2.2) *are trivially satisfied. For $\epsilon^\pm_{3,m} < 0$ we have to select a small enough initial value $\boldsymbol{U}^{(0),\pm}$.*

Corollary 6.1.2 (Existence of a Solution of (6.1.1))
Let $\eta > 0$ and $\Omega_\pm \subset \mathbb{R}^3$ as in Remark 6.1.1 such that $\widetilde{S} \in \mathcal{ML}^{3,3}_{\eta,\mathrm{cv}}\left(\mathbb{R}^2, \Omega_\pm\right)$. For an initial value $\boldsymbol{U}^{(0)} \in \mathcal{H}^3(\mathbb{R}^2)^3$ that satisfies the nonlinear compatibility conditions of order 3 *and $\overline{\operatorname{im} \boldsymbol{U}^{(0),\pm}} \subset \Omega_\pm$ there exists a unique solution $\boldsymbol{U} \in \mathcal{G}^3(\mathbb{R}^2 \times (0, t_M))^3$ of* (6.1.1)*, where $t_M > 0$ is the maximal existence time.*

6.1.1. Idea of Error Estimate

Before we prove (6.1.3) in the next section, let us study the main idea behind our approach, by establishing the estimates formally in the $\mathcal{G}^0\left(\mathbb{R}^2 \times \left(0, T_0\varepsilon^{-2}\right)\right)^3$-norm instead of the $\mathcal{G}^3\left(\mathbb{R}^2 \times \left(0, T_0\varepsilon^{-2}\right)\right)^3$-norm. For the purpose of this section, let us assume that $\boldsymbol{R}$ is a sufficiently regular solution of (6.1.2) on a time interval $J := (0, T')$, such that all the estimates are possible.

Our goal is the estimate $\|\boldsymbol{R}(\cdot,t)\|_{L^2(\mathbb{R}^2)^3} \leqslant C$ on a long time interval $\left[0, T_0\varepsilon^{-2}\right]$. To this end, we start with the differential equation in (6.1.2), i.e.

$$S(\boldsymbol{R})\partial_t \boldsymbol{R} + \sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{R} + W(\boldsymbol{R})\boldsymbol{R} = -\varepsilon^{-a}\mathbf{Res}, \qquad \boldsymbol{x} \in \mathbb{R}^2_{\pm}, \quad t \in J,$$

and test it with $\boldsymbol{R}$, which gives us:

$$\int_0^t \int_{\mathbb{R}^2} S(\boldsymbol{R})\partial_t \boldsymbol{R} \cdot \boldsymbol{R} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s = \int_0^t \int_{\mathbb{R}^2} \left(-\sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{R} \cdot \boldsymbol{R} - W(\boldsymbol{R})\boldsymbol{R} \cdot \boldsymbol{R} - \varepsilon^{-a}\mathbf{Res} \cdot \boldsymbol{R} \right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}s. \tag{6.1.4}$$

The integrand on the left-hand side of (6.1.4) can be written as

$$S(\boldsymbol{R})\partial_t \boldsymbol{R} \cdot \boldsymbol{R} = \partial_t \left(\frac{1}{2} S(\boldsymbol{R})\boldsymbol{R} \cdot \boldsymbol{R} \right) - \frac{1}{2}\partial_t S(\boldsymbol{R})\boldsymbol{R} \cdot \boldsymbol{R}$$

since $S(\boldsymbol{R})$ is symmetric. Using the definiteness of $S(\boldsymbol{R})$ it follows

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^2} S(\boldsymbol{R})\partial_t \boldsymbol{R} \cdot \boldsymbol{R} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s &= \int_0^t \int_{\mathbb{R}^2} \left(\partial_t \left(\frac{1}{2} S(\boldsymbol{R})\boldsymbol{R} \cdot \boldsymbol{R} \right) - \frac{1}{2}\partial_t S(\boldsymbol{R})\boldsymbol{R} \cdot \boldsymbol{R} \right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\geqslant \frac{1}{2} \int_{\mathbb{R}^2} \left(\eta \boldsymbol{R}(t) \cdot \boldsymbol{R}(t) - S\left(\boldsymbol{R}^{(0)}\right) \boldsymbol{R}^{(0)} \cdot \boldsymbol{R}^{(0)} \right) \mathrm{d}\boldsymbol{x} \\
&\quad - \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} \partial_t S(\boldsymbol{R})\boldsymbol{R} \cdot \boldsymbol{R} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.
\end{aligned} \tag{6.1.5}$$

A combination of (6.1.4) and (6.1.5) yields

$$\begin{aligned}
\frac{\eta}{2} \|\boldsymbol{R}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)^3} \leqslant{}& \frac{1}{2} \left\| S\left(\boldsymbol{R}^{(0)}\right) \right\|^2_{L^\infty(\mathbb{R}^2)^3} \left\| \boldsymbol{R}^{(0)} \right\|^2_{L^2(\mathbb{R}^2)^3} \\
&+ \int_0^t \int_{\mathbb{R}^2} \left(\frac{1}{2}\partial_t S(\boldsymbol{R})\boldsymbol{R} \cdot \boldsymbol{R} - \sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{R} \cdot \boldsymbol{R} - W(\boldsymbol{R})\boldsymbol{R} \cdot \boldsymbol{R} - \varepsilon^{-a}\mathbf{Res} \cdot \boldsymbol{R} \right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}s.
\end{aligned} \tag{6.1.6}$$

We now have to analyze the remaining integral. But first, we need some more assumptions:

i) Assume that
$$J \subset (0, T_0 \varepsilon^{-2}); \tag{6.1.7}$$

ii) let $\boldsymbol{U}_{\text{ext}}$ be such that
$$\|\boldsymbol{U}_{\text{ext}}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)^3}, \|\partial_t \boldsymbol{U}_{\text{ext}}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)^3} \leqslant C\varepsilon, \qquad \forall\, t \in J, \tag{6.1.8}$$
see Remark 4.2.7;

iii) let $\boldsymbol{R}$ be such that
$$\|\boldsymbol{R}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)^3}, \|\partial_t \boldsymbol{R}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)^3}, \|\boldsymbol{R}(\cdot,t)\|_{L^2(\mathbb{R}^2)^3} \leqslant C, \qquad \forall\, t \in J. \tag{6.1.9}$$

First, we use integration by parts, see Lemma 5.4.7, to show:
$$\begin{aligned} -\int_{\mathbb{R}^2} \sum_{j=1}^2 A_j \partial_{x_j} \boldsymbol{R} \cdot \boldsymbol{R}\, \mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^2} (\partial_{x_2} R_3 R_1 - \partial_{x_1} R_3 R_2 + \partial_{x_2} R_1 R_3 - \partial_{x_1} R_2 R_3)\, \mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}} \left(\mathrm{Tr}_\Gamma\, R_3^+\, \mathrm{Tr}_\Gamma\, R_2^+\right)(0,x_2)\, \mathrm{d}x_2 - \int_{\mathbb{R}} \left(\mathrm{Tr}_\Gamma\, R_3^-\, \mathrm{Tr}_\Gamma\, R_2^-\right)(0,x_2)\, \mathrm{d}x_2 \\ &= 0. \end{aligned} \tag{6.1.10}$$

The last equality in (6.1.10) holds for solutions $\boldsymbol{R}$ that satisfy the interface conditions $[\![R_2]\!]_{2\text{D}} = [\![R_3]\!]_{2\text{D}} = 0$ in the sense of traces.
Second, the Cauchy-Schwarz inequality together with the estimate for the residual (4.2.6) and the Assumptions (6.1.7), (6.1.9) give us
$$\begin{aligned} \int_0^t \int_{\mathbb{R}^2} \varepsilon^{-a} \mathbf{Res} \cdot \boldsymbol{R}\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}s &\leqslant \varepsilon^{-a} \int_0^t \|\mathbf{Res}(\cdot,s)\|_{L^2(\mathbb{R}^2)^3} \|\boldsymbol{R}(\cdot,s)\|_{L^2(\mathbb{R}^2)^3}\, \mathrm{d}s \\ &\leqslant Ct\varepsilon^{\frac{7}{2}-a} \leqslant C\varepsilon^{\frac{7}{2}-2-a} \end{aligned}$$
for $t \in J$.
All that is left is the term $\int_0^t \int_{\mathbb{R}^2} (\partial_t S(\boldsymbol{R}) - W(\boldsymbol{R}))\boldsymbol{R} \cdot \boldsymbol{R}\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}s$, which contains products between components of $\boldsymbol{R}$ and $\boldsymbol{U}_{\text{ext}}$. Instead of estimating all the terms, we will only give examples for the four qualitatively different kinds of terms that appear in this expression. The typical terms are the following:

1. Four $\boldsymbol{R}$-factors and no $\boldsymbol{U}_{\text{ext}}$-factors:
   This is the easiest case since all such terms are contained in $\varepsilon^{2a}\epsilon_3 \partial_t \theta(\boldsymbol{R})\boldsymbol{R} \cdot \boldsymbol{R}$, which can be estimated as follows
$$\begin{aligned} \varepsilon^{2a}\epsilon_3 \partial_t \theta(\boldsymbol{R})\boldsymbol{R} \cdot \boldsymbol{R} &= 6\varepsilon^{2a}\epsilon_3 \left(\left(R_1^3 + R_1 R_2^2\right)\partial_t R_1 + \left(R_2^3 + R_1^2 R_2\right)\partial_t R_2\right) \\ &\leqslant C\varepsilon^{2a} |\partial_t \boldsymbol{R}| |\boldsymbol{R}|^3. \end{aligned}$$

With (6.1.9) we get

$$\varepsilon^{2a}\epsilon_3 \int_0^t \int_{\mathbb{R}^2} \partial_t \theta(\boldsymbol{R})\boldsymbol{R} \cdot \boldsymbol{R} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \leqslant C\varepsilon^{2a} \int_0^t \|\boldsymbol{R}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)^3} \,\mathrm{d}s, \qquad \forall\, t \in J.$$

2. Three $\boldsymbol{R}$-factors, one $\boldsymbol{U}_{\mathrm{ext}}$-factor and one temporal derivative on one of the $\boldsymbol{R}$-factors: Due to the special structure of the nonlinearity we can always rewrite the terms of this type as $\varepsilon^a U_{\mathrm{ext},i}\left(2\partial_t R_j R_j R_k + R_j^2 \partial_t R_k\right) = \varepsilon^a U_{\mathrm{ext},i}\partial_t\left(R_j^2 R_k\right)$. Now we can apply partial integration and get with (6.1.8), (6.1.9)

$$\begin{aligned}
&\varepsilon^a \int_{\mathbb{R}^2}\int_0^t U_{\mathrm{ext},i}\partial_t\left(R_j^2 R_k\right)\mathrm{d}s\,\mathrm{d}\boldsymbol{x}\\
&= \varepsilon^a\left(-\int_0^t\int_{\mathbb{R}^2} \partial_t U_{\mathrm{ext},i} R_j^2 R_k \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \int_{\mathbb{R}^2}\left[U_{\mathrm{ext},i} R_j^2 R_k\right]_0^t \mathrm{d}\boldsymbol{x}\right)\\
&\leqslant C\varepsilon^{1+a}\left(1 + \left\|\boldsymbol{R}^{(0)}\right\|^2_{L^2(\mathbb{R}^2)^3} + \int_0^t \|\boldsymbol{R}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)^3}\,\mathrm{d}s\right)
\end{aligned}$$

for all $t \in J$, $i,j,k \in \{1,2\}$.

3. Three $\boldsymbol{R}$-factors, one $\boldsymbol{U}_{\mathrm{ext}}$-factor and one temporal derivative on the $\boldsymbol{U}_{\mathrm{ext}}$-factor: Here we get with (6.1.8), (6.1.9) that

$$\int_0^t\int_{\mathbb{R}^2} \varepsilon^a \partial_t U_{\mathrm{ext},i} R_j R_k R_l \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \leqslant C\varepsilon^{1+a}\int_0^t\int_{\mathbb{R}^2} |\boldsymbol{R}||\boldsymbol{R}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \leqslant C\varepsilon^{1+a}\int_0^t \|\boldsymbol{R}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)^3}\,\mathrm{d}s$$

for all $t \in J$, $i,j,k,l \in \{1,2\}$.

4. Two $\boldsymbol{R}$-factors, two $\boldsymbol{U}_{\mathrm{ext}}$-factors and one temporal derivative on one of the $\boldsymbol{U}_{\mathrm{ext}}$-factors: The same arguments as before show us

$$\int_0^t\int_{\mathbb{R}^2} \partial_t U_{\mathrm{ext},i} U_{\mathrm{ext},j} R_k R_l \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \leqslant C\varepsilon^2\int_0^t\int_{\mathbb{R}^2} |\boldsymbol{R}|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \leqslant C\varepsilon^2\int_0^t \|\boldsymbol{R}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)^3}\,\mathrm{d}s$$

for all $t \in J$, $i,j,k,l \in \{1,2\}$.

All in all, we get

$$\begin{aligned}
\eta\,\|\boldsymbol{R}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)^3} \leqslant{}& C\left(\varepsilon^{\frac{3}{2}-a} + \varepsilon^{1+a} + \left(1+\varepsilon^{1+a}\right)\left\|\boldsymbol{R}^{(0)}\right\|^2_{L^2(\mathbb{R}^2)^3}\right)\\
&+ C\left(\varepsilon^{2a} + \varepsilon^{1+a} + \varepsilon^2\right)\int_0^t \|\boldsymbol{R}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)^3}\,\mathrm{d}s
\end{aligned}$$

for $t \in J$. This simplifies for $a \geqslant 1$ to

$$\eta\,\|\boldsymbol{R}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)^3} \leqslant C\left(\varepsilon^{\frac{3}{2}-a} + \left\|\boldsymbol{R}^{(0)}\right\|^2_{L^2(\mathbb{R}^2)^3} + \varepsilon^2\int_0^t \|\boldsymbol{R}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)^3}\,\mathrm{d}s\right).$$

Gronwall's inequality for measurable functions, see e.g. [58], implies

$$\begin{aligned}\|\boldsymbol{R}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)^3} &\leqslant C\left(\varepsilon^{\frac{3}{2}-a}+\left\|\boldsymbol{R}^{(0)}\right\|^2_{L^2(\mathbb{R}^2)^3}\right)\mathrm{e}^{C\int_0^t\varepsilon^2\,\mathrm{d}s}\\ &= C\left(\varepsilon^{\frac{3}{2}-a}+\left\|\boldsymbol{R}^{(0)}\right\|^2_{L^2(\mathbb{R}^2)^3}\right)\mathrm{e}^{Ct\varepsilon^2}\\ &\leqslant C\left(\varepsilon^{\frac{3}{2}-a}+\left\|\boldsymbol{R}^{(0)}\right\|^2_{L^2(\mathbb{R}^2)^3}\right),\end{aligned}\tag{6.1.11}$$

as long as $t \in J$.
Now we additionally demand $a < \frac{3}{2}$ such that $\varepsilon^{3/2-a}$ converges to zero for $\varepsilon \to 0$.
To achieve (6.1.11) we had to assume that $\|\boldsymbol{R}(\cdot,t)\|_{L^2(\mathbb{R}^2)^3} \leqslant C$ on an ε-independent interval J, but now we have a much better estimate since $\|\boldsymbol{R}(\cdot,t)\|_{L^2(\mathbb{R}^2)^3}$ gets as small as we want on this interval for small enough $\boldsymbol{R}^{(0)}$ and ε provided $t \leqslant T_0\varepsilon^{-2}$.
This is the main ingredient for a bootstrapping argument. All we need is a small time interval where we can estimate $\|\boldsymbol{R}(\cdot,t)\|_{L^2(\mathbb{R}^2)^3} \leqslant C$ and the bootstrapping argument will show us that this estimate holds true as long as $t \leqslant T_0\varepsilon^{-2}$.
For the estimates in the $\mathcal{G}^3\left(\mathbb{R}^2\times\left(0,T_0\varepsilon^{-2}\right)\right)^3$-norm we have to estimate the derivatives of $\boldsymbol{R}$ as well. For the tangential derivatives a similar strategy as in this section will be used. For the normal derivatives, i.e. the x_1-derivatives, we will also use the divergence equation $\nabla\cdot\boldsymbol{\mathcal{D}} = \varrho_0$.

6.2. Bootstrapping Argument

In this section we will use a bootstrapping argument to prove (6.1.3).

Assume that Assumptions (A1) – (A7) hold. Take a solution $A \in \bigcap_{k=0}^4 C^{4-k}([0,T_0],H^{2+k}(\mathbb{R}))$ of the effective nonlinear Schrödinger equation (4.1.13) for some $T_0 > 0$ and construct $\boldsymbol{U}_{\text{ext}}$ as discussed in Chapter 4.
Choose $\boldsymbol{R}^{(0)} \in \mathcal{H}^3(\mathbb{R}^2)^3$ and $\varepsilon_* > 0$ small enough such that $\boldsymbol{U}^{(0)} := \boldsymbol{U}_{\text{ext}}(\cdot,0) + \varepsilon_*^a\boldsymbol{R}^{(0)}$ satisfies $\overline{\operatorname{im}\boldsymbol{U}^{(0),\pm}} \subset \Omega_\pm$ and the nonlinear compatibility conditions of order 3, see Definition 5.3.1. Then Corollary 6.1.2 yields a maximal existence time $t_M > 0$ and a solution $\boldsymbol{U} \in \mathcal{G}^3\left(\mathbb{R}^2\times(0,t_M)\right)^3$ of (6.1.1).
With Remark 4.2.7 it follows that $\boldsymbol{R} \in \mathcal{G}^3\left(\mathbb{R}^2\times\left(0,\min\left\{t_M,T_0\varepsilon^{-2}\right\}\right)\right)^3$.
For $t \in \left[0,\min\left\{t_M,T_0\varepsilon^{-2}\right\}\right)$ we set

$$z(t) := \sum_{k=0}^{3}\left\|\partial_t^k\boldsymbol{R}(\cdot,t)\right\|^2_{\mathcal{H}^{3-k}(\mathbb{R}^2)^3}.$$

By the Sobolev embeddings $H^2\left(\mathbb{R}^2_\pm\right) \hookrightarrow L^\infty\left(\mathbb{R}^2_\pm\right)$ we have that

$$\|\boldsymbol{R}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)^3}, \|\partial_t\boldsymbol{R}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)^3} \leqslant c_S z(t)^{\frac{1}{2}}$$

for a constant $c_S > 1$. Take $\varpi > 0$ with

$$\varpi^2 < \frac{1}{c_S^2} \min\left\{ \frac{\eta - \epsilon_{1,m}^-}{3\min\{\epsilon_{3,m}^-, 0\}}, \frac{\eta - \epsilon_{1,m}^+}{3\min\{\epsilon_{3,m}^+, 0\}} \right\}, \tag{6.2.1}$$

where $\frac{c}{0} := +\infty$.

The major part of the rest of the proof is a bootstrapping argument to prove the statement

$$\begin{cases} \text{There exist } \rho \in (0,1],\ \rho_0 \in (0,\rho),\ \varepsilon_0 = \varepsilon_0(\rho) \in (0,\varepsilon_*) \text{ and } t^* \in (0,t_M) \text{ such that} \\ \text{for all } \varepsilon \in (0,\varepsilon_0) \text{ we have } \varepsilon^a\rho + \|\boldsymbol{U}_{\mathrm{ext}}\|_{L^\infty(\mathbb{R}^2\times[0,t^*])^3} \leqslant \varpi \\ \text{and if } \ z(0) \leqslant \rho_0^2 \text{ and } t^* \leqslant T_0\varepsilon^{-2}, \text{ then } \ z(t) \leqslant \rho^2 \ \text{ for all } t \in [0,t^*]. \end{cases} \tag{6.2.2}$$

Remark 6.2.1

Note that (6.2.2) *contains the two conditions that guarantee us a long existence time, see Theorem 5.2.3. The first condition* $\varepsilon^a\rho + \|\boldsymbol{U}_{\mathrm{ext}}\|_{L^\infty(\mathbb{R}^2\times[0,T_0\varepsilon^{-2}))^3} \leqslant \varpi$ *guarantees us that we do not leave the domains* $\Omega_\pm$ *and can be achieved for* $\varepsilon_0 = \varepsilon_0(\rho)$ *small enough. Indeed,* $\|\boldsymbol{U}_{\mathrm{ext}}\|_{L^\infty(\mathbb{R}^2\times[0,t^*])^3} \leqslant C\varepsilon$ *by* (4.2.11).

The second condition $z(t) \leqslant \rho^2$ *shows us that there is no blow-up.*

Also note that under these assumptions (6.1.7), (6.1.8) *and* (6.1.9) *from Section 6.1.1 are satisfied.*

To establish (6.2.2), we define for $1 \geqslant \rho > \rho_0 > 0$

$$\begin{aligned} T_{\rho_0,\varepsilon_0} &:= \sup\Big\{ t^* \in \big[0, T_0\varepsilon^{-2}\big] \ \Big|\ \varepsilon^a\rho + \|\boldsymbol{U}_{\mathrm{ext}}\|_{L^\infty(\mathbb{R}^2\times[0,t^*])} \leqslant \varpi, \\ &\qquad z(t) \leqslant \rho^2 \text{ for all } t \in [0,t^*),\ z(0) = \rho_0^2 \Big\}, \\ J_{\rho_0,\varepsilon_0} &:= [0, T_{\rho_0,\varepsilon_0}). \end{aligned} \tag{6.2.3}$$

For $\boldsymbol{U}^{(0)}$ and ϵ_0 small enough the time interval J_{ρ_0,ε_0} is not-empty and the conditions

$$\forall\, t \in J_{\rho_0,\varepsilon_0}: \quad \operatorname{dist}\left(\overline{\operatorname{im} \boldsymbol{U}^\pm(\cdot,t)}, \partial\Omega_\pm\right) > \kappa > 0, \quad \|\boldsymbol{U}(\cdot,t)\|_{\mathcal{H}^3(\mathbb{R}^2)^3} \leqslant C < \infty$$

are guaranteed for some $\kappa > 0$.

We will now prove that

$$z(t) \leqslant \rho^2/2, \qquad t \in J_{\rho_0,\varepsilon_0}$$

for suitable ε_0 and ρ_0 and hence $T_{\rho_0,\varepsilon_0} = T_0\varepsilon^{-2}$ and (6.2.2) is true. This yields the estimate

$$\begin{aligned} \|\boldsymbol{U} - \boldsymbol{U}_{\mathrm{ext}}\|_{\mathcal{G}^3(\mathbb{R}^2\times(0,T_0\varepsilon^{-2}))^3} &= \varepsilon^a \|\boldsymbol{R}\|_{\mathcal{G}^3(\mathbb{R}^2\times(0,T_0\varepsilon^{-2}))^3} \\ &= \varepsilon^a \sup_{t\in[0,T_0\varepsilon^{-2}]} z(t)^{1/2} \\ &\leqslant \rho\varepsilon^a. \end{aligned} \tag{6.2.4}$$

Main approach:
Let $\boldsymbol{\beta} := (\beta_1, \beta_2, \beta_t)^\top \in \mathbb{N}_0^3$, $|\boldsymbol{\beta}| \leqslant 3$. Applying $\partial^{\boldsymbol{\beta}} = \partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}\partial_t^{\beta_t}$ to (6.1.2) yields

$$\begin{cases} S(\boldsymbol{x},t,\boldsymbol{R})\partial_t \boldsymbol{r}_{\boldsymbol{\beta}} + \sum_{j=1}^{2} A_j \partial_{x_j}\boldsymbol{r}_{\boldsymbol{\beta}} \\ \qquad = \boldsymbol{s}_{\boldsymbol{\beta}}(\boldsymbol{x},t,\boldsymbol{R}) + \boldsymbol{w}_{\boldsymbol{\beta}}(\boldsymbol{x},t,\boldsymbol{R}) - \varepsilon^{-a}\partial^{\boldsymbol{\beta}}\mathbf{Res}(\boldsymbol{x},t), & \boldsymbol{x} \in \mathbb{R}^2\backslash\Gamma_2, \quad t \in J_{\rho_0,\varepsilon_0}, \\ \boldsymbol{r}_{\boldsymbol{\beta}}(\cdot,0) = \boldsymbol{r}^{(0)} := \partial^{\boldsymbol{\beta}}\boldsymbol{R}(\cdot,0), & \boldsymbol{x} \in \mathbb{R}^2_{\pm}, \end{cases} \tag{6.2.5}$$

with

$$\begin{aligned} \boldsymbol{r}_{\boldsymbol{\beta}}(\boldsymbol{x},t) &:= \partial^{\boldsymbol{\beta}}\boldsymbol{R}(\boldsymbol{x},t), \\ \boldsymbol{s}_{\boldsymbol{\beta}}(\boldsymbol{x},t,\boldsymbol{R}) &:= -\left(\partial_{\boldsymbol{\beta}}(S(\boldsymbol{x},t,\boldsymbol{R})\partial_t\boldsymbol{R}) - S(\boldsymbol{x},t,\boldsymbol{R})\partial_t\boldsymbol{r}_{\boldsymbol{\beta}}\right) \\ &= -\sum_{\substack{\gamma\in\mathbb{N}^3\backslash\{\mathbf{0}\} \\ \gamma\leqslant\boldsymbol{\beta}}} \binom{\boldsymbol{\beta}}{\gamma}\partial^{\gamma}S(\boldsymbol{x},t,\boldsymbol{R})\partial^{\boldsymbol{\beta}-\gamma}\partial_t\boldsymbol{R}, \\ \boldsymbol{w}_{\boldsymbol{\beta}}(\boldsymbol{x},t,\boldsymbol{R}) &:= -\partial^{\boldsymbol{\beta}}(W(\boldsymbol{x},t,\boldsymbol{R})\boldsymbol{R}). \end{aligned}$$

Remark 6.2.2
Note that the interface conditions cannot be simply differentiated for all $\boldsymbol{\beta}$, since normal derivatives, i.e. x_1-derivatives, do in general not commute with the jump-brackets $[\![\cdot]\!]_{2D}$. Therefore, we will treat normal derivatives different from the tangential and temporal derivatives. For the estimation of the normal derivatives a more involved method will be necessary and the structure of Maxwell's equations will be essential.

Remark 6.2.3
For $\boldsymbol{R} \in \mathcal{G}^3\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)^3$ we have $S(\boldsymbol{R}) \in \mathcal{F}^{3,3}_{\eta,\mathrm{cv}}\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)$. Indeed,

$$S(\boldsymbol{x},t,\boldsymbol{R}(\boldsymbol{x},t)) = \Lambda(\boldsymbol{x}) + \epsilon_3(x_1)\varepsilon^{2a}\theta(\boldsymbol{R}(\boldsymbol{x},t)) + \varphi(\boldsymbol{R}(\boldsymbol{x},t)),$$

the Banach algebra property of $\mathcal{H}^3(\mathbb{R}^2)$ and the regularity of $\boldsymbol{U}_{\mathrm{ext}}$, ϵ_1, ϵ_3 give us that $S(\boldsymbol{R}) \in \mathcal{F}^{3,3}\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)$. With the algebra property it also follows that $W(\boldsymbol{R}) \in \mathcal{H}^3\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)^{3\times 3}$. Furthermore, let $\widetilde{\varepsilon} > 0$ and take arbitrary but sufficiently regular domains $M_{\pm} \subset \mathbb{R}^2_{\pm}$ such that for $\operatorname{dist}\{M_{\pm},\{\mathbf{0}\}\}$ large enough

$$\|\boldsymbol{U}_{\mathrm{ext}}(\cdot,t)\|_{H^3(M_{\pm})^3}, \|\boldsymbol{R}(\cdot,t)\|_{H^3(M_{\pm})^3} \leqslant \widetilde{\varepsilon}.$$

With the Sobolev embedding $H^3(M_{\pm}) \hookrightarrow C^1(M_{\pm})$ it follows that

$$\sup_{\boldsymbol{x}\in M_{\pm}} |\boldsymbol{U}_{\mathrm{ext}}(\boldsymbol{x},t)|, \sup_{\boldsymbol{x}\in M_{\pm}} |\boldsymbol{R}(\boldsymbol{x},t)| \leqslant C\widetilde{\varepsilon}.$$

Hence, $|\boldsymbol{U}_{\text{ext}}(\cdot,t)+\varepsilon^a\boldsymbol{R}(\cdot,t)| \to \mathbf{0}$ *for* $|\boldsymbol{x}| \to \infty$*. Now* $S(\boldsymbol{x},t,\boldsymbol{R}) = \widetilde{S}(\boldsymbol{x},\boldsymbol{U}_{\text{ext}}(\boldsymbol{x},t)+\varepsilon^a\boldsymbol{R})$ *and Remark 6.1.1 imply that* $S(\boldsymbol{R})$ *is positive definite and convergent for* $|\boldsymbol{x}| \to \infty$*.*
We also have that

$$\boldsymbol{f}_{\boldsymbol{\beta}}(\boldsymbol{R}) := \boldsymbol{s}_{\boldsymbol{\beta}}(\boldsymbol{R}) + \boldsymbol{w}_{\boldsymbol{\beta}}(\boldsymbol{R}) - \varepsilon^{-a}\partial^{\boldsymbol{\beta}}\mathbf{Res} \in \mathcal{G}^0\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)^3.$$

Indeed, the residual satisfies $\mathbf{Res}(\cdot,t) \in \mathcal{H}^3(\mathbb{R}^2)^3$ *by Lemma 4.2.4. By the definition of* $\boldsymbol{U}_{\text{ext}}$*,* $\boldsymbol{R}$ *and the algebra property of* $\mathcal{H}^3(\mathbb{R}_\pm)$ *it follows that* $W(\cdot,t,\boldsymbol{R}(\cdot,t))\boldsymbol{R}(\cdot,t) \in \mathcal{H}^3(\mathbb{R}^2)^3$*. Finally,* $\boldsymbol{s}_{\boldsymbol{\beta}}(\cdot,t,\boldsymbol{R}) \in L^2(\mathbb{R}^2)^3$ *follows from i), iv) and v) in Lemma 5.1.4.*

The temporal derivatives $\partial_t^k\boldsymbol{R}(\cdot,0)$ have to be interpreted as one-sided derivatives that satisfy

$$\partial_t^j\boldsymbol{R}(\cdot,0) = \boldsymbol{V}^{(j)}(\boldsymbol{R}(\cdot,0)) = \boldsymbol{V}^{(j)}\left(\boldsymbol{R}^{(0)}\right), \tag{6.2.6}$$

with $\boldsymbol{V}^{(j)}$ as defined in Section 5.3.
Similar to the procedure in Section 6.1.1 we test (6.2.5) with $\boldsymbol{r}_{\boldsymbol{\beta}}$ and get

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^2}\left(S(\boldsymbol{R})\partial_t\boldsymbol{r}_{\boldsymbol{\beta}}\cdot\boldsymbol{r}_{\boldsymbol{\beta}} + \sum_{j=1}^2 A_j\partial_{x_j}\boldsymbol{r}_{\boldsymbol{\beta}}\cdot\boldsymbol{r}_{\boldsymbol{\beta}}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}s &= \int_0^t\int_{\mathbb{R}^2}\left(\boldsymbol{w}_{\boldsymbol{\beta}}(\boldsymbol{R})\cdot\boldsymbol{r}_{\boldsymbol{\beta}} + \boldsymbol{s}_{\boldsymbol{\beta}}(\boldsymbol{R})\cdot\boldsymbol{r}_{\boldsymbol{\beta}}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}s \\ &\quad - \varepsilon^{-a}\int_0^t\int_{\mathbb{R}^2}\partial^{\boldsymbol{\beta}}\mathbf{Res}\cdot\boldsymbol{r}_{\boldsymbol{\beta}}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s.\end{aligned} \tag{6.2.7}$$

For $t \in J_{\rho_0,\varepsilon_0}$ the main steps of our bootstrapping argument are:

I. Use (6.2.6) to estimate $\left\|\boldsymbol{r}_{\boldsymbol{\beta}}^{(0)}\right\|_{L^2(\mathbb{R}^2)}$ for all $\boldsymbol{\beta} \in \mathbb{N}_0^3$ with $|\beta| \leqslant 3$.

II. Based on (6.2.7), estimate $\sum_{|\gamma|\leqslant 3,\gamma_1=0}\|\partial^\gamma\boldsymbol{R}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)^3}$ using that, for $\beta_1 = 0$,

$$\int_{\mathbb{R}^2}\sum_{j=1}^2 A_j\partial_{x_j}\boldsymbol{r}_{\boldsymbol{\beta}}\cdot\boldsymbol{r}_{\boldsymbol{\beta}}\,\mathrm{d}\boldsymbol{x} = 0.$$

III. Rewrite (6.2.5) to analyze $\partial^{\boldsymbol{\beta}}R_2$ and $\partial^{\boldsymbol{\beta}}R_3$ for $\beta_1 = 1$ and then iterate the process for $\beta_1 = 2$ and $\beta_1 = 3$.

IV. Use $\nabla\cdot\partial_t\boldsymbol{\mathcal{D}}(\boldsymbol{U}_E) = 0$ to estimate $\partial^{\boldsymbol{\beta}}R_1$ for $\beta_1 = 1$, where we start with $\boldsymbol{\beta} = (1,0,0)^\top$, and then iterate to increase β_t and β_2. Finally, we have to iterate the process again for $\beta_1 = 2$ and $\beta_1 = 3$.

Remark 6.2.4
This approach follows the proof of the local a priori estimates in [67]. The main difference is that, using the structure of our ansatz, we can derive the estimates on a large time interval $\left(0, T_0\varepsilon^{-2}\right)$ *with the desired dependence on* ε*.*

Step I: Estimates of the Initial Values

In this section we want to estimate $\left\|\boldsymbol{r}_{\boldsymbol{\beta}}(\cdot,0)\right\|_{L^2(\mathbb{R}^2)^3}$ for all $\boldsymbol{\beta} \in \mathbb{N}_0^3$ with $|\boldsymbol{\beta}| \leqslant 3$. For $\beta_t = 0$ we already have by assumption that

$$\|\boldsymbol{R}(\cdot,0)\|_{\mathcal{H}^3(\mathbb{R}^2)^3} = \left\|\boldsymbol{R}^{(0)}\right\|_{\mathcal{H}^3(\mathbb{R}^2)^3} < \rho_0.$$

If $\beta_t \neq 0$ we use (6.2.6) to estimate $\partial_t^p \boldsymbol{R}(\cdot,0)$ in $\mathcal{H}^{3-p}(\mathbb{R}^2)^3$ for $p \in \{1,2,3\}$.
Since $\boldsymbol{U}^{(0)}$ satisfies the nonlinear compatibility conditions of order 3, we know from Section 5.3 that (suppressing the $\boldsymbol{x}$-dependence)

$$\partial_t^j \boldsymbol{U}(t) = \boldsymbol{V}^{(j)}(\boldsymbol{U}(t))$$

for all $t \in [0,t_M)$ and $j \in \{0,1,2\}$. With $\boldsymbol{U} = \varepsilon^a \boldsymbol{R} + \boldsymbol{U}_{\text{ext}}$ we can rewrite these three equations as

$$\begin{aligned}
\partial_t \boldsymbol{R} = &-\widetilde{S}(\boldsymbol{U})^{-1}\Bigg(\sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{R} + \varepsilon^{-a}\mathbf{Res} + \varepsilon^{-a}\left(\widetilde{S}(\boldsymbol{U}) - \widetilde{S}(\boldsymbol{U}_{\text{ext}})\right)\partial_t \boldsymbol{U}_{\text{ext}}\Bigg),\\
\partial_t^2 \boldsymbol{R} = &-\widetilde{S}(\boldsymbol{U})^{-1}\Bigg(\sum_{j=1}^{2} A_j \partial_{x_j}\partial_t \boldsymbol{R} + \partial_t \widetilde{S}(\boldsymbol{U})\partial_t \boldsymbol{R} + \varepsilon^{-a}\partial_t \mathbf{Res}\Bigg)\\
&-\varepsilon^{-a}\widetilde{S}(\boldsymbol{U})^{-1}\left(\partial_t\left(\widetilde{S}(\boldsymbol{U}) - \widetilde{S}(\boldsymbol{U}_{\text{ext}})\right)\partial_t \boldsymbol{U}_{\text{ext}} + \left(\widetilde{S}(\boldsymbol{U}) - \widetilde{S}(\boldsymbol{U}_{\text{ext}})\right)\partial_t^2 \boldsymbol{U}_{\text{ext}}\right),\\
\partial_t^3 \boldsymbol{R} = &-\widetilde{S}(\boldsymbol{U})^{-1}\Bigg(\sum_{j=1}^{2} A_j \partial_{x_j}\partial_t^2 \boldsymbol{R} + 2\partial_t\left(\widetilde{S}(\boldsymbol{U})\right)\partial_t^2 \boldsymbol{R} + \partial_t^2\left(\widetilde{S}(\boldsymbol{U})\right)\partial_t \boldsymbol{R} + \varepsilon^{-a}\partial_t^2 \mathbf{Res}\Bigg)\\
&-\varepsilon^{-a}\widetilde{S}(\boldsymbol{U})^{-1}\Bigg(\partial_t^2\left(\widetilde{S}(\boldsymbol{U}) - \widetilde{S}(\boldsymbol{U}_{\text{ext}})\right)\partial_t \boldsymbol{U}_{\text{ext}} + 2\partial_t\left(\widetilde{S}(\boldsymbol{U}) - \widetilde{S}(\boldsymbol{U}_{\text{ext}})\right)\partial_t^2 \boldsymbol{U}_{\text{ext}}\\
&\qquad + \left(\widetilde{S}(\boldsymbol{U}) - \widetilde{S}(\boldsymbol{U}_{\text{ext}})\right)\partial_t^3 \boldsymbol{U}_{\text{ext}}\Bigg).
\end{aligned} \tag{6.2.8}$$

The following lemma collects some properties of the matrix function $\widetilde{S}$. The result and the proof are similar to Lemma 2.23, Lemma 7.1 and Corollary 7.2 in [75].

Lemma 6.2.5 (Properties of $\widetilde{S}$)
Let $T', \eta_0, R > 0$, $\Omega_\pm \subset \mathbb{R}^3$ and $\widetilde{S} \in B_R(0) \subset \mathcal{ML}^{3,3}_{\eta_0,\text{cv}}\left(\mathbb{R}^2, \Omega_\pm\right)$.
Then for all $\boldsymbol{U}, \boldsymbol{V} \in \mathcal{G}^3\left(\mathbb{R}^2 \times [0,T']\right)^3$ with $\overline{\operatorname{im} \boldsymbol{U}^\pm}, \overline{\operatorname{im} \boldsymbol{V}^\pm} \subset \Omega_\pm$ there exists a constant $C > 0$ such that

i) $\left\|\widetilde{S}(\boldsymbol{U}(t))^{-1}\right\|_{\mathcal{W}^{2,\infty}(\mathbb{R}^2)^{3\times3}+\mathcal{H}^2(\mathbb{R}^2)^{3\times3}} \leqslant C,$

ii) $\left\|\partial_t^k \widetilde{S}(\boldsymbol{U}(t))\right\|_{\mathcal{W}^{3-k,\infty}(\mathbb{R}^2)^{3\times3}+\mathcal{H}^{3-k}(\mathbb{R}^2)^{3\times3}} \leqslant C,$

iii) $\left\|\partial_t^k\left(\widetilde{S}(\boldsymbol{U}(t)) - \widetilde{S}(\boldsymbol{V}(t))\right)\right\|_{\mathcal{H}^{2-k}(\mathbb{R}^2)^{3\times3}} \leqslant C\sum_{j=0}^{k}\left\|\partial_t^j \boldsymbol{U}(t) - \partial_t^j \boldsymbol{V}(t)\right\|_{\mathcal{H}^{2-k}(\mathbb{R}^2)^3}$

for all $k \in \{0,1,2\}$ and $t \in [0,T']$.

PROOF: i) We have

$$\widetilde{S}(\boldsymbol{U})^{-1} = \begin{pmatrix} d(U_1,U_2)(\epsilon_1+\epsilon_3(U_1^2+3U_2^2)) & -2\epsilon_3 d(U_1,U_2)U_1U_2 & 0 \\ -2\epsilon_3 d(U_1,U_2)U_1U_2 & d(U_1,U_2)(\epsilon_1+\epsilon_3(3U_1^2+U_2^2)) & 0 \\ 0 & 0 & \mu_0^{-1} \end{pmatrix},$$

with

$$d(U_1,U_2) = \left(\epsilon_1^2 + 4\epsilon_1\epsilon_3\left(U_1^2+U_2^2\right) + 3\epsilon_3^2\left(U_1^2+U_2^2\right)^2\right)^{-1} =: p^{-1}(\boldsymbol{x},t).$$

Since $d(U_1,U_2)$ is the inverse of the product of the eigenvalues of $\widetilde{S}(\boldsymbol{U})$, it follows that $0 < d(U_1,U_2) \leqslant \eta_0^{-3}$ and therefore $d(U_1(t),U_2(t)) \in L^\infty(\mathbb{R}^2)$ for all $t \in [0,T']$.
Since $\epsilon_1,\epsilon_3 \in \mathcal{W}^{3,\infty}(\mathbb{R}^2)$ and $\boldsymbol{U}(t) \in \mathcal{H}^3(\mathbb{R}^2)^3$, it follows that $p(\boldsymbol{x},t) \in \mathcal{H}^3(\mathbb{R}^2)+\mathcal{W}^{3,\infty}(\mathbb{R}^2)$. With the algebra property of $\mathcal{H}^2(\mathbb{R}^2)$ the spatial derivatives of d have the following regularity:

$$\begin{aligned} \partial_{x_i} d(t) &= -d(t)^2\partial_{x_i}p(t) \in \mathcal{H}^2(\mathbb{R}^2)+\mathcal{W}^{2,\infty}(\mathbb{R}^2), \\ \partial_{x_i}\partial_{x_j} d(t) &= 2d(t)^3\partial_{x_i}p(t)\partial_{x_j}p(t) - d^2\partial_{x_i}\partial_{x_j}p(t) \in \mathcal{H}^1(\mathbb{R}^2)+\mathcal{W}^{1,\infty}(\mathbb{R}^2). \end{aligned}$$

It follows that $\widetilde{S}(\boldsymbol{U}(t))^{-1} \in \mathcal{W}^{2,\infty}(\mathbb{R}^2)^{3\times 3}+\mathcal{H}^2(\mathbb{R}^2)^{3\times 3}$ for all $t\in[0,T']$.
ii) For $\epsilon_1,\epsilon_2 \in \mathcal{W}^{3,\infty}(\mathbb{R}^2)$ and $\boldsymbol{U} \in \mathcal{G}^3(\mathbb{R}^2\times[0,T'])^3$ we can use the algebra property of $\mathcal{H}^3(\mathbb{R}^2)$ to see that all components of

$$\widetilde{S}(\boldsymbol{U}) = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \mu_0 \end{pmatrix} + \epsilon_3 \begin{pmatrix} 3U_1^2+U_2^2 & 2U_1U_2 & 0 \\ 2U_1U_2 & U_1^2+3U_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are in $\mathcal{W}^{3,\infty}(\mathbb{R}^2)+\mathcal{H}^3(\mathbb{R}^2)$. Since U_1 and U_2 appear polynomially in $\widetilde{S}(\boldsymbol{U})$, it is clear that all $\boldsymbol{U}$-derivatives of $\widetilde{S}(\boldsymbol{U})$, e.g. $\partial_{U_1}\partial_{U_2}\widetilde{S}(\boldsymbol{U})$, are even in $\mathcal{H}^3(\mathbb{R}^2)^{3\times3}$. To prove the statement for $k=1$ note that

$$\partial_t\left(\widetilde{S}(\boldsymbol{U}(t))\right) = \partial_{U_1}\widetilde{S}(\boldsymbol{U})\partial_t U_1 + \partial_{U_2}\widetilde{S}(\boldsymbol{U})\partial_t U_2,$$

where all factors are at least in $\mathcal{H}^2(\mathbb{R}^2)$ and the statement follows from the algebra property. For $k=2$ we have

$$\begin{aligned} \partial_t^2\left(\widetilde{S}(\boldsymbol{U}(t))\right) &= \partial_{U_1}^2\widetilde{S}(\boldsymbol{U})\partial_t U_1\partial_t U_1 + 2\partial_{U_1}\partial_{U_2}\widetilde{S}(\boldsymbol{U})\partial_t U_2\partial_t U_2 + \partial_{U_2}^2\widetilde{S}(\boldsymbol{U})\partial_t U_2\partial_t U_2 \\ &\quad + \partial_{U_1}\widetilde{S}(\boldsymbol{U})\partial_t^2 U_1 + \partial_{U_2}\widetilde{S}(\boldsymbol{U})\partial_t^2 U_2. \end{aligned} \tag{6.2.9}$$

We can again use the algebra property and for the terms $\partial_{U_1}\widetilde{S}(\boldsymbol{U})\partial_t^2 U_1 + \partial_{U_2}\widetilde{S}(\boldsymbol{U})\partial_t^2 U_2$ we use part ii) of Lemma 5.1.4 with $m_1 = 3$ and $m_2 = 1$.
iii) We restrict us to the case $k=2$ and show the estimate explicitly for the term corresponding to the fourth term on the right-hand side in (6.2.9). For other values of k and for all other

terms the estimates are analogous. We have

$$\begin{aligned}
&\left\|\partial_{U_1}\widetilde{S}(\boldsymbol{U})\partial_t^2 U_1-\partial_{U_1}\widetilde{S}(\boldsymbol{V})\partial_t^2 V_1\right\|_{L^2(\mathbb{R}^2)^{3\times 3}}\\
&\quad\leqslant\left\|\left(\partial_{U_1}\widetilde{S}(\boldsymbol{U})-\partial_{U_1}\widetilde{S}(\boldsymbol{V})\right)\partial_t^2 U_1\right\|_{L^2(\mathbb{R}^2)^{3\times 3}}+\left\|\partial_{U_1}\widetilde{S}(\boldsymbol{V})\left(\partial_t^2 U_1-\partial_t^2 V_1\right)\right\|_{L^2(\mathbb{R}^2)^{3\times 3}}\\
&\quad\leqslant C_1\left(\|\boldsymbol{U}-\boldsymbol{V}\|_{L^\infty(\mathbb{R}^2)^3}\left\|\partial_t^2 U_1\right\|_{L^2(\mathbb{R}^2)}+\left\|\partial_t^2 U_1-\partial_t^2 V_1\right\|_{L^2(\mathbb{R}^2)}\right)\\
&\quad\leqslant C\left(\|\boldsymbol{U}-\boldsymbol{V}\|_{\mathcal{H}^2(\mathbb{R}^2)^3}+\left\|\partial_t^2 U_1-\partial_t^2 V_1\right\|_{L^2(\mathbb{R}^2)}\right),
\end{aligned}$$

where

$$C_1:=\sup_{\boldsymbol{u}^\pm\in\Omega_\pm}\left|\partial_{U_1}\widetilde{S}(\boldsymbol{u})\right|+\sup_{\boldsymbol{u}^\pm\in\Omega_\pm}\left|D\partial_{U_1}\widetilde{S}(\boldsymbol{u})\right|,$$

with $D\partial_{U_1}\widetilde{S}$ being the Jacobian (tensor) of $\partial_{U_1}\widetilde{S}$. Above we used the mean value theorem, the Sobolev embedding $\mathcal{H}^2(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$ and the fact that $\widetilde{S}\in\mathcal{ML}^{3,3}_{\eta_0,\mathrm{cv}}\left(\mathbb{R}^2,\Omega_\pm\right)$, which guarantees that the suprema in C_1 are finite. □

We can now go back to equations (6.2.8) and use Lemma 5.1.4, Lemma 6.2.5 and $\varepsilon^a\boldsymbol{R}=\boldsymbol{U}-\boldsymbol{U}_{\mathrm{ext}}$ to show

$$\begin{aligned}
&\|\partial_t\boldsymbol{R}(\cdot,0)\|_{\mathcal{H}^2(\mathbb{R}^2)^3}\\
&\quad\leqslant C\left\|\widetilde{S}(\boldsymbol{U}(\cdot,0))^{-1}\right\|_{\mathcal{W}^{2,\infty}(\mathbb{R}^2)^{3\times3}+\mathcal{H}^2(\mathbb{R}^2)^{3\times3}}\left(\|\boldsymbol{R}(\cdot,0)\|_{\mathcal{H}^3(\mathbb{R}^2)^3}+\varepsilon^{-a}\|\mathbf{Res}(\cdot,0)\|_{\mathcal{H}^2(\mathbb{R}^2)^3}\right)\\
&\quad+C\varepsilon^{-a}\|\varepsilon^a\boldsymbol{R}(\cdot,0)\|_{\mathcal{H}^2(\mathbb{R}^2)^3}\|\partial_t\boldsymbol{U}_{\mathrm{ext}}(\cdot,0)\|_{\mathcal{H}^2(\mathbb{R}^2)^3}\\
&\quad\leqslant C\left(\left\|\boldsymbol{R}^{(0)}\right\|_{\mathcal{H}^3(\mathbb{R}^2)^3}+\varepsilon^{-a}\|\mathbf{Res}(\cdot,0)\|_{\mathcal{H}^2(\mathbb{R}^2)^3}\right).
\end{aligned}$$

The remaining two estimates follow analogously:

$$\begin{aligned}
\left\|\partial_t^2\boldsymbol{R}(\cdot,0)\right\|_{\mathcal{H}^1(\mathbb{R}^2)^3}&=C\left(\|\boldsymbol{R}(\cdot,0)\|_{\mathcal{H}^3(\mathbb{R}^2)^3}+\|\partial_t\boldsymbol{R}(\cdot,0)\|_{\mathcal{H}^2(\mathbb{R}^2)^3}+\varepsilon^{-a}\|\partial_t\mathbf{Res}(\cdot,0)\|_{\mathcal{H}^1(\mathbb{R}^2)^3}\right),\\
\left\|\partial_t^3\boldsymbol{R}(\cdot,0)\right\|_{L^2(\mathbb{R}^2)^3}&=C\left(\|\boldsymbol{R}(\cdot,0)\|_{\mathcal{H}^3(\mathbb{R}^2)^3}+\|\partial_t\boldsymbol{R}(\cdot,0)\|_{\mathcal{H}^2(\mathbb{R}^2)^3}+\left\|\partial_t^2\boldsymbol{R}(\cdot,0)\right\|_{\mathcal{H}^1(\mathbb{R}^2)^3}\right)\\
&\quad+C\left(\varepsilon^{-a}\left\|\partial_t^2\mathbf{Res}(\cdot,0)\right\|_{L^2(\mathbb{R}^2)^3}\right).
\end{aligned}$$

Finally, we use the recursive structure of the estimates to obtain

$$\left\|\partial_t^p\boldsymbol{R}(\cdot,0)\right\|_{\mathcal{H}^{3-p}(\mathbb{R}^2)^3}\leqslant C\left(\left\|\boldsymbol{R}^{(0)}\right\|_{\mathcal{H}^3(\mathbb{R}^2)^3}+\varepsilon^{-a}\sum_{j=0}^{p-1}\left\|\partial_t^j\mathbf{Res}(\cdot,0)\right\|_{\mathcal{H}^{2-j}(\mathbb{R}^2)^3}\right)$$

for all $p\in\{1,2,3\}$. With our estimate for the residual, see Lemma 4.2.4, we get

$$\left\|\boldsymbol{r}^{(0)}_{\boldsymbol{\beta}}\right\|_{L^2(\mathbb{R}^2)^3}\leqslant C\left(\rho_0+\varepsilon^{\frac{7}{2}-a}\right) \tag{6.2.10}$$

for all $\boldsymbol{\beta}\in\mathbb{N}_0^3$ with $|\boldsymbol{\beta}|\leqslant 3$.

Step II: Estimates of the β-derivatives of $R(\cdot, t)$ with $\beta_1 = 0$, $|\beta| \leqslant 3$

We first show an energy estimate similar to (6.1.6) for the t- and x_2-derivatives of $\boldsymbol{R}$.

Lemma 6.2.6 (Energy Estimate for $\boldsymbol{r}_\beta$)
Let $\boldsymbol{R} \in \mathcal{G}^3\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)^3$ be a solution of (6.1.2) *and let $\boldsymbol{\beta} \in \mathbb{N}_0^3, |\boldsymbol{\beta}| \leqslant 3, \beta_1 = 0$. Then $\boldsymbol{r}_\beta = \partial^\beta \boldsymbol{R}$ satisfies*

$$\begin{aligned} \frac{\eta}{2}\left\|\boldsymbol{r}_\beta(\cdot,t)\right\|^2_{L^2(\mathbb{R}^2)^3} &\leqslant C\left\|\boldsymbol{r}_\beta^{(0)}\right\|^2_{L^2(\mathbb{R}^2)^3} \\ &+ \int_0^t\int_{\mathbb{R}^2}\left(\boldsymbol{w}_\beta(\boldsymbol{R})\cdot\boldsymbol{r}_\beta + \boldsymbol{s}_\beta(\boldsymbol{R})\cdot\boldsymbol{r}_\beta + \frac{1}{2}\partial_t S(\boldsymbol{R})\boldsymbol{r}_\beta\cdot\boldsymbol{r}_\beta - \varepsilon^{-a}\partial^\beta\mathbf{Res}\cdot\boldsymbol{r}_\beta\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \end{aligned} \tag{6.2.11}$$

for every $t \in J_{\rho_0,\varepsilon_0}$.

Proof: The proof is divided into two parts. First, we prove (6.2.11) for $|\boldsymbol{\beta}| < 3$. In this case, we have enough regularity to do calculations similar to Section 6.1.1. Second, we use Lemma 5.1.10 to prove the energy estimate for $|\boldsymbol{\beta}| = 3$.

Step 1: Let us first study the case $|\boldsymbol{\beta}| < 3$.
Since $\boldsymbol{R} \in \mathcal{G}^3\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)^3$, we have $\boldsymbol{r}_\beta = \partial^\beta\boldsymbol{R} \in \mathcal{G}^1\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)^3$. To employ (6.2.7), we compute

$$\int_0^t\int_{\mathbb{R}^2} S(\boldsymbol{R})\partial_t\boldsymbol{r}_\beta\cdot\boldsymbol{r}_\beta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s = \frac{1}{2}\int_0^t\partial_t\left(\int_{\mathbb{R}^2} S(\boldsymbol{R})\boldsymbol{r}_\beta\cdot\boldsymbol{r}_\beta\,\mathrm{d}\boldsymbol{x}\right)\mathrm{d}s - \frac{1}{2}\int_0^t\int_{\mathbb{R}^2}\partial_t S(\boldsymbol{R})\boldsymbol{r}_\beta\cdot\boldsymbol{r}_\beta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.$$

Using that $S(\boldsymbol{R})$ is positive definite, we estimate

$$\int_{\mathbb{R}^2} S(\boldsymbol{R})(t)\boldsymbol{r}_\beta(t)\cdot\boldsymbol{r}_\beta(t)\,\mathrm{d}\boldsymbol{x} \geqslant \eta\left\|\boldsymbol{r}_\beta(\cdot,t)\right\|^2_{L^2(\mathbb{R}^2)^3}.$$

Moreover, we have

$$\int_{\mathbb{R}^2} S(\boldsymbol{R})(0)\boldsymbol{r}_\beta(0)\cdot\boldsymbol{r}_\beta(0)\,\mathrm{d}\boldsymbol{x} \leqslant \left\|S\left(\boldsymbol{R}^{(0)}\right)\right\|_{L^\infty(\mathbb{R}^2)^{3\times3}}\left\|\boldsymbol{r}_\beta^{(0)}\right\|^2_{L^2(\mathbb{R}^2)^3}.$$

Since $\boldsymbol{R}^{(0)}, \boldsymbol{U}_{\mathrm{ext}}^{(0)} \in L^\infty(\mathbb{R}^2)^3$, this leads to

$$\frac{\eta}{2}\left\|\boldsymbol{r}_\beta(\cdot,t)\right\|^2_{L^2(\mathbb{R}^2)^3} \leqslant C\left\|\boldsymbol{r}_\beta^{(0)}\right\|^2_{L^2(\mathbb{R}^2)^3} + \int_0^t\int_{\mathbb{R}^2} S(\boldsymbol{R})\partial_t\boldsymbol{r}_\beta\cdot\boldsymbol{r}_\beta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s + \frac{1}{2}\int_0^t\int_{\mathbb{R}^2}\partial_t S(\boldsymbol{R})\boldsymbol{r}_\beta\cdot\boldsymbol{r}_\beta\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.$$

An integration by parts yields

$$\int_{\mathbb{R}^2}\sum_{j=1}^2 A_j\partial_{x_j}\boldsymbol{r}_\beta\cdot\boldsymbol{r}_\beta\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^2}\left(-\partial_{x_2}r_{\beta,3}r_{\beta,1} + \partial_{x_1}r_{\beta,3}r_{\beta,2} - \partial_{x_2}r_{\beta,1}r_{\beta,3} + \partial_{x_1}r_{\beta,2}r_{\beta,3}\right)\mathrm{d}\boldsymbol{x} = 0,$$

employing the differentiated interface conditions

$$[\![r_{\beta,2}]\!]_{\mathrm{2D}} = [\![r_{\beta,3}]\!]_{\mathrm{2D}} = 0 \tag{6.2.12}$$

in the x_1-integral. The interface conditions can be differentiated since $\beta_1 = 0$, see Lemma 5.4.6. Now (6.2.11) is a consequence of (6.1.2) and the above formulas.
Step 2: Next, we consider the remaining case $|\boldsymbol{\beta}| = 3, \beta_1 = 0$.
Let $\boldsymbol{f} := \boldsymbol{s}_{\boldsymbol{\beta}}(\boldsymbol{R}) + \boldsymbol{w}_{\boldsymbol{\beta}}(\boldsymbol{R}) - \varepsilon^{-a}\partial^{\boldsymbol{\beta}}\mathbf{Res}$. The differential equation in (6.2.5) becomes

$$S(\boldsymbol{R})\partial_t \boldsymbol{r}_{\boldsymbol{\beta}} + \sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{r}_{\boldsymbol{\beta}} = \boldsymbol{f}, \qquad \boldsymbol{x} \in \mathbb{R}^2 \backslash \Gamma_2, \quad t \in J_{\rho_0,\varepsilon_0}. \tag{6.2.13}$$

Since $S(\boldsymbol{R}) \in \mathcal{F}^{3,3}_{\eta,\mathrm{cv}}\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)$ and $\boldsymbol{f} \in \mathcal{G}^0\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)^3$, see Remark 6.2.3, we can apply Lemma 5.1.10 to (6.2.13) (setting $A_{\mathrm{t}} := S(\boldsymbol{R})$ and $M := 0$). Because $\boldsymbol{r}_{\boldsymbol{\beta}}$ is a weak solution of (6.2.13) with the initial conditions from (6.2.5) and the interface conditions (6.2.12), the lemma provides sequences $\left(\boldsymbol{r}^{(0)}_{\boldsymbol{\beta},n}\right)_n \subset \mathcal{D}_\Gamma(\mathbb{R}^2)^3$, $\left(\boldsymbol{f}_n\right)_n \subset \mathcal{H}^1\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)^3$ and $\left(\boldsymbol{r}_{\boldsymbol{\beta},n}\right)_n \subset \mathcal{G}^1\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)^3$ with $\boldsymbol{r}^{(0)}_{\boldsymbol{\beta},n} \to \boldsymbol{r}^{(0)}_{\boldsymbol{\beta}}$ in $L^2(\mathbb{R}^2)^3$, $\boldsymbol{f}_n \to \boldsymbol{f}$ in $L^2\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)^3$ and $\boldsymbol{r}_{\boldsymbol{\beta},n} \to \boldsymbol{r}_{\boldsymbol{\beta}}$ in $\mathcal{G}^0\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)^3$ for $n \to \infty$, and $\boldsymbol{r}_{\boldsymbol{\beta},n}$ is a weak solution of (6.2.13) with data $\left(\boldsymbol{f}_n, \boldsymbol{r}^{(0)}_{\boldsymbol{\beta},n}\right)$ for all $n \in \mathbb{N}$.
Now all functions are sufficiently regular and the same calculation as in Step 1 shows that

$$\frac{\eta}{2}\left\|\boldsymbol{r}_{\boldsymbol{\beta},n}(\cdot,t)\right\|^2_{L^2(\mathbb{R}^2)^3} \leqslant C\left\|\boldsymbol{r}^{(0)}_{\boldsymbol{\beta},n}\right\|^2_{L^2(\mathbb{R}^2)^3} + \int_0^t \int_{\mathbb{R}^2}\left(\boldsymbol{f}_n \cdot \boldsymbol{r}_{\boldsymbol{\beta},n} + \frac{1}{2}\partial_t S(\boldsymbol{R})\boldsymbol{r}_{\boldsymbol{\beta},n} \cdot \boldsymbol{r}_{\boldsymbol{\beta},n}\,\mathrm{d}\boldsymbol{x}\right)\mathrm{d}s. \tag{6.2.14}$$

The Cauchy-Schwarz inequality shows that for almost every $t \in J_{\rho_0,\varepsilon_0}$

$$\begin{aligned}\int_{\mathbb{R}^2} \boldsymbol{f}_n \cdot \boldsymbol{r}_{\boldsymbol{\beta},n}\,\mathrm{d}\boldsymbol{x} \leqslant& \int_{\mathbb{R}^2} \boldsymbol{f} \cdot \boldsymbol{r}_{\boldsymbol{\beta}}\,\mathrm{d}\boldsymbol{x} + \|\boldsymbol{f}(\cdot,t)\|_{L^2(\mathbb{R}^2)^3}\left\|(\boldsymbol{r}_{\boldsymbol{\beta},n} - \boldsymbol{r}_{\boldsymbol{\beta}})(\cdot,t)\right\|_{L^2(\mathbb{R}^2)^3}\\ &+ \|(\boldsymbol{f}_n - \boldsymbol{f})(\cdot,t)\|_{L^2(\mathbb{R}^2)^3}\left\|\boldsymbol{r}_{\boldsymbol{\beta},n}(\cdot,t)\right\|_{L^2(\mathbb{R}^2)^3}\\ &\to \int_{\mathbb{R}^2} \boldsymbol{f} \cdot \boldsymbol{r}_{\boldsymbol{\beta}}\,\mathrm{d}\boldsymbol{x} \qquad (n \to \infty).\end{aligned}$$

Since $\partial_t S(\boldsymbol{R}) \in L^\infty\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)^{3\times 3}$, we also get for almost every $t \in J_{\rho_0,\varepsilon_0}$

$$\begin{aligned}\int_{\mathbb{R}^2} \partial_t S(\boldsymbol{R})\boldsymbol{r}_{\boldsymbol{\beta},n} \cdot \boldsymbol{r}_{\boldsymbol{\beta},n}\,\mathrm{d}\boldsymbol{x} \leqslant& \int_{\mathbb{R}^2} \partial_t S(\boldsymbol{R})\boldsymbol{r}_{\boldsymbol{\beta}} \cdot \boldsymbol{r}_{\boldsymbol{\beta}}\,\mathrm{d}\boldsymbol{x} + C\left\|\boldsymbol{r}_{\boldsymbol{\beta},n}(\cdot,t) - \boldsymbol{r}_{\boldsymbol{\beta}}(\cdot,t)\right\|_{L^2(\mathbb{R}^2)^3}\\ &\to \int_{\mathbb{R}^2} \partial_t S(\boldsymbol{R})\boldsymbol{r}_{\boldsymbol{\beta}} \cdot \boldsymbol{r}_{\boldsymbol{\beta}}\,\mathrm{d}\boldsymbol{x} \qquad (n \to \infty).\end{aligned}$$

Lebesgue's dominated convergence theorem allows us to do the limit process for (6.2.14) and gives us the statement.

□

We now have to estimate each part of the right-hand side in (6.2.11). The main ideas for the estimates will be similar to the ones in Section 6.1.1.

Let

$$\widetilde{z}(t) := \sum_{\substack{\gamma \in \mathbb{N}_0^3 \\ |\gamma| \leqslant 3, \gamma_1 = 0}} \|\partial^\gamma \boldsymbol{R}(\cdot, t)\|^2_{L^2(\mathbb{R}^2)^3}.$$

From Step I we know that

$$\left\| \boldsymbol{r}_{\boldsymbol{\beta}}^{(0)} \right\|_{L^2(\mathbb{R}^2)^3} \leqslant C \left(\rho + \varepsilon^{\frac{7}{2} - a} \right).$$

Let us start with the term

$$\int_0^t \int_{\mathbb{R}^2} \frac{1}{2} \partial_t S(\boldsymbol{R}) \boldsymbol{r}_{\boldsymbol{\beta}} \cdot \boldsymbol{r}_{\boldsymbol{\beta}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}s = \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} \left(\epsilon_3(x_1) \varepsilon^{2a} \partial_t \theta(\boldsymbol{R}) + \partial_t \varphi(\boldsymbol{R}) \right) \boldsymbol{r}_{\boldsymbol{\beta}} \cdot \boldsymbol{r}_{\boldsymbol{\beta}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}s.$$

First, we have

$$\frac{1}{2} \varepsilon^{2a} \epsilon_3 \int_0^t \int_{\mathbb{R}^2} \partial_t \theta(\boldsymbol{R}) \boldsymbol{r}_{\boldsymbol{\beta}} \cdot \boldsymbol{r}_{\boldsymbol{\beta}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}s \leqslant C \varepsilon^{2a} \int_0^t \left\| \boldsymbol{r}_{\boldsymbol{\beta}}(\cdot, s) \right\|^2_{L^2(\mathbb{R}^2)^3} \mathrm{d}s \leqslant C \varepsilon^{2a} \int_0^t \widetilde{z}(s) \, \mathrm{d}s$$

since $\boldsymbol{R}, \partial_t \boldsymbol{R} \in L^\infty \left(\mathbb{R}^2 \times J_{\rho_0, \varepsilon_0} \right)^3$ and $\theta(\boldsymbol{R})$ only contains quadratic terms in $\boldsymbol{R}$. Similarly, using that

$$\|\boldsymbol{U}_{\mathrm{ext}}\|_{L^\infty \left(\mathbb{R}^2 \times J_{\rho_0, \varepsilon_0} \right)^3}, \|\partial_t \boldsymbol{U}_{\mathrm{ext}}\|_{L^\infty \left(\mathbb{R}^2 \times J_{\rho_0, \varepsilon_0} \right)^3} \leqslant C\varepsilon,$$

see (4.2.11), (4.2.13), we derive

$$\frac{1}{2} \int_0^t \int_{\mathbb{R}^2} \partial_t \varphi(\boldsymbol{R}) \boldsymbol{r}_{\boldsymbol{\beta}} \cdot \boldsymbol{r}_{\boldsymbol{\beta}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}s \leqslant C \left(\varepsilon^2 + \varepsilon^{1+a} \right) \int_0^t \left\| \boldsymbol{r}_{\boldsymbol{\beta}}(\cdot, s) \right\|^2_{L^2(\mathbb{R}^2)^3} \mathrm{d}s \leqslant C \varepsilon^2 \int_0^t \widetilde{z}(s) \, \mathrm{d}s.$$

Note that $\partial_t \varphi(\boldsymbol{R})$ only contains terms that are linear or quadratic in $\boldsymbol{U}_{\mathrm{ext}}$ and $\partial_t \boldsymbol{U}_{\mathrm{ext}}$.
For the residual term, (4.2.9) yields $\left\| \partial^{\boldsymbol{\beta}} \operatorname{Res}(\cdot, t) \right\|_{L^2(\mathbb{R}^2)^3} \leqslant C \varepsilon^{7/2}$. The Cauchy-Schwarz inequality and $\left\| \boldsymbol{r}_{\boldsymbol{\beta}}(\cdot, t) \right\|^2_{L^2(\mathbb{R}^2)^3} \leqslant z(t) \leqslant \rho \leqslant 1$ for $t \in J_{\rho_0, \varepsilon_0}$ then give us

$$\int_0^t \int_{\mathbb{R}^2} \varepsilon^{-a} \partial^{\boldsymbol{\beta}} \mathbf{Res} \cdot \boldsymbol{r}_{\boldsymbol{\beta}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}s \leqslant \int_0^t \varepsilon^{-a} \left\| \partial^{\boldsymbol{\beta}} \mathbf{Res}(\cdot, s) \right\|_{L^2(\mathbb{R}^2)^3} \left\| \boldsymbol{r}_{\boldsymbol{\beta}}(\cdot, s) \right\|_{L^2(\mathbb{R}^2)^3} \mathrm{d}s \leqslant C \varepsilon^{\frac{3}{2} - a}.$$

The remaining term $\int_0^t \int_{\mathbb{R}^2} (\boldsymbol{w}_{\boldsymbol{\beta}} \cdot \boldsymbol{r}_{\boldsymbol{\beta}} + \boldsymbol{s}_{\boldsymbol{\beta}} \cdot \boldsymbol{r}_{\boldsymbol{\beta}}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}s$ mainly consist of integrals of the type

$$I_1 := \int_0^t \int_{\mathbb{R}^2} \partial^{\boldsymbol{a}} f(\boldsymbol{x}, s) \partial^{\boldsymbol{b}} g(\boldsymbol{x}, s) \partial^{\boldsymbol{c}} h(\boldsymbol{x}, s) k(\boldsymbol{x}, s) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}s, \tag{6.2.15}$$

where $f, g, h \in \mathcal{G}^3(\mathbb{R}^2 \times J)$, $k \in \mathcal{G}^0(\mathbb{R}^2 \times J)$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{N}_0^3$ with $|\boldsymbol{a}|, |\boldsymbol{b}|, |\boldsymbol{c}| < 4$ and $s := |\boldsymbol{a}| + |\boldsymbol{b}| + |\boldsymbol{c}| \leqslant 4$. For $s = 4$ we only have integrals where at least one time-derivative is present, i.e. $a_t = b_t = c_t = 0$ is not possible.
The case where four derivatives fall on one component of $\boldsymbol{U}_{\mathrm{ext}}$ also occurs and will be discussed separately.

Due to the symmetry in (6.2.15), there are the following two classes of terms and associated estimates:

i) $|\boldsymbol{a}| \leqslant 3$, $|\boldsymbol{b}| \leqslant 1$, $|\boldsymbol{c}| \leqslant 1$:
Here $\partial^{\boldsymbol{a}} f(\cdot,t), k(\cdot,t) \in L^2(\mathbb{R}^2)$ and $\partial^{\boldsymbol{b}} g(\cdot,t), \partial^{\boldsymbol{c}} h(\cdot,t) \in L^\infty(\mathbb{R}^2)$, by Sobolev embedding. With the Cauchy-Schwarz inequality we obtain

$$I_1 \leqslant C \int_0^t \|\partial^{\boldsymbol{a}} f(\cdot,s)\|_{L^2(\mathbb{R}^2)} \left\|\partial^{\boldsymbol{b}} g(\cdot,s)\right\|_{L^\infty(\mathbb{R}^2)} \|\partial^{\boldsymbol{c}} h(\cdot,s)\|_{L^\infty(\mathbb{R}^2)} \|k(\cdot,s)\|_{L^2(\mathbb{R}^2)} \,\mathrm{d}s.$$

ii) $|\boldsymbol{a}| \leqslant 2$, $|\boldsymbol{b}| \leqslant 2$, $|\boldsymbol{c}| = 0$:
Now $k(\cdot,t) \in L^2(\mathbb{R}^2)$, $\partial^{\boldsymbol{c}} h(\cdot,t) \in L^\infty(\mathbb{R}^2)$ and $\partial^{\boldsymbol{a}} f(\cdot,t), \partial^{\boldsymbol{b}} g(\cdot,t) \in L^p(\mathbb{R}^2)$ for all $p \in [1,\infty)$. This follows from the Sobolev embedding $H^1(\mathbb{R}^2_\pm) \hookrightarrow L^p(\mathbb{R}^2_\pm)$ for all $1 \leqslant p < \infty$. The generalized Hölder inequality then yields

$$I_1 \leqslant C \int_0^t \|\partial^{\boldsymbol{a}} f(\cdot,s)\|_{L^3(\mathbb{R}^2)} \left\|\partial^{\boldsymbol{b}} g(\cdot,s)\right\|_{L^6(\mathbb{R}^2)} \|\partial^{\boldsymbol{c}} h(\cdot,s)\|_{L^\infty(\mathbb{R}^2)} \|k(\cdot,s)\|_{L^2(\mathbb{R}^2)} \,\mathrm{d}s.$$

Note that for some indices the cases overlap.
The role of the function k in (6.2.15) will always be played by a component of $\boldsymbol{r}_{\boldsymbol{\beta}} = \partial^{\boldsymbol{\beta}} \boldsymbol{R}$. Recall that $S(\boldsymbol{R}) = \Lambda + \epsilon_3 \varepsilon^{2a} \theta(\boldsymbol{R}) + \varphi(\boldsymbol{R})$. Hence, to estimate $\int_0^t \int_{\mathbb{R}^2} \boldsymbol{s}_{\boldsymbol{\beta}} \cdot \boldsymbol{r}_{\boldsymbol{\beta}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s$, we first analyze $\varepsilon^{2a} \epsilon_3 \partial^{\boldsymbol{\gamma}} \theta(\boldsymbol{R}) \partial^{\boldsymbol{\beta}-\boldsymbol{\gamma}} \partial_t \boldsymbol{R} \cdot \boldsymbol{r}_{\boldsymbol{\beta}}$ where $\beta_1 = \gamma_1 = 0$. This sum consists of terms of the form

$$C\varepsilon^{2a} \partial^{\boldsymbol{\gamma}'} R_i \partial^{\boldsymbol{\gamma}''} R_j \partial^{\boldsymbol{\beta}-\boldsymbol{\gamma}} \partial_t R_k \partial^{\boldsymbol{\beta}} R_l,$$

with $\boldsymbol{\gamma} = \boldsymbol{\gamma}' + \boldsymbol{\gamma}''$, $\gamma_1' = \gamma_1'' = 0$ and $i,j,k,l \in \{1,2\}$. We therefore have to estimate

$$I_2 := C\varepsilon^{2a} \int_0^t \int_{\mathbb{R}^2} \partial^{\boldsymbol{\gamma}'} R_i \partial^{\boldsymbol{\gamma}''} R_j \partial^{\boldsymbol{\beta}-\boldsymbol{\gamma}} \partial_t R_k \partial^{\boldsymbol{\beta}} R_l \,\mathrm{d}\boldsymbol{x}\mathrm{d}s. \tag{6.2.16}$$

The case i) above applies if $|\boldsymbol{\beta} - \boldsymbol{\gamma}| = 0$, where we may take $|\boldsymbol{\gamma}''| \leqslant 1$. We then estimate

$$\begin{aligned} I_2 &\leqslant C\varepsilon^{2a} \int_0^t \left\|\partial^{\boldsymbol{\beta}} R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \left\|\partial^{\boldsymbol{\gamma}'} R_i(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \mathrm{d}s \\ &\leqslant C\varepsilon^{2a} \int_0^t \widetilde{z}(s)\,\mathrm{d}s. \end{aligned}$$

A representative of type ii) is any term with $|\boldsymbol{\beta}-\boldsymbol{\gamma}| = 1$, $|\boldsymbol{\gamma}'| = 2$ and $|\boldsymbol{\gamma}''| = 0$, which is estimated via

$$\begin{aligned} I_2 &\leqslant C\varepsilon^{2a} \int_0^t \left\|\partial^{\boldsymbol{\beta}} R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \left\|\partial^{\boldsymbol{\gamma}'} R_i(\cdot,s)\right\|_{L^6(\mathbb{R}^2)} \left\|\partial^{\boldsymbol{\beta}-\boldsymbol{\gamma}}\partial_t R_k(\cdot,s)\right\|_{L^3(\mathbb{R}^2)} \,\mathrm{d}s \\ &\leqslant C\varepsilon^{2a} \int_0^t \left(\left\|\partial^{\boldsymbol{\beta}} R_l(\cdot,s)\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\partial^{\boldsymbol{\gamma}'} R_i(\cdot,s)\right\|^2_{L^6(\mathbb{R}^2)} \left\|\partial^{\boldsymbol{\beta}-\boldsymbol{\gamma}}\partial_t R_k(\cdot,s)\right\|^2_{L^3(\mathbb{R}^2)} \right) \mathrm{d}s \\ &\leqslant C\varepsilon^{2a} \int_0^t \left(\widetilde{z}(s) + (z(s))^2\right) \,\mathrm{d}s \\ &\leqslant C\varepsilon^{2a} \int_0^t \widetilde{z}(s)\,\mathrm{d}s + C\rho^2\varepsilon^{2a} t, \end{aligned}$$

using $H^1(\mathbb{R}^2_\pm) \hookrightarrow L^p(\mathbb{R}^2_\pm)$ for $1 \leqslant p < \infty$. The remaining cases can be treated similarly.
Next, we study $\partial^{\boldsymbol{\gamma}}\varphi(\boldsymbol{R})\partial^{\boldsymbol{\beta}-\boldsymbol{\gamma}}\partial_t \boldsymbol{R} \cdot \partial^{\boldsymbol{\beta}}\boldsymbol{R}$ with $|\boldsymbol{\beta}-\boldsymbol{\gamma}| \leqslant 2$ and $\beta_1 = \gamma_1 = 0$. We use (4.2.11) which provides the inequality $\|\partial^{\boldsymbol{\alpha}}\boldsymbol{U}_{\mathrm{ext}}\|_{L^\infty\left(\mathbb{R}^2\times J_{\rho_0,\varepsilon_0}\right)^3} \leqslant C\varepsilon$ for all $|\boldsymbol{\alpha}| \leqslant 3$ with $\alpha_1 \leqslant 2$. For terms quadratic in $\boldsymbol{U}_{\mathrm{ext}}$ case i) applies:

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^2} \partial^{\boldsymbol{\gamma}}(U_{\mathrm{ext},i}U_{\mathrm{ext},j})\partial^{\boldsymbol{\beta}-\boldsymbol{\gamma}}\partial_t R_k \partial^{\boldsymbol{\beta}} R_l \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s &\leqslant C\varepsilon^2 \int_0^t \left\|\partial^{\boldsymbol{\beta}} R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \left\|\partial^{\boldsymbol{\beta}-\boldsymbol{\gamma}}\partial_t R_k(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \mathrm{d}s \\ &\leqslant C\varepsilon^2 \int_0^t \widetilde{z}(s)\,\mathrm{d}s. \end{aligned}$$

For terms linear in $\boldsymbol{U}_{\mathrm{ext}}$, i.e.

$$I_3 := \varepsilon^a \int_0^t \int_{\mathbb{R}^2} \partial^{\boldsymbol{\gamma}'} R_i \partial^{\boldsymbol{\gamma}''} U_{\mathrm{ext},j} \partial^{\boldsymbol{\beta}-\boldsymbol{\gamma}}\partial_t R_k \partial^{\boldsymbol{\beta}} R_l \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s, \tag{6.2.17}$$

we distinguish the three cases $|\boldsymbol{\beta}-\boldsymbol{\gamma}| = 0, 1$, and 2. For $|\boldsymbol{\beta}-\boldsymbol{\gamma}| = 0$ we compute

$$I_3 \leqslant C\varepsilon^{1+a} \int_0^t \left\|\partial^{\boldsymbol{\beta}} R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \left\|\partial^{\boldsymbol{\gamma}'} R_i(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \mathrm{d}s \leqslant C\varepsilon^{2a} \int_0^t \widetilde{z}(s)\,\mathrm{d}s$$

by means of the estimate type i) and the fact that $\partial_t R_k \in L^\infty\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)$. For $|\boldsymbol{\beta}-\boldsymbol{\gamma}| = 1$ the estimate of type ii) applies and we have

$$\begin{aligned} I_3 &\leqslant C\varepsilon^{1+a} \int_0^t \left\|\partial^{\boldsymbol{\beta}} R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \left\|\partial^{\boldsymbol{\gamma}'} R_i(\cdot,s)\right\|_{L^6(\mathbb{R}^2)} \left\|\partial^{\boldsymbol{\beta}-\boldsymbol{\gamma}}\partial_t R_k(\cdot,s)\right\|_{L^3(\mathbb{R}^2)} \mathrm{d}s \\ &\leqslant C\varepsilon^{1+a} \int_0^t \widetilde{z}(s)\,\mathrm{d}s + C\varepsilon^{1+a} t \end{aligned}$$

as $\partial^{\boldsymbol{\gamma}'} R_i, \partial^{\boldsymbol{\beta}-\boldsymbol{\gamma}}\partial_t R_k \in L^\infty(J_{\rho_0,\varepsilon_0}, \mathcal{H}^1(\mathbb{R}^2))$. Finally, for $|\boldsymbol{\beta}-\boldsymbol{\gamma}| = 2$ case i) again yields

$$I_3 \leqslant C\varepsilon^{1+a} \int_0^t \left\|\partial^{\boldsymbol{\beta}-\boldsymbol{\gamma}}\partial_t R_k(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \left\|\partial^{\boldsymbol{\beta}} R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \mathrm{d}s \leqslant C\varepsilon^{1+a} \int_0^t \widetilde{z}(s)\,\mathrm{d}s,$$

where we have used $\partial^{\boldsymbol{\gamma}'} R_i \in L^\infty\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)$ because $|\boldsymbol{\gamma}'| \leqslant 1$.

Remark 6.2.7

Let us collect all possible cases for the estimates above with $|\boldsymbol{\beta}| = 3$ in one table. For $|\boldsymbol{\beta}| \leqslant 2$ we have enough regularity to always apply estimates of type i).

For the estimates above we have only used that $\partial^{\boldsymbol{\beta}}\boldsymbol{R} \in L^2(\mathbb{R}^2)^3$ and $\partial^{\gamma''}\boldsymbol{U}_{\text{ext}} \in L^\infty(\mathbb{R}^2)^3$. The other components have to satisfy certain regularity assumptions to use case i) or case ii) for the estimates. In the first column of Table 6.1 we see which cases apply when $\partial^{\gamma''}$ is applied to a component of $\boldsymbol{R}$, see (6.2.16)*, in the tenth column we see the cases when $\partial^{\gamma''}$ is applied to a component of $\boldsymbol{U}_{\text{ext}}$, see* (6.2.17).

cases	$\|\gamma\|$	$\|\boldsymbol{\beta}-\gamma\|$	$\|\gamma'\|$	$\|\gamma''\|$	$\partial^{\boldsymbol{\beta}}\boldsymbol{R}$	$\partial^{\boldsymbol{\beta}-\gamma}\partial_t\boldsymbol{R}$	$\partial^{\gamma'}\boldsymbol{R}$	$\partial^{\gamma''}\boldsymbol{R}$	cases	$\partial^{\gamma''}\boldsymbol{U}_{\text{ext}}$
i)	1	2	0	1	L^2	L^2	H^3	H^2	i)	L^∞
i)	1	2	1	0	L^2	L^2	H^2	H^3	i)	L^∞
ii)	2	1	0	2	L^2	H^1	H^3	H^1	i),ii)	L^∞
i),ii)	2	1	1	1	L^2	H^1	H^2	H^2	i),ii)	L^∞
ii)	2	1	2	0	L^2	H^1	H^1	H^3	ii)	L^∞
i)	3	0	0	3	L^2	H^2	H^3	L^2	i)	L^∞
i),ii)	3	0	1	2	L^2	H^2	H^2	H^1	i),ii)	L^∞
i),ii)	3	0	2	1	L^2	H^2	H^1	H^2	i),ii)	L^∞
i)	3	0	3	0	L^2	H^2	L^2	H^3	i)	L^∞

Table 6.1.: Regularity of factors for $|\boldsymbol{\beta}| = 3$

At last, we treat $\partial^{\boldsymbol{\beta}}(W(\boldsymbol{R})\boldsymbol{R}) \cdot \partial^{\boldsymbol{\beta}}\boldsymbol{R}$. Terms quadratic in $\boldsymbol{U}_{\text{ext}}$ are estimated as follows, where $\boldsymbol{\beta} = \boldsymbol{\beta}' + \boldsymbol{\beta}''$. If $|\boldsymbol{\beta}'| < 3$ or if not all three derivatives fall on $\partial_t U_{\text{ext},j}$, we obtain

$$I_4 := \int_0^t\int_{\mathbb{R}^2} \partial^{\boldsymbol{\beta}'}(U_{\text{ext},i}\partial_t U_{\text{ext},j})\partial^{\boldsymbol{\beta}''}R_k\partial^{\boldsymbol{\beta}}R_l \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \leqslant C\varepsilon^2\int_0^t \left\|\partial^{\boldsymbol{\beta}}R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \left\|\partial^{\boldsymbol{\beta}''}R_k(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \mathrm{d}s$$
$$\leqslant C\varepsilon^2\int_0^t \tilde{z}(s)\,\mathrm{d}s$$

as $\left\|\partial^{\boldsymbol{\beta}'}(U_{\text{ext},i}\partial_t U_{\text{ext},j})\right\|_{L^\infty\left(\mathbb{R}^2\times J_{\rho_0,\varepsilon_0}\right)} \leqslant C\varepsilon^2$ by (4.2.11) and (4.2.13). If $|\boldsymbol{\beta}'| = 3$ and $\partial^{\boldsymbol{\beta}'}$ is only applied to $\partial_t U_{\text{ext},j}$, i.e. $\beta'' = 0$, we use (4.2.15) with $\partial^{\boldsymbol{\beta}'}\partial_t U_{\text{ext},j} = \mathcal{A}_j + \mathcal{B}_j$. Sobolev's embedding for $x_2 \mapsto R_k(x_1,x_2,s)$ implies that

$$
\begin{aligned}
I_4 &\leqslant \left|\int_0^t\int_{\mathbb{R}^2} U_{\mathrm{ext},i}\mathcal{A}_j\partial^{\beta''}R_k\partial^{\beta}R_l\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\right| + \left|\int_0^t\int_{\mathbb{R}^2} U_{\mathrm{ext},i}\mathcal{B}_j\partial^{\beta''}R_k\partial^{\beta}R_l\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\right| \\
&\leqslant C\varepsilon^2\int_0^t \tilde{z}(s)\,\mathrm{d}s + \int_0^t \left\|\partial^{\beta}R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)}\left[\int_{\mathbb{R}^2}|U_{\mathrm{ext},i}\mathcal{B}_jR_k|^2\,\mathrm{d}\boldsymbol{x}\right]^{\frac{1}{2}}\mathrm{d}s \\
&\leqslant C\varepsilon^2\int_0^t \tilde{z}(s)\,\mathrm{d}s \\
&\quad + C\varepsilon^2\int_0^t \left\|\partial^{\beta}R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)}\left[\int_{\mathbb{R}}\sup_{x_1\in\mathbb{R}}|\mathcal{B}_j(x_1,x_2,s)|^2\,\mathrm{d}x_2\int_{\mathbb{R}}\sup_{x_2\in\mathbb{R}}|R_k(x_1,x_2,s)|^2\,\mathrm{d}x_1\right]^{\frac{1}{2}}\mathrm{d}s \\
&\leqslant C\varepsilon^2\int_0^t \tilde{z}(s)\,\mathrm{d}s + C\varepsilon^2\int_0^t \left\|\partial^{\beta}R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)}\left[\int_{\mathbb{R}^2}\left(|R_k(x_1,x_2,s)|^2 + |\partial_{x_2}R_k(x_1,x_2,s)|^2\right)\mathrm{d}\boldsymbol{x}\right]^{\frac{1}{2}}\mathrm{d}s \\
&\leqslant C\varepsilon^2\int_0^t \tilde{z}(s)\,\mathrm{d}s.
\end{aligned}
\tag{6.2.18}
$$

In the same way we treat terms linear in $\boldsymbol{U}_{\mathrm{ext}}$. Let $\boldsymbol{\beta} = \boldsymbol{\beta}' + \boldsymbol{\beta}'' + \boldsymbol{\beta}'''$ and let us study

$$
I_5 := \varepsilon^a\int_0^t\int_{\mathbb{R}^2}\partial^{\beta'}\partial_t U_{\mathrm{ext},i}\partial^{\beta''}R_j\partial^{\beta'''}R_k\partial^{\beta}R_l\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s.
$$

If $|\boldsymbol{\beta}'| \leqslant 2$, either $\partial^{\beta''}R_j$ or $\partial^{\beta'''}R_k$ is in $\mathcal{H}^2(\mathbb{R}^2)$ and therefore in $L^\infty(\mathbb{R}^2)$. W.l.o.g for $|\boldsymbol{\beta}'''| \leqslant 1$, it follows

$$
\begin{aligned}
I_5 &\leqslant C\varepsilon^{1+a}\int_0^t \left\|\partial^{\beta}R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)}\left\|\partial^{\beta''}R_j(\cdot,s)\right\|_{L^2(\mathbb{R}^2)}\mathrm{d}s \\
&\leqslant C\varepsilon^{1+a}\int_0^t \tilde{z}(s)\,\mathrm{d}s,
\end{aligned}
$$

using $\partial^{\beta'''}R_k, \partial^{\beta'}\partial_t U_{\mathrm{ext},i} \in L^\infty\left(\mathbb{R}^2\times J_{\rho_0,\varepsilon_0}\right)$.
If $|\boldsymbol{\beta}'| = 3$, an estimate similar to (6.2.18) yields

$$
\begin{aligned}
I_5 &\leqslant \varepsilon^a\left|\int_0^t\int_{\mathbb{R}^2}\mathcal{A}_iR_jR_k\partial^{\beta}R_l\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\right| + \varepsilon^a\left|\int_0^t\int_{\mathbb{R}^2}\mathcal{B}_iR_jR_k\partial^{\beta}R_l\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\right| \\
&\leqslant C\varepsilon^{1+a}\int_0^t \tilde{z}(s)\,\mathrm{d}s + C\varepsilon^a\int_0^t \left\|\partial^{\beta}R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)}\left(\int_{\mathbb{R}^2}|\mathcal{B}_iR_jR_k|^2\,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}}\mathrm{d}s \\
&\leqslant C\varepsilon^{1+a}\int_0^t \tilde{z}(s)\,\mathrm{d}s + C\varepsilon^{1+a}\int_0^t \left\|\partial^{\beta}R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)}\left(\int_{\mathbb{R}}\sup_{x_2\in\mathbb{R}}|R_k(x_1,x_2,s)|^2\,\mathrm{d}x_1\right)^{\frac{1}{2}}\mathrm{d}s \\
&\leqslant C\varepsilon^{1+a}\int_0^t \tilde{z}(s)\,\mathrm{d}s.
\end{aligned}
\tag{6.2.19}
$$

Note that these are the only cases where four derivatives can fall on one function in this step.

Collecting the above partial estimates, we finally get with (6.2.11)

$$\widetilde{z}(t) \leqslant C\left(\rho_0^2 + \left(\varepsilon^2 + \varepsilon^{1+a}\right)\int_0^t \widetilde{z}(s)\,\mathrm{d}s + \varepsilon^{1+a}t + \varepsilon^{\frac{3}{2}-a} + \varepsilon^{7-2a}\right).$$

If $a \in \left(1, \frac{11}{2}\right)$, we have $1+a > 2$ and $7-2a > \frac{3}{2}-a$ and Gronwall's inequality yields

$$\widetilde{z}(t) \leqslant C\left(\rho_0^2 + \varepsilon^{\frac{3}{2}-a} + \varepsilon^{1+a}t\right)\mathrm{e}^{C\varepsilon^2 t} \leqslant C\left(\rho_0^2 + \varepsilon^{\frac{3}{2}-a} + \varepsilon^{a-1}\right)$$

for all $t \in J_{\rho_0,\varepsilon_0}$ if $\left\|\boldsymbol{R}^{(0)}\right\|_{\mathcal{H}^3(\mathbb{R}^2)^3} \leqslant \rho_0$.

Step III: Analysis of $\partial^{\beta} R_{2,3}$ for $|\beta| \leqslant 3, \beta_1 \neq 0$

We will now use the structure of Maxwell's equations to get expressions for $\partial^{\beta} R_2$, $\partial^{\beta} R_3$ where $\beta_1 \neq 0$. The estimates will then follow with the estimates of Step II and by iteration.

We first consider $\beta_1 = 1$. Setting $\boldsymbol{\alpha} := (0, \beta_2, \beta_t)^\top$, we have $\boldsymbol{\beta} = (1,0,0)^\top + \boldsymbol{\alpha}$. We now take the differential equation in (6.2.5) with $\boldsymbol{\alpha}$ instead of $\boldsymbol{\beta}$ and rearrange the terms into

$$\begin{cases} \partial^{\beta} R_2 = \partial_{x_1}\partial^{\alpha} R_2 = \partial_{x_2}\partial^{\alpha} R_1 - \left(S(\boldsymbol{R})\partial_t\partial^{\alpha}\boldsymbol{R} + \boldsymbol{s}_{\alpha}(\boldsymbol{R}) + \boldsymbol{w}_{\alpha}(\boldsymbol{R}) + \varepsilon^{-a}\partial^{\alpha}\mathbf{Res}\right)_3, \\ \partial^{\beta} R_3 = \partial_{x_1}\partial^{\alpha} R_3 = -\left(S(\boldsymbol{R})\partial_t\partial^{\alpha}\boldsymbol{R} + \boldsymbol{s}_{\alpha}(\boldsymbol{R}) + \boldsymbol{w}_{\alpha}(\boldsymbol{R}) + \varepsilon^{-a}\partial^{\alpha}\mathbf{Res}\right)_2. \end{cases} \tag{6.2.20}$$

Note that this is possible for our matrix A_1.
Each term on the right-hand side in (6.2.20) has derivatives ∂^{γ} with $|\gamma| \leqslant 3$ and $\gamma_1 = 0$ and can hence be bounded by Step II.
From Remark 6.2.3 we know that $S(\boldsymbol{R}) \in \mathcal{F}^{3,3}_{\eta,\mathrm{cv}}\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)$ and $W(\boldsymbol{R}) \in \mathcal{H}^3\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)^{3\times 3}$, now with the estimates in Lemma 5.1.4 it follows

$$\begin{aligned}
\|\partial_{x_2}\partial^{\alpha} R_1(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} &\leqslant \widetilde{z}(t), \\
\|(S(\boldsymbol{R})\partial_t\partial^{\alpha}\boldsymbol{R})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)^3} &\leqslant \|S(\boldsymbol{R})(\cdot,t)\|^2_{L^\infty(\mathbb{R}^2)^{3\times 3}}\|\partial_t\partial^{\alpha}\boldsymbol{R}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)^3} \leqslant C\widetilde{z}(t), \\
\|\boldsymbol{s}_{\alpha}(\boldsymbol{R})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)^3} &\leqslant C\sum_{0\neq\gamma\leqslant\alpha}\left\|\partial^{\gamma} S(\cdot,t,\boldsymbol{R})\partial^{\alpha-\gamma}\partial_t\boldsymbol{R}(\cdot,t)\right\|^2_{L^2(\mathbb{R}^2)^3} \leqslant C\widetilde{z}(t), \\
\left\|\boldsymbol{w}_{\beta}(\cdot,t,\boldsymbol{R})\right\|^2_{L^2(\mathbb{R}^2)^3} &\leqslant \left\|\partial^{\beta} W(\cdot,t,\boldsymbol{R})\boldsymbol{R}(\cdot,t)\right\|^2_{L^2(\mathbb{R}^2)^3} + \left\|W(\cdot,t,\boldsymbol{R})\partial^{\beta}\boldsymbol{R}(\cdot,t)\right\|^2_{L^2(\mathbb{R}^2)^3} \leqslant C\widetilde{z}(t), \\
\varepsilon^{-2a}\|\partial^{\alpha}\mathbf{Res}\|^2_{L^2(\mathbb{R}^2)^3} &\leqslant C\varepsilon^{7-2a}.
\end{aligned}$$

In summary, we get

$$\left\|\partial^{\beta} R_2\right\|^2_{L^2(\mathbb{R}^2)}, \left\|\partial^{\beta} R_3\right\|^2_{L^2(\mathbb{R}^2)} \leqslant C\left(\widetilde{z}(t) + \varepsilon^{7-2a}\right) \leqslant C\left(\rho_0^2 + \varepsilon^{7-2a} + \varepsilon^{a-1}\right)$$

for all $|\boldsymbol{\beta}| \leqslant 3$, $\beta_1 = 1$ and all $t \in J_{\rho_0,\varepsilon_0}$ if $a \in \left(1, \frac{11}{2}\right)$ and $\left\|\boldsymbol{R}^{(0)}\right\|_{\mathcal{H}^3(\mathbb{R}^2)^3} \leqslant \rho_0$.

For larger values of β_1 we iterate the process. For $\beta_1 = 2$ we have (6.2.20) with $\boldsymbol{\beta} = (1,0,0)^\top + \boldsymbol{\alpha}$ and $\boldsymbol{\alpha} := (1, \beta_2, \beta_t)^\top$ and using the previous step, all terms in the right-hand side can be estimated in $L^2(\mathbb{R}^2)$. For $\beta_1 = 3$ the same process applies, with $\boldsymbol{\beta} = (1,0,0)^\top + \boldsymbol{\alpha}$ and $\boldsymbol{\alpha} = (2,0,0)^\top$. Altogether, we arrive at

$$\sum_{\substack{|\boldsymbol{\beta}|\leqslant 3,\\ \beta_1=0}} \left\| \partial^{\boldsymbol{\beta}} \boldsymbol{R}(\cdot,t) \right\|^2_{L^2(\mathbb{R}^2)^3} + \sum_{|\boldsymbol{\beta}|\leqslant 3} \left(\left\| \partial^{\boldsymbol{\beta}} R_2(\cdot,t) \right\|^2_{L^2(\mathbb{R}^2)} + \left\| \partial^{\boldsymbol{\beta}} R_3(\cdot,t) \right\|^2_{L^2(\mathbb{R}^2)} \right) \leqslant C \left(\rho_0^2 + \varepsilon^{\frac{3}{2}-a} + \varepsilon^{a-1} \right) \tag{6.2.21}$$

for all $t \in J_{\rho_0,\varepsilon_0}$ if $a \in \left(1, \frac{11}{2}\right)$ and $\left\| \boldsymbol{R}^{(0)} \right\|_{\mathcal{H}^3(\mathbb{R}^2)^3} \leqslant \rho_0$.

Step IV: Analysis of $\partial^{\boldsymbol{\beta}} \boldsymbol{R_1}$, $|\boldsymbol{\beta}| \leqslant 3$

In this final step we exploit the divergence equation $\nabla \cdot \boldsymbol{\mathcal{D}}(\boldsymbol{U}_E) = \nabla \cdot \boldsymbol{\mathcal{D}}\left(\boldsymbol{U}_E^{(0)}\right) = \varrho_0$ in order to estimate $\partial^{\boldsymbol{\beta}} R_1$, remember that $\boldsymbol{U}_E = (U_1, U_2, 0)^\top$. We will again use an iteration to prove the estimates for all $\boldsymbol{\beta} \in \mathbb{N}_0^3$ with $|\boldsymbol{\beta}| \leqslant 3$.

First, for $\boldsymbol{\alpha} \in \mathbb{N}_0^3$, $|\boldsymbol{\alpha}| \leqslant 2$ and $\boldsymbol{r}_{\boldsymbol{\alpha}} = \partial^{\boldsymbol{\alpha}} \boldsymbol{R}$ we have

$$\begin{aligned}
\varepsilon^{-a} \partial^{\boldsymbol{\alpha}} \partial_t \widetilde{\boldsymbol{\mathcal{D}}}(\varepsilon^a \boldsymbol{R}_E + \boldsymbol{U}_{\mathrm{ext},E}) &= \partial_t \left(\left(\epsilon_1 + \varepsilon^{2a} \epsilon_3 |\widetilde{\boldsymbol{R}}|^2 \right) \widetilde{\boldsymbol{r}}_{\boldsymbol{\alpha}} \right) + \partial^{\boldsymbol{\alpha}} \left(\widetilde{\varphi}(\boldsymbol{R}_E) \partial_t \widetilde{\boldsymbol{R}} \right) + \partial^{\boldsymbol{\alpha}} \left(\widetilde{W}(\boldsymbol{R}_E) \widetilde{\boldsymbol{R}} \right) \\
&\quad + \partial_t \left(\sum_{\boldsymbol{0} \neq \boldsymbol{\gamma} \leqslant \boldsymbol{\alpha}} \partial^{\boldsymbol{\gamma}} \left(\epsilon_1 + \varepsilon^{2a} \epsilon_3 |\widetilde{\boldsymbol{R}}|^2 \right) \partial^{\boldsymbol{\alpha}-\boldsymbol{\gamma}} \widetilde{\boldsymbol{R}} \right) + \varepsilon^{-a} \partial^{\boldsymbol{\alpha}} \widetilde{\mathbf{Res}} \\
&\quad + \varepsilon^{-a} \partial^{\boldsymbol{\alpha}} \begin{pmatrix} \partial_{x_2} U_{\mathrm{ext},3} \\ -\partial_{x_1} U_{\mathrm{ext},3} \end{pmatrix}
\end{aligned} \tag{6.2.22}$$

on $\mathbb{R}^2_\pm \times J_{\rho_0,\varepsilon_0}$, where $\widetilde{\cdot}$ of a (3×3)-matrix denotes the restriction to the upper left (2×2)-submatrix and $\widetilde{\cdot}$ of a vector in $\mathbb{R}^3$ denotes the first two components of this vector. The calculation to obtain (6.2.22) uses that $\varphi(\boldsymbol{R}_E)$ and $W(\boldsymbol{R}_E)$ have a block structure and that by definition of $\mathbf{Res}$ it follows that

$$\varepsilon^{-a} \partial_t \widetilde{\boldsymbol{\mathcal{D}}}(\boldsymbol{U}_{\mathrm{ext},E}) = \varepsilon^{-a} \partial^{\boldsymbol{\alpha}} \widetilde{\mathbf{Res}} + \varepsilon^{-a} \partial^{\boldsymbol{\alpha}} \begin{pmatrix} \partial_{x_2} U_{\mathrm{ext},3} \\ -\partial_{x_1} U_{\mathrm{ext},3} \end{pmatrix}.$$

Note that by definition $\varphi(\boldsymbol{R}_E) = \varphi(\boldsymbol{R})$ and $W(\boldsymbol{R}_E) = W(\boldsymbol{R})$.
An integration by parts yields

$$\begin{aligned}
\int_0^t \partial^{\boldsymbol{\alpha}} \left(\widetilde{\varphi}(\boldsymbol{R}_E) \partial_t \widetilde{\boldsymbol{R}} \right) \,\mathrm{d}s &= \int_0^t \left(\widetilde{\varphi}(\boldsymbol{R}_E) \partial^{\boldsymbol{\alpha}} \partial_t \widetilde{\boldsymbol{R}} + \sum_{\boldsymbol{0} \neq \boldsymbol{\gamma} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\gamma}} \partial^{\boldsymbol{\gamma}} \widetilde{\varphi}(\boldsymbol{R}_E) \partial^{\boldsymbol{\alpha}-\boldsymbol{\gamma}} \partial_t \widetilde{\boldsymbol{R}} \right) \mathrm{d}s \\
&= \int_0^t \left(-\partial_t \widetilde{\varphi}(\boldsymbol{R}_E) \partial^{\boldsymbol{\alpha}} \widetilde{\boldsymbol{R}} + \sum_{\boldsymbol{0} \neq \boldsymbol{\gamma} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\gamma}} \partial^{\boldsymbol{\gamma}} \widetilde{\varphi}(\boldsymbol{R}_E) \partial^{\boldsymbol{\alpha}-\boldsymbol{\gamma}} \partial_t \widetilde{\boldsymbol{R}} \right) \mathrm{d}s + \left[\widetilde{\varphi}(\boldsymbol{R}_E) \partial^{\boldsymbol{\alpha}} \widetilde{\boldsymbol{R}} \right]_0^t.
\end{aligned}$$

By integrating (6.2.22) in time, we then deduce

$$
\begin{aligned}
&\left[\varepsilon^{-a}\partial^{\boldsymbol{\alpha}}\widetilde{\boldsymbol{\mathcal{D}}}(\varepsilon^a \boldsymbol{R}_E + \boldsymbol{U}_{\mathrm{ext},E})\right]_0^t \\
&\quad = \left[\left(\epsilon_1 + \varepsilon^{2a}\epsilon_3|\widetilde{\boldsymbol{R}}|^2\right)\widetilde{\boldsymbol{r}}_{\boldsymbol{\alpha}} + \widetilde{\varphi}(\boldsymbol{R}_E)\widetilde{\boldsymbol{r}}_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{0}\neq\boldsymbol{\gamma}\leqslant\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\gamma}}\partial^{\boldsymbol{\gamma}}\left(\epsilon_1 + \varepsilon^{2a}\epsilon_3|\widetilde{\boldsymbol{R}}|^2\right)\partial^{\boldsymbol{\alpha}-\boldsymbol{\gamma}}\widetilde{\boldsymbol{R}}\right]_0^t \\
&\qquad + \int_0^t \left(-\partial_t\widetilde{\varphi}(\boldsymbol{R}_E)\widetilde{\boldsymbol{r}}_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{0}\neq\boldsymbol{\gamma}\leqslant\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\gamma}}\partial^{\boldsymbol{\gamma}}\widetilde{\varphi}(\boldsymbol{R}_E)\partial^{\boldsymbol{\alpha}-\boldsymbol{\gamma}}\partial_t\widetilde{\boldsymbol{R}} + \partial^{\boldsymbol{\alpha}}\left(\widetilde{W}(\boldsymbol{R}_E)\widetilde{\boldsymbol{R}}\right)\right)\mathrm{d}s \\
&\qquad + \varepsilon^{-a}\int_0^t\left(\partial^{\boldsymbol{\alpha}}\widetilde{\mathbf{Res}} + \partial^{\boldsymbol{\alpha}}\begin{pmatrix}\partial_{x_2}U_{\mathrm{ext},3}\\ -\partial_{x_1}U_{\mathrm{ext},3}\end{pmatrix}\right)\mathrm{d}s.
\end{aligned}
\tag{6.2.23}
$$

Note that the divergence of the last term vanishes.

Substep 1: $\boldsymbol{\beta_1 = 1}$.
We write $\boldsymbol{\beta} = (1,0,0)^\top + \boldsymbol{\alpha}$, where $\boldsymbol{\alpha} = (0,\beta_2,\beta_t)^\top$. We have that $\nabla\cdot\partial^{\boldsymbol{\alpha}}\widetilde{\boldsymbol{\mathcal{D}}}(\boldsymbol{U}_E)$ is constant in time because

$$
\nabla\cdot\partial^{\boldsymbol{\alpha}}\widetilde{\boldsymbol{\mathcal{D}}}(\boldsymbol{U}_E) = \partial^{\boldsymbol{\alpha}}(\nabla\cdot\widetilde{\boldsymbol{\mathcal{D}}}(\boldsymbol{U}_E)) = \partial^{\boldsymbol{\alpha}}\tilde{\varrho}_0, \quad \tilde{\varrho}_0 := \nabla\cdot\widetilde{\boldsymbol{\mathcal{D}}}\left(\boldsymbol{U}_E^{(0)}\right).
$$

Note that $\tilde{\varrho}\in\mathcal{H}^2(\mathbb{R}^2)$ because of the algebra property of $\mathcal{H}^2(\mathbb{R}^2)$ and $\boldsymbol{U}^{(0)}\in\mathcal{H}^3(\mathbb{R}^2)^3$. Hence, taking the divergence of (6.2.23), the first term vanishes and we have

$$
\begin{aligned}
&\left[\left(\epsilon_1 + \varepsilon^{2a}\epsilon_3|\widetilde{\boldsymbol{R}}|^2\right)(\partial_{x_1}r_{\boldsymbol{\alpha},1} + \partial_{x_2}r_{\boldsymbol{\alpha},2}) + \nabla\left(\epsilon_1 + \varepsilon^{2a}\epsilon_3|\widetilde{\boldsymbol{R}}|^2\right)\cdot\widetilde{\boldsymbol{r}}_{\boldsymbol{\alpha}}\right]_0^t \\
&\quad = -\left[\nabla\cdot\left(\widetilde{\varphi}(\boldsymbol{R}_E)\partial^{\boldsymbol{\alpha}}\widetilde{\boldsymbol{R}} + \sum_{\boldsymbol{0}\neq\boldsymbol{\gamma}\leqslant\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\gamma}}\partial^{\boldsymbol{\gamma}}(\epsilon_1 + \varepsilon^{2a}\epsilon_3|\widetilde{\boldsymbol{R}}|^2)\partial^{\boldsymbol{\alpha}-\boldsymbol{\gamma}}\widetilde{\boldsymbol{R}}\right)\right]_0^t \\
&\qquad - \int_0^t \nabla\cdot\left(-\partial_t\widetilde{\varphi}(\boldsymbol{R}_E)\partial^{\boldsymbol{\alpha}}\widetilde{\boldsymbol{R}} + \partial^{\boldsymbol{\alpha}}\left(\widetilde{W}(\boldsymbol{R}_E)\widetilde{\boldsymbol{R}}\right) + \varepsilon^{-a}\partial^{\boldsymbol{\alpha}}\widetilde{\mathbf{Res}} + \sum_{\boldsymbol{0}\neq\boldsymbol{\gamma}\leqslant\boldsymbol{\alpha}}\binom{\boldsymbol{\alpha}}{\boldsymbol{\gamma}}\partial^{\boldsymbol{\gamma}}\widetilde{\varphi}(\boldsymbol{R}_E)\partial^{\boldsymbol{\alpha}-\boldsymbol{\gamma}}\partial_t\widetilde{\boldsymbol{R}}\right)\mathrm{d}s.
\end{aligned}
$$

Because of $\widetilde{\boldsymbol{R}}\in C\left(J_{\rho_0,\varepsilon_0}, L^\infty(\mathbb{R}^2)\right)^2$, there exists a number $\vartheta > 0$ with $(\epsilon_1 + \varepsilon^{2a}\epsilon_3|\widetilde{\boldsymbol{R}}|^2)(\boldsymbol{x},t)\geqslant\vartheta$ for small enough ε, all $t\in J_{\rho_0,\varepsilon_0}$ and almost all $\boldsymbol{x}\in\mathbb{R}^2$.
Since $\partial_{x_1}\widetilde{\boldsymbol{R}}, \partial_{x_2}\widetilde{\boldsymbol{R}}\in C\left(J_{\rho_0,\varepsilon_0}, L^\infty(\mathbb{R}^2)\right)^2$, we can also estimate

$$
\left\|\nabla\left(\epsilon_1 + \varepsilon^{2a}\epsilon_3|\widetilde{\boldsymbol{R}}|^2\right)(\cdot,t)\right\|_{L^\infty(\mathbb{R}^2)^3}\leqslant C
$$

for all $t\in J_{\rho_0,\varepsilon_0}$.

These facts yield the central inequality of this step:

$$
\begin{aligned}
&\vartheta \left\| \partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,t) \right\|_{L^2(\mathbb{R}^2)} \\
&\quad \leqslant C \left(\left\| \partial_{x_2} r_{\boldsymbol{\alpha},2}(\cdot,t) \right\|_{L^2(\mathbb{R}^2)} + \left\| \nabla \cdot \widetilde{\boldsymbol{r}}_{\boldsymbol{\alpha}}(\cdot,0) \right\|_{L^2(\mathbb{R}^2)} + \left\| \widetilde{\boldsymbol{r}}_{\boldsymbol{\alpha}}(\cdot,t) \right\|_{L^2(\mathbb{R}^2)^2} + \left\| \widetilde{\boldsymbol{r}}_{\boldsymbol{\alpha}}(\cdot,0) \right\|_{L^2(\mathbb{R}^2)^2} \right) \\
&\quad + \left\| \left[\nabla \cdot \left(\widetilde{\varphi}(\boldsymbol{R}_E) \partial^{\boldsymbol{\alpha}} \widetilde{\boldsymbol{R}} + \sum_{\boldsymbol{0} \neq \boldsymbol{\gamma} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\gamma}} \partial^{\boldsymbol{\gamma}} (\epsilon_1 + \varepsilon^{2a} \epsilon_3 |\widetilde{\boldsymbol{R}}|^2) \partial^{\boldsymbol{\alpha}-\boldsymbol{\gamma}} \widetilde{\boldsymbol{R}} \right)(\cdot,s) \right]_0^t \right\|_{L^2(\mathbb{R}^2)} \\
&\quad + \left\| \int_0^t \nabla \cdot \left(-\partial_t \widetilde{\varphi}(\boldsymbol{R}_E) \partial^{\boldsymbol{\alpha}} \widetilde{\boldsymbol{R}} + \partial^{\boldsymbol{\alpha}} \left(\widetilde{W}(\boldsymbol{R}_E) \widetilde{\boldsymbol{R}} \right) + \sum_{\boldsymbol{0} \neq \boldsymbol{\gamma} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\gamma}} \partial^{\boldsymbol{\gamma}} \widetilde{\varphi}(\boldsymbol{R}_E) \partial^{\boldsymbol{\alpha}-\boldsymbol{\gamma}} \partial_t \widetilde{\boldsymbol{R}} \right)(\cdot,s) \,\mathrm{d}s \right\|_{L^2(\mathbb{R}^2)} \\
&\quad + \varepsilon^{-a} \left\| \int_0^t \nabla \cdot \partial^{\boldsymbol{\alpha}} \widetilde{\mathbf{Res}}(\cdot,s) \,\mathrm{d}s \right\|_{L^2(\mathbb{R}^2)} .
\end{aligned}
\tag{6.2.24}
$$

We next iterate over β_t and β_2.
i) We start with $\boldsymbol{\alpha} = (0,0,0)^\top$.
Here (6.2.24) simplifies to

$$
\begin{aligned}
&\vartheta \left\| \partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,t) \right\|_{L^2(\mathbb{R}^2)} \leqslant \varepsilon^{-a} \left\| \int_0^t \nabla \cdot \widetilde{\mathbf{Res}}(\cdot,s) \,\mathrm{d}s \right\|_{L^2(\mathbb{R}^2)} \\
&\quad + C \left(\left\| \partial_{x_2} r_{\boldsymbol{\alpha},2}(\cdot,t) \right\|_{L^2(\mathbb{R}^2)} + \left\| \nabla \cdot \widetilde{\boldsymbol{r}}_{\boldsymbol{\alpha}}(\cdot,0) \right\|_{L^2(\mathbb{R}^2)} + \left\| \widetilde{\boldsymbol{r}}_{\boldsymbol{\alpha}}(\cdot,t) \right\|_{L^2(\mathbb{R}^2)^2} + \left\| \widetilde{\boldsymbol{r}}_{\boldsymbol{\alpha}}(\cdot,0) \right\|_{L^2(\mathbb{R}^2)^2} \right) \\
&\quad + \left\| \left[\nabla \cdot \left(\widetilde{\varphi}(\boldsymbol{R}_E) \widetilde{\boldsymbol{R}} \right)(\cdot,s) \right]_0^t \right\|_{L^2(\mathbb{R}^2)} + \left\| \int_0^t \nabla \cdot \left(-\partial_t \widetilde{\varphi}(\boldsymbol{R}_E) \widetilde{\boldsymbol{R}} + \widetilde{W}(\boldsymbol{R}_E) \widetilde{\boldsymbol{R}} \right)(\cdot,s) \,\mathrm{d}s \right\|_{L^2(\mathbb{R}^2)} .
\end{aligned}
\tag{6.2.25}
$$

The residual term on the right-hand side is bounded by $C\varepsilon^{\frac{3}{2}-a}$ due to (4.2.6).
The second and fourth term on the right-hand side of (6.2.25) are estimated by (6.2.21) and the third and fifth term by (6.2.10).
In the first norm on the last line of (6.2.25) all terms have been treated in Steps I, II or III except for those of the type

$$
\varepsilon^a \epsilon_3 \partial_{x_1} r_{\boldsymbol{\alpha},1} R_j U_{\mathrm{ext},k} \quad \text{and} \quad \epsilon_3 \partial_{x_1} r_{\boldsymbol{\alpha},1} U_{\mathrm{ext},j} U_{\mathrm{ext},k}.
$$

Using $\boldsymbol{R} \in L^\infty \left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0} \right)^3$ and $\|\boldsymbol{U}_{\mathrm{ext}}\|_{L^\infty \left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0} \right)^3} \leqslant C\varepsilon$, we have

$$
\begin{aligned}
\left\| \varepsilon^a \epsilon_3 \left(\partial_{x_1} r_{\boldsymbol{\alpha},1} R_j U_{\mathrm{ext},k} \right)(\cdot,t) \right\|_{L^2(\mathbb{R}^2)} &\leqslant C\varepsilon^{1+a} \left\| \partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,t) \right\|_{L^2(\mathbb{R}^2)}, \\
\left\| \epsilon_3 \left(\partial_{x_1} r_{\boldsymbol{\alpha},1} U_{\mathrm{ext},j} U_{\mathrm{ext},k} \right)(\cdot,t) \right\|_{L^2(\mathbb{R}^2)} &\leqslant C\varepsilon^2 \left\| \partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,t) \right\|_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

In the last norm of the right-hand side of (6.2.25), the terms which have not been estimated so far are of the type

$$
\partial_{x_1} r_{\boldsymbol{\alpha},1} \partial_t U_{\mathrm{ext},j} U_{\mathrm{ext},k}, \qquad \varepsilon^a \partial_t (U_{\mathrm{ext},j} R_k) \partial_{x_1} r_{\boldsymbol{\alpha},1} \quad \text{and} \quad \varepsilon^a U_{\mathrm{ext},j} R_k \partial_t \partial_{x_1} r_{\boldsymbol{\alpha},1}
$$

for $j,k \in \{1,2,3\}$. Using $\boldsymbol{R}, \partial_t \boldsymbol{R} \in L^\infty\left(\mathbb{R}^2 \times J_{\rho_0,\varepsilon_0}\right)^3$ and $\|\partial_t \boldsymbol{U}_{\mathrm{ext}}\|_{L^\infty\left(\mathbb{R}^2\times J_{\rho_0,\varepsilon_0}\right)^3} \leqslant C\varepsilon$, we obtain

$$\int_0^t \left\|\epsilon_3\left(\partial_{x_1} r_{\boldsymbol{\alpha},1}\partial_t U_{\mathrm{ext},j} U_{\mathrm{ext},k}\right)(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \mathrm{d}s \leqslant C\varepsilon^2 \int_0^t \|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,s)\|_{L^2(\mathbb{R}^2)} \mathrm{d}s,$$
$$\int_0^t \left\|\varepsilon^a\epsilon_3\left(\partial_{x_1} r_{\boldsymbol{\alpha},1}\partial_t (R_k U_{\mathrm{ext},j})\right)(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \mathrm{d}s \leqslant C\varepsilon^{1+a} \int_0^t \|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,s)\|_{L^2(\mathbb{R}^2)} \mathrm{d}s,$$

and, integrating by parts in time,

$$\begin{aligned}
&\left\|\int_0^t \varepsilon^a \epsilon_3\left(U_{\mathrm{ext},j} R_k \partial_t \partial_{x_1} r_{\boldsymbol{\alpha},1}\right)(\cdot,s)\,\mathrm{d}s\right\|_{L^2(\mathbb{R}^2)} \\
&\leqslant \left\|\left[\varepsilon^a\left(U_{\mathrm{ext},j} R_k \partial_{x_1} r_{\boldsymbol{\alpha},1}\right)(\cdot,s)\right]_0^t - \int_0^t \varepsilon^a\left(\partial_t(U_{\mathrm{ext},j}R_k)\partial_{x_1} r_{\boldsymbol{\alpha},1}\right)(\cdot,s)\,\mathrm{d}s\right\|_{L^2(\mathbb{R}^2)} \\
&\leqslant C\varepsilon^{1+a}\left(\|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,t)\|_{L^2(\mathbb{R}^2)} + \|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,0)\|_{L^2(\mathbb{R}^2)} + \int_0^t \|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,s)\|_{L^2(\mathbb{R}^2)} \mathrm{d}s\right) \\
&\leqslant C\varepsilon^{1+a}\left(\rho_0 + \|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,t)\|_{L^2(\mathbb{R}^2)} + \int_0^t \|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,s)\|_{L^2(\mathbb{R}^2)} \mathrm{d}s\right).
\end{aligned}$$

We can therefore estimate the two last terms on the right-hand side of (6.2.25) by

$$\begin{aligned}
\left\|\left[\nabla\cdot\left(\widetilde{\varphi}(\boldsymbol{R})\widetilde{\boldsymbol{R}}\right)(\cdot,s)\right]_0^t\right\|_{L^2(\mathbb{R}^2)} &\leqslant C\left(\rho_0 + \varepsilon^{\frac{1}{2}\left(\frac{3}{2}-a\right)} + \varepsilon^{\frac{1}{2}(a-1)}\right) \\
&\quad + C(\varepsilon^2+\varepsilon^{1+a})\|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,t)\|_{L^2(\mathbb{R}^2)}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\int_0^t \nabla\cdot\left(-\partial_t\widetilde{\varphi}(\boldsymbol{R})\widetilde{\boldsymbol{R}} + \widetilde{W}(\boldsymbol{R})\widetilde{\boldsymbol{R}}\right)(\cdot,s)\,\mathrm{d}s\right\|_{L^2(\mathbb{R}^2)} \\
&\quad\leqslant C\left(\rho_0 + \varepsilon^{\frac{1}{2}\left(\frac{3}{2}-a\right)} + \varepsilon^{\frac{1}{2}(a-1)}\right) + C\left(\varepsilon^2+\varepsilon^{1+a}\right)\int_0^t\left(\rho_0 + \varepsilon^{\frac{1}{2}\left(\frac{3}{2}-a\right)} + \varepsilon^{\frac{1}{2}(a-1)}\right)\mathrm{d}s \\
&\quad + C(\varepsilon^2+\varepsilon^{1+a})\|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,t)\|_{L^2(\mathbb{R}^2)} + C\left(\varepsilon^2+\varepsilon^{1+a}\right)\int_0^t \|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,s)\|_{L^2(\mathbb{R}^2)} \mathrm{d}s.
\end{aligned}$$

Combining the above inequalities, for $a \in \left(1, \frac{11}{2}\right)$ and $0 \leqslant t \leqslant T_{\rho_0,\varepsilon_0} \leqslant T_0\varepsilon^{-2}$ we infer

$$\begin{aligned}
\vartheta\|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leqslant\, & C\left(\rho_0 + \varepsilon^{\frac{1}{2}\left(\frac{3}{2}-a\right)} + \varepsilon^{\frac{1}{2}(a-1)} + \varepsilon^2\|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,t)\|_{L^2(\mathbb{R}^2)} + \varepsilon^{\frac{3}{2}-a}\right) \\
& + C\varepsilon^2\int_0^t\left(\rho_0 + \varepsilon^{\frac{1}{2}\left(\frac{3}{2}-a\right)} + \varepsilon^{\frac{1}{2}(a-1)}\right)\mathrm{d}s + C\varepsilon^2\int_0^t \|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,s)\|_{L^2(\mathbb{R}^2)} \mathrm{d}s.
\end{aligned}$$

For ε small enough and $a \in \left[\frac{5}{4}, \frac{11}{2}\right)$ (so that $\frac{3}{2} - a \leqslant a - 1$) it follows

$$\|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leqslant C\left(\rho_0 + \varepsilon^{\frac{1}{2}\left(\frac{3}{2}-a\right)} + \varepsilon^2\int_0^t \|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,s)\|_{L^2(\mathbb{R}^2)} \mathrm{d}s\right).$$

Finally, Gronwall's inequality yields

$$\|\partial_{x_1} r_{\boldsymbol{\alpha},1}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leqslant C\left(\rho_0 + \varepsilon^{\frac{1}{2}(\frac{3}{2}-a)}\right) \mathrm{e}^{C\varepsilon^2 t} \leqslant C\left(\rho_0 + \varepsilon^{\frac{1}{2}(\frac{3}{2}-a)}\right).$$

ii) We iterate the process from i) for higher $\alpha_2 = \beta_2$ and $\alpha_t = \beta_t$ (keeping $\beta_1 = 1$).
For instance, the following sequence of $\boldsymbol{\alpha}$'s can be chosen:

$$\boldsymbol{\alpha} = (0,1,0)^\top,\ (0,0,1)^\top,\ (0,2,0)^\top,\ (0,0,2)^\top,\ (0,1,1)^\top.$$

Note that $|\boldsymbol{\alpha}| = \beta_t + \beta_2 \leqslant 2$, therefore we can always use integration by parts and Lemma 5.1.4. In the terms with $\widetilde{W}$ again three derivatives can fall on $\partial_t U_{\text{ext},k}$. If ∂_{x_1} is included, then one can proceed as above by means of (4.2.11) and (4.2.13). Otherwise, one uses (4.2.15) and argues as in (6.2.18).

Substep 2: $\beta_1 > 1$.
In this last step we have to iterate over β_1 and increase it to 3.
For $\beta_1 = 2$ we set $\boldsymbol{\beta} = (1,0,0)^\top + \boldsymbol{\alpha}$ with $\boldsymbol{\alpha} = (1,\beta_2,\beta_t)^\top$. The estimates work like in Substep 1 i) since $|\boldsymbol{\alpha}| \leqslant 2$. Finally, for $\beta_1 = 3$ we have $\boldsymbol{\beta} = (3,0,0)^\top = (1,0,0)^\top + \boldsymbol{\alpha}$ with $\boldsymbol{\alpha} = (2,0,0)^\top$ and apply Substep 1 i) again. Here, factors $\partial_{x_1}^3 U_{\text{ext},j}$, $\partial_{x_1}^3 \partial_t U_{\text{ext},j}$ occur in the terms with $\widetilde{W}$, which are treated with (4.2.12), (4.2.14), similarly to the estimate of the $\boldsymbol{\mathcal{B}}$-terms in (6.2.18) and (6.2.19), e.g. with the Sobolev embedding for $x_1 \mapsto R_k(x_1,x_2,s)$ we get

$$\begin{aligned}
&\int_0^t\!\!\int_{\mathbb{R}^2} U_{\text{ext},i}\partial_{x_1}^3\partial_t U_{\text{ext},j} R_k \partial^{\boldsymbol{\beta}} R_l \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s \\
&\leqslant \int_0^t \left\|\partial^{\boldsymbol{\beta}} R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \left[\int_{\mathbb{R}^2} |U_{\text{ext},i}\partial_{x_1}^3\partial_t U_{\text{ext},j} R_k|^2 \,\mathrm{d}\boldsymbol{x}\right]^{\frac{1}{2}} \mathrm{d}s \\
&\leqslant C\varepsilon^2 \int_0^t \left\|\partial^{\boldsymbol{\beta}} R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \left[\int_{\mathbb{R}} \sup_{x_1\in\mathbb{R}} |R_k(x_1,x_2,s)|^2 \,\mathrm{d}x_2\right]^{\frac{1}{2}} \mathrm{d}s \\
&\leqslant C\varepsilon^2 \int_0^t \left\|\partial^{\boldsymbol{\beta}} R_l(\cdot,s)\right\|_{L^2(\mathbb{R}^2)} \left[\int_{\mathbb{R}^2} \left(|R_k(x_1,x_2,s)|^2 + |\partial_{x_2} R_k(x_1,x_2,s)|^2\right) \mathrm{d}\boldsymbol{x}\right]^{\frac{1}{2}} \mathrm{d}s \\
&\leqslant C\varepsilon^2 \int_0^t \widetilde{z}(s)\,\mathrm{d}s.
\end{aligned}$$

In summary, collecting all the above estimates, one concludes

$$z(t) \leqslant C\left(\rho_0^2 + \varepsilon^{\frac{3}{2}-a}\right)$$

for every $t \in J_{\rho_0,\varepsilon_0}$ and $\varepsilon \in (0,\varepsilon_0)$ if $a \in \left[\frac{5}{4}, \frac{11}{2}\right)$ and ε_0 is small enough.

Next, we keep ρ fixed, choose $a \in \left[\frac{5}{4}, \frac{3}{2}\right)$ and ρ_0, ε_0 so small that

$$C\left(\rho_0^2 + \varepsilon_0^{\frac{3}{2}-a}\right) < \tfrac{1}{2}\rho^2$$

and

$$\varepsilon_0^a \rho + \|\boldsymbol{U}_{\mathrm{ext}}\|_{L^\infty(\mathbb{R}^2\times[0,T_0\varepsilon^{-2}))^3} \leqslant \varpi,$$

where we recall (6.2.1) and that $\|\boldsymbol{U}_{\mathrm{ext}}\|_{L^\infty(\mathbb{R}^2\times[0,T_0\varepsilon^{-2}))^3} \leqslant C\varepsilon \leqslant C\varepsilon_0$. With this choice we have

$$z(t) < \frac{1}{2}\rho^2$$

for every $t \in J_{\rho_0,\varepsilon_0}$ and $\varepsilon \in (0, \varepsilon_0)$ if $a \in \left[\frac{5}{4}, \frac{3}{2}\right)$. Definition (6.2.3) of T_{ρ_0,ε_0} now implies that $T_{\rho_0,\varepsilon_0} = T_0\varepsilon^{-2} < t_M$ and that (6.2.2) holds with $t_* = T_0\varepsilon^{-2}$.
Estimate (6.2.4), $\varepsilon < 1$ and the monotonicity of the exponential function therefore imply that

$$\|\boldsymbol{U} - \boldsymbol{U}_{\mathrm{ext}}\|_{\mathcal{G}^3(\mathbb{R}^2\times[0,T_0\varepsilon^{-2}])^3} \leqslant \rho\varepsilon^a \tag{6.2.26}$$

for all $a < \frac{3}{2}$.

Remark 6.2.8
A careful examination of the bootstrapping argument shows that the limitation $a < \frac{3}{2}$ comes from the estimate of the residual $\mathbf{Res}(\boldsymbol{U}_{\mathrm{ext}})$. *For another ansatz with a smaller residual an improved version of estimate* (6.2.26) *can be achieved.*

6.3. Approximation Result

In this section we will finalize our approximation theorem.

First, note that we could formulate the approximation result with (6.2.26), but $\boldsymbol{U}_{\mathrm{ext}}$ contains many corrections terms that are not easily determined explicitly or numerically. Instead, we want to compare $\boldsymbol{U}$ with $\boldsymbol{U}_{\mathrm{ans}}$, which is much simpler to calculate, see (4.2.1) and (4.1.1).
To this end, we use that $\boldsymbol{U}_{\mathrm{ext}}$ only contains higher order correction terms and that the regularity of A, $\partial_k \boldsymbol{w}(k_0)$, $\partial_k^2 \boldsymbol{w}(k_0)$, $\boldsymbol{p}$, $\boldsymbol{h}$ is given by (4.2.7) and (4.2.8) and estimate

$$\begin{aligned}
&\|\boldsymbol{U}_{\mathrm{ext}}(\cdot,t) - \boldsymbol{U}_{\mathrm{ans}}(\cdot,t)\|_{\mathcal{H}^3(\mathbb{R}^2)^3} \\
&\quad = \left\| \left(\left(-\varepsilon^2 \mathrm{i}\partial_{X_2} A \partial_k \boldsymbol{w}(k_0) - \varepsilon^3 \frac{1}{2}\partial_{X_2}^2 A \partial_k^2 \boldsymbol{w}(k_0) + \varepsilon^3 |A|^2 A \boldsymbol{p} \right) F_1 + \varepsilon^3 A^3 \boldsymbol{h} F_1^3 \right)(\cdot,t) + \text{c.c.} \right\|_{\mathcal{H}^3(\mathbb{R}^2)^3} \\
&\quad \leqslant C\varepsilon^{3/2} \left\| \partial_{X_2} A(\cdot,T) \right\|_{H^3(\mathbb{R})} \left\| \partial_k \boldsymbol{w}(k_0) \right\|_{\mathcal{H}^3(\mathbb{R})^3} \\
&\qquad + C\varepsilon^{5/2} \left(\left\| \partial_{X_2}^2 A(\cdot,T) \right\|_{H^3(\mathbb{R})} \left\| \partial_k^2 \boldsymbol{w}(k_0) \right\|_{\mathcal{H}^3(\mathbb{R})^3} + \|A(\cdot,T)\|_{H^3(\mathbb{R})}^3 \left(\|\boldsymbol{p}\|_{\mathcal{H}^3(\mathbb{R})^3} + \|\boldsymbol{h}\|_{\mathcal{H}^3(\mathbb{R})^3} \right) \right) \\
&\quad \leqslant C\varepsilon^{3/2}
\end{aligned}$$

for all $t \in [0, T_0\varepsilon^{-2}]$. Note that we again lose half an order of ε since A depends on $X_2 = \varepsilon(x_2 - \nu_1 t)$.
Now we use (6.2.26) and the triangle inequality to conclude

$$\begin{aligned}\|\boldsymbol{U} - \boldsymbol{U}_{\text{ans}}\|_{\mathcal{G}^3(\mathbb{R}^2\times J_{\rho_0,\varepsilon_0})^3} &\leqslant \|\boldsymbol{U} - \boldsymbol{U}_{\text{ext}}\|_{\mathcal{G}^3(\mathbb{R}^2\times J_{\rho_0,\varepsilon_0})^3} + \|\boldsymbol{U}_{\text{ext}} - \boldsymbol{U}_{\text{ans}}\|_{\mathcal{G}^3(\mathbb{R}^2\times J_{\rho_0,\varepsilon_0})^3} \\ &\leqslant \rho\varepsilon^a + C\varepsilon^{3/2} \\ &\leqslant C\varepsilon^a\end{aligned} \tag{6.3.1}$$

for all $a < \frac{3}{2}$. Note that ρ, $\boldsymbol{U}_{\text{ans}}$ and $\boldsymbol{U}_{\text{ext}}$ are independent of a and that therefore the constant C in (6.3.1) is also independent of a.
With this we can formulate the main result of this thesis.

Theorem 6.3.1 (Approximation Theorem)
Assume (A1) – (A7) *and let* $A \in \bigcap_{k=0}^{4} C^{4-k}\left([0, T_0], H^{3+k}(\mathbb{R})\right)$ *be a solution of the effective nonlinear Schrödinger equation* (4.1.13) *for some* $T_0 > 0$*. Assume that the initial value* $\boldsymbol{U}^{(0)} := \boldsymbol{U}(\cdot, 0) \in \mathcal{H}^3(\mathbb{R}^2)^3$ *satisfies the nonlinear compatibility conditions of order* 3*, see Definition 5.3.1.*
Then there exist constants $\varepsilon_0 > 0$ *and* $C > 0$ *such that if* $\varepsilon \in (0, \varepsilon_0)$ *and if* $\boldsymbol{U}^{(0)}$ *fulfills*

$$\left\|\boldsymbol{U}^{(0)} - \boldsymbol{U}_{\text{ans}}(\cdot, 0)\right\|_{\mathcal{H}^3(\mathbb{R}^2)^3} \leqslant c\varepsilon^{\frac{3}{2}}, \tag{6.3.2}$$

with $c > 0$*, there exists a solution* $\boldsymbol{U} \in \mathcal{G}^3\left(\mathbb{R}^2 \times \left(0, T_0\varepsilon^{-2}\right)\right)^3$ *of* (4.0.2), (4.0.3) *and* (4.0.4) *such that*

$$\|\boldsymbol{U} - \boldsymbol{U}_{\text{ans}}\|_{\mathcal{G}^3(\mathbb{R}^2\times(0,T_0\varepsilon^{-2}))^3} \leqslant C\varepsilon^{\frac{3}{2}-\delta} \tag{6.3.3}$$

for all $\delta > 0$*.*
If, in addition, $\boldsymbol{U}_E^{(0)}$ *satisfies* (4.0.5) *and* (4.0.6)*, then we have* $\nabla \cdot \boldsymbol{\mathcal{D}}(\boldsymbol{U}_E) = \varrho_0$ *on* $\left(\mathbb{R}^2\backslash\Gamma_2\right) \times \left(0, T_0\varepsilon^{-2}\right)$ *and* $[\![\mathcal{D}_1(\boldsymbol{U}_E)]\!]_{2\text{D}} = \varrho_\Gamma$ *on* $\Gamma_2 \times \left(0, T_0\varepsilon^{-2}\right)$*.*

Remark 6.3.2
Let us collect some remarks to Theorem 6.3.1.

1. *Theorem 6.3.1 and the proof in the previous chapters was published in [27].*

2. *The existence of initial data* $\boldsymbol{U}^{(0)}$ *which satisfy* (6.3.2) *and the nonlinear compatibility conditions of order* 3 *is an open problem. Similarly, the existence of initial data* $\boldsymbol{U}^{(0)}$ *which satisfy* (6.3.2) *as well as* (4.0.5) *and* (4.0.6) *for a prescribed* ϱ_0 *and* ϱ_Γ *is an open problem.*
   We will discuss some ideas and first results concerning these problems in Chapter 7.

3. *In order to apply Theorem 5.2.3 the condition* $\overline{\operatorname{im} \boldsymbol{U}^{(0),\pm}} \subset \Omega_\pm$ *has to be satisfied. For* $\epsilon_{3,m}^{\pm} \geqslant 0$ *and* $\Omega_\pm = \mathbb{R}^3$ *this is always the case. For* $\epsilon_{3,m}^{\pm}$ *negative, a sufficient small initial value* $\boldsymbol{U}^{(0)}$ *would be necessary. This follows from* (6.3.2) *and the definition of* $\boldsymbol{U}_{\text{ans}}$ *for* ε *small enough.*

4. *Due to Sobolev's embedding, the components* $\boldsymbol{\mathcal{E}} := (U_1, U_2, 0)^\top$ *and* $\boldsymbol{\mathcal{H}} := (0, 0, U_3)^\top$ *of the solution* $\boldsymbol{U}$ *of Theorem 6.3.1 satisfy* (2.2.1), (2.2.2), (2.2.3) *and* (2.2.8) *on* $(\mathbb{R}^2\backslash\Gamma_2) \times (0, T_0\varepsilon^{-2})$ *in the classical sense.*

5. *In the case* $\varrho_\Gamma = 0$ *the regularity of* $\boldsymbol{U}$ *produced by Theorem 6.3.1 guarantees that we have* $\boldsymbol{\mathcal{E}} \in H_{\text{curl}}(\mathbb{R}^2)$, $\boldsymbol{\mathcal{D}} \in H_{\text{div}}(\mathbb{R}^2)$ *and* $\boldsymbol{\mathcal{H}} \in H^1(\mathbb{R}^2)$ *at each point in time, compare Remark 2.2.2.*

6. *If we replace* (6.3.2) *by*
$$\left\| \boldsymbol{U}^{(0)} - \boldsymbol{U}_{\text{ans}}(\cdot, 0) \right\|_{\mathcal{H}^3(\mathbb{R}^2)^3} \leqslant c\varepsilon^b,$$
for a $b \in \left(\frac{1}{2}, \frac{3}{2}\right)$*, the approximation result* (6.3.3) *would change to*
$$\| \boldsymbol{U} - \boldsymbol{U}_{\text{ans}} \|_{\mathcal{G}^3(\mathbb{R}^2 \times (0, T_0\varepsilon^{-2}))^3} \leqslant C\varepsilon^b.$$
This is still a meaningful approximation result since $\|\boldsymbol{U}_{\text{ans}}\|_{\mathcal{G}^3(\mathbb{R}^2 \times (0, T_0\varepsilon^{-2}))^3} \leqslant C\varepsilon^{1/2}$.

A combination of the numerical results of Section 2.4 and Section 3.4 allows us to calculate $\boldsymbol{U}_{\text{ans}}$.

Example 6.3.3 (Numerical Calculation of $\boldsymbol{U}_{\text{ans}}$)
We again choose $\epsilon_1(x_1) = 1\chi_{\mathbb{R}_-} + (1 + \mathrm{e}^{-x_1})\chi_{\mathbb{R}_+}$ *as in Example 3.4.1, and for the nonlinear part we choose* $\epsilon_3 \equiv -1$*. We now construct* $\boldsymbol{U}_{\text{ans}}$ *for* $k_0 = 0.5$ *and* $\varepsilon = 0.1$*. Thereby, we use the methods described in Section 3.4 and Section 2.4.*
First, we calculate the eigenfunction $\boldsymbol{m}$*, see Figure 3.2* (a)*, and determine* $\nu_0 \approx 0.494$, $\nu_1 \approx 0.964$, $\nu_2 \approx -0.115$ *from the dispersion relation, see Remark 3.4.2.*
Second, the envelope A has to satisfy the nonlinear Schrödinger equation (4.1.13) *with* $\kappa \approx 0.012$*. For the initial value of the envelope we set*
$$A^{(0)}(\varepsilon x_2) = \beta_1^{-1}\sqrt{2}\,\tilde{\eta}\,\text{sech}\,(\tilde{\eta}(\beta_2\varepsilon x_2 - x_0))\,\mathrm{e}^{\mathrm{i}(-2c\beta_2\varepsilon x_2+\gamma)/4},$$
with $\tilde{\eta} = 2$, $c = 1$, $x_0 = \gamma = 0$, $\beta_1 \approx 0.231$ *and* $\beta_2 \approx 1.999$*. We can now use the explicit solution* (2.4.2) *of the nonlinear Schrödinger equation. Note that this explicit solution satisfies the regularity assumptions of Theorem 6.3.1.*
The first component of $\boldsymbol{U}_{\text{ans}}$ *at time* $t = 0$ *and at time* $t = \varepsilon^{-2} = 100$ *can be seen in Figure 6.1. Note that we illustrate the interface* Γ_2 *as a semi-transparent hyperplane.*
For our example we have $\epsilon^{\pm}_{1,m} = 1$, $\epsilon^{\pm}_{3,m} = -1$ *and we can calculate*
$$\max\left\{\left(U^{(0)}_{\text{ans},1}\right)^2 + \left(U^{(0)}_{\text{ans},2}\right)^2\right\} \approx 0.207.$$
For $\eta = \frac{1}{10}$ *we have*
$$0.207 < \frac{3}{10} = \frac{\eta - \epsilon^{\pm}_{1,m}}{3\epsilon^{\pm}_{3,m}}$$
and therefore $\overline{\text{im}\,\boldsymbol{U}^{(0),\pm}_{\text{ans}}} \subset \Omega_\pm$*, see Remark 6.1.1.*

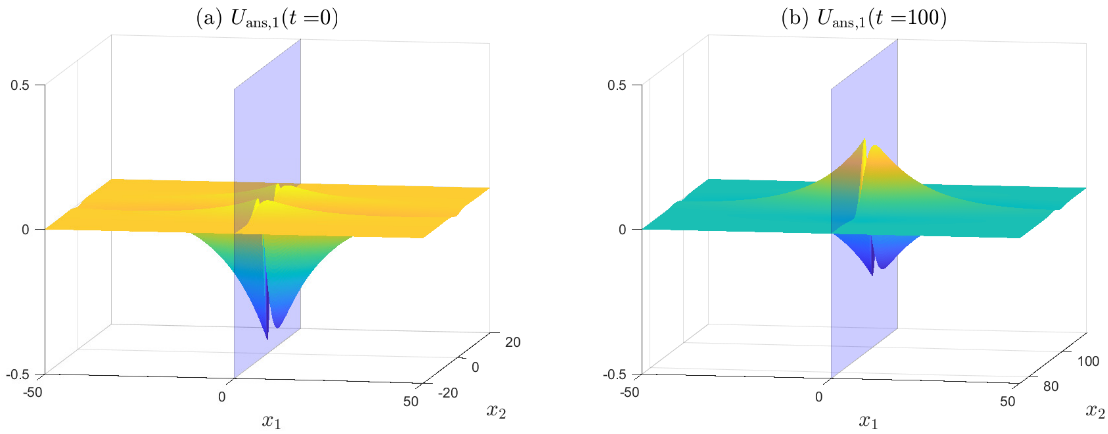

Figure 6.1.: (a) Plot of $U_{\text{ans},1}(t=0)$. (b) Plot of $U_{\text{ans},1}(t=100)$.

To illustrate that the calculated wave packet travels along the x_2-axis we plotted $|U_{\text{ans},1}(t)|$ for $t \in \{0, 33, 66, 100\}$ in one picture, see Figure 6.2.

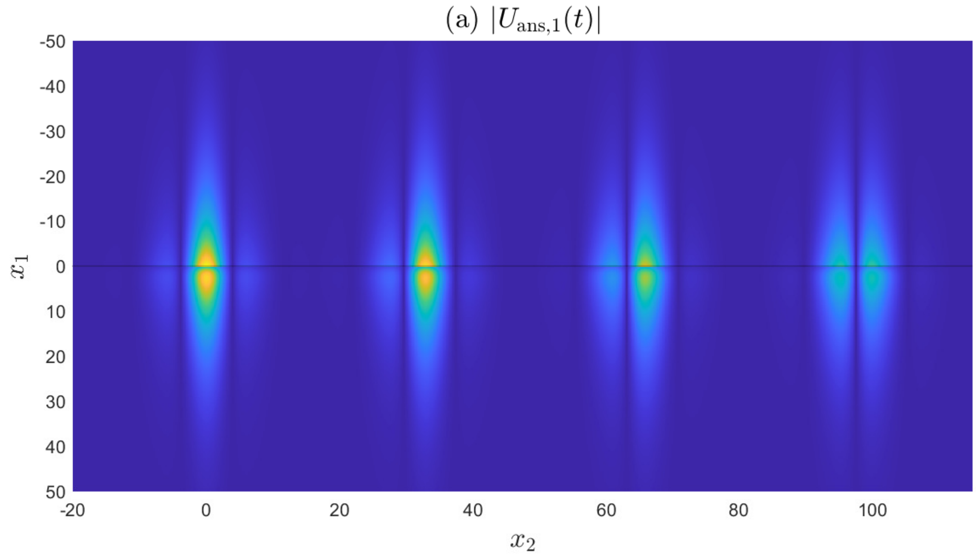

Figure 6.2.: (a) Plot of $|U_{\text{ans},1}(t)|$ for $t \in \{0, 33, 66, 100\}$.

7. Construction of Suitable Initial Values

To apply Theorem 6.3.1 the initial value $\boldsymbol{U}^{(0)}$ has to satisfy multiple conditions. In this chapter we want to discuss the following three open problems:

i) Determine $\boldsymbol{U}^{(0)}$ such that a $\mathcal{G}^3$-solution of Maxwell's equations can exist, i.e. the compatibility conditions of order 3 are satisfied;

ii) determine $\boldsymbol{U}^{(0)}$ such that the additional conditions for a solution of Maxwell's equations are satisfied, i.e. (4.0.5) and (4.0.6) are satisfied;

iii) determine $\boldsymbol{U}^{(0)}$ such that the initial values of $\boldsymbol{U}_{\text{ans}}$ are close, i.e. (6.3.2) holds true.

Since problems i) and ii) come with some difficulties, we will discuss them separately in the following sections.

Remark 7.0.1
There are two different view points for this problem. The first can be summarized as follows. One starts with an initial value $\boldsymbol{U}^{(0)}$ that satisfies conditions i) and ii) and Theorem 5.2.3 provides the local existence of a solution of Maxwell's equations. We now want to construct a suitable initial value

$$\boldsymbol{U}_{\text{ans}}^{(0)}(x_1, x_2) = \varepsilon A^{(0)}(\varepsilon x_2)\boldsymbol{m}(x_1)\mathrm{e}^{\mathrm{i}k_0 x_2} + \text{c.c.}$$

such that we can use our approximation result, which provides the long time existence and an approximative solution. To this end, we have to find a small correction $\boldsymbol{\Psi} : \mathbb{R}^2\backslash\Gamma_2 \to \mathbb{R}^3$ and a suitable initial value $\boldsymbol{U}_{\text{ans}}^{(0)}$ such that

$$\boldsymbol{U}^{(0)} + \boldsymbol{\Psi} = \boldsymbol{U}_{\text{ans}}^{(0)}, \qquad \|\boldsymbol{\Psi}\|_{\mathcal{H}^3(\mathbb{R}^2)^3} \leqslant C\varepsilon^{3/2}.$$

To solve this problem for a concrete case, information about the chosen $\boldsymbol{U}^{(0)}$ and the material functions ϵ_1, ϵ_3 are necessary.
We however proceed by the following complementary strategy. Assume that $\boldsymbol{U}_{\text{ans}}$ is given and the existence of a solution $\boldsymbol{U}$ has to be shown with Theorem 6.3.1. Therefore, we will have to find a suitable small correction $\boldsymbol{\Phi} : \mathbb{R}^2\backslash\Gamma_2 \to \mathbb{R}^3$ such that

$$\boldsymbol{U}^{(0)} := \boldsymbol{U}_{\text{ans}}^{(0)} + \boldsymbol{\Phi}, \qquad \|\boldsymbol{\Phi}\|_{\mathcal{H}^3(\mathbb{R}^2)^3} \leqslant C\varepsilon^{3/2}, \tag{7.0.1}$$

and conditions i) and ii) are satisfied.

7.1. Compatibility Conditions for $\boldsymbol{U}^{(\mathbf{0})}$

The goal for this section is the construction of initial values of the form (7.0.1) such that the nonlinear compatibility conditions of order 3 are satisfied.

Remark 7.1.1

The existence of general initial values that satisfy the nonlinear compatibility conditions is shown in Lemma 6.1 of [67]. The proof is based on Lemma 2.34 of [75] and an extension theorem for Sobolev functions [38, Theorem 2.5.7]. To find initial values that are also of the form (7.0.1) *a more involved strategy is necessary.*

With the notation of Section 5.3 applied to system (6.1.1) we get

$$
\begin{aligned}
\boldsymbol{U} &= \boldsymbol{V}^{(0)}(\boldsymbol{U}),\\
\partial_t \boldsymbol{U} &= -\widetilde{S}(\boldsymbol{U})^{-1}\left(\sum_{j=1}^{2} A_j \partial_{x_j} \boldsymbol{U}\right) = \widetilde{\boldsymbol{V}}^{(1)}(\boldsymbol{U}),\\
\partial_t^2 \boldsymbol{U} &= -\widetilde{S}(\boldsymbol{U})^{-1}\left(\sum_{j=1}^{2} A_j \partial_{x_j}\partial_t \boldsymbol{U} + \partial_t\left(\widetilde{S}(\boldsymbol{U})\right)\partial_t \boldsymbol{U}\right) = \widetilde{\boldsymbol{V}}^{(2)}(\boldsymbol{U}).
\end{aligned}
$$

For

$$
\widetilde{S}(\boldsymbol{U}) = \begin{pmatrix} \epsilon_1 & 0 & 0\\ 0 & \epsilon_1 & 0\\ 0 & 0 & \mu_0 \end{pmatrix} + \epsilon_3 \begin{pmatrix} 3U_1^2 + U_2^2 & 2U_1U_2 & 0\\ 2U_1U_2 & U_1^2 + 3U_2^2 & 0\\ 0 & 0 & 0 \end{pmatrix}
$$

we get

$$
\widetilde{S}(\boldsymbol{U})^{-1} = \begin{pmatrix} d(U_1,U_2)(\epsilon_1 + \epsilon_3(U_1^2 + 3U_2^2)) & -2\epsilon_3 d(U_1,U_2)U_1U_2 & 0\\ -2\epsilon_3 d(U_1,U_2)U_1U_2 & d(U_1,U_2)(\epsilon_1 + \epsilon_3(3U_1^2 + U_2^2)) & 0\\ 0 & 0 & \mu_0^{-1} \end{pmatrix},
$$

with

$$
d(U_1,U_2) := \left(\epsilon_1^2 + 4\epsilon_1\epsilon_3\left(U_1^2 + U_2^2\right) + 3\epsilon_3^2\left(U_1^2 + U_2^2\right)^2\right)^{-1},
$$

and

$$
\partial_t \widetilde{S}(\boldsymbol{U}) = \epsilon_3 \begin{pmatrix} 6U_1\partial_t U_1 + 2U_2\partial_t U_2 & 2\partial_t U_1 U_2 + 2U_1\partial_t U_2 & 0\\ 2\partial_t U_1 U_2 + 2U_1\partial_t U_2 & 2U_1\partial_t U_1 + 6U_2\partial_t U_2 & 0\\ 0 & 0 & 0 \end{pmatrix}.
$$

With this we can write the operators $\boldsymbol{V}^{(j)}$ without temporal derivatives:

$$
\begin{aligned}
&\boldsymbol{V}^{(0)}(\boldsymbol{U}) = \boldsymbol{U},\\
&\boldsymbol{V}^{(1)}(\boldsymbol{U}) = \begin{pmatrix} 2\epsilon_3 d(U_1,U_2)U_1U_2\partial_{x_1}U_3 + d(U_1,U_2)\left(\epsilon_1 + \epsilon_3\left(U_1^2+3U_2^2\right)\right)\partial_{x_2}U_3 \\ -2\epsilon_3 d(U_1,U_2)U_1U_2\partial_{x_2}U_3 - d(U_1,U_2)\left(\epsilon_1+\epsilon_3\left(3U_1^2+U_2^2\right)\right)\partial_{x_1}U_3 \\ \mu_0^{-1}\left(\partial_{x_2}U_1 - \partial_{x_1}U_2\right)\end{pmatrix},\\
&\boldsymbol{V}^{(2)}(\boldsymbol{U}) = -\widetilde{S}(\boldsymbol{U})^{-1}\\
&\cdot\begin{pmatrix} 6\epsilon_3U_1\left(V_1^{(1)}(\boldsymbol{U})\right)^2 + 4\epsilon_3U_2V_1^{(1)}(\boldsymbol{U})V_2^{(1)}(\boldsymbol{U}) + 2\epsilon_3U_1\left(V_2^{(1)}(\boldsymbol{U})\right)^2 - \partial_{x_2}V_3^{(1)}(\boldsymbol{U}) \\ 2\epsilon_3U_2\left(V_1^{(1)}(\boldsymbol{U})\right)^2 + 4\epsilon_3U_1V_1^{(1)}(\boldsymbol{U})V_2^{(1)}(\boldsymbol{U}) + 6\epsilon_3U_2\left(V_2^{(1)}(\boldsymbol{U})\right)^2 + \partial_{x_1}V_3^{(1)}(\boldsymbol{U}) \\ \partial_{x_1}V_2^{(1)}(\boldsymbol{U}) - \partial_{x_2}V_1^{(1)}(\boldsymbol{U})\end{pmatrix}.
\end{aligned}
$$

The nonlinear compatibility conditions of order 3 are now given by

$$
\left[\!\left[V_2^{(j)}\left(\boldsymbol{U}^{(0)}\right)\right]\!\right]_{2\mathrm{D}} = \left[\!\left[V_3^{(j)}\left(\boldsymbol{U}^{(0)}\right)\right]\!\right]_{2\mathrm{D}} = 0, \qquad j \in \{0,1,2\}, \tag{7.1.1}
$$

and have to be satisfied so that a solution $\boldsymbol{U} \in \mathcal{G}^3(\mathbb{R}^2 \times (0, T_0\varepsilon^{-2}))^3$ can exist.
For

$$
\boldsymbol{U}^{(0)}(\boldsymbol{x}) = \boldsymbol{U}^{(0)}_{\mathrm{ans}}(\boldsymbol{x}) + \begin{pmatrix}\Phi_1(\boldsymbol{x})\\ \Phi_2(\boldsymbol{x})\\ \Phi_3(\boldsymbol{x})\end{pmatrix}
$$

we now have to find a function $\boldsymbol{\Phi} : \mathbb{R}^2 \to \mathbb{R}^3$ such that (7.1.1) is satisfied.
Note that $\boldsymbol{U}^{(0)}_{\mathrm{ans}}$ satisfies the nonlinear compatibility conditions of order 1 since $[\![m_2]\!]_{1\mathrm{D}} = [\![m_3]\!]_{1\mathrm{D}} = 0$, see Section 3.2, therefore we get from (7.1.1) for $j = 0$ the following conditions:

$$
[\![\Phi_2]\!]_{1\mathrm{D}} = [\![\Phi_3]\!]_{1\mathrm{D}} = 0.
$$

To rewrite equation $\left[\!\left[V_3^{(1)}\left(\boldsymbol{U}^{(0)}\right)\right]\!\right]_{2\mathrm{D}} = 0$ we use that $\mathrm{i}k_0m_1 - \partial_{x_1}m_2 = -\mathrm{i}\mu_0\nu_0m_3$ and that m_3 is continuous at $x_1 = 0$. Hence,

$$
[\![\mathrm{i}k_0m_1 - \partial_{x_1}m_2]\!]_{2\mathrm{D}} = [\![-\mathrm{i}\mu_0\nu_0m_3]\!]_{2\mathrm{D}} = 0
$$

and $\boldsymbol{\Phi}$ has to satisfy

$$
\left[\!\left[\mu^{-1}\left(m_1\varepsilon^2\partial_{X_2}A(\varepsilon x_2)\mathrm{e}^{\mathrm{i}k_0x_2} + \partial_{x_2}\Phi_1 - \partial_{x_1}\Phi_2\right)\right]\!\right]_{2\mathrm{D}} = 0. \tag{7.1.2}
$$

The remaining nonlinear compatibility conditions give us rather complicated expressions. From $\left[\!\left[V_2^{(1)}\left(\boldsymbol{U}^{(0)}\right)\right]\!\right]_{2\mathrm{D}} = 0$ we get a nonlinear partial differential equation involving

$$
\Phi_1,\ \Phi_2,\ \partial_{x_1}\Phi_3,\ \partial_{x_2}\Phi_3.
$$

Finally, we get from the remaining two nonlinear compatibility conditions two nonlinear equations in

$$\Phi_1,\ \Phi_2,\ \partial_{x_1}\Phi_3,\ \partial_{x_2}\Phi_3,\ \partial_{x_2}\partial_{x_1}\Phi_1,\ \partial_{x_1}^2\Phi_2$$

and

$$\Phi_1,\ \Phi_2,\ \partial_{x_1}\Phi_1,\ \partial_{x_2}\Phi_1,\ \partial_{x_1}\Phi_2,\ \partial_{x_2}\Phi_2,\ \partial_{x_1}\Phi_3,\ \partial_{x_2}\Phi_3,\ \partial_{x_1}^2\Phi_3,\ \partial_{x_2}^2\Phi_3,\ \partial_{x_2}\partial_{x_1}\Phi_3.$$

One now needs to find a solution $\boldsymbol{\Phi}$ of these equations such that $\|\boldsymbol{\Phi}\|_{\mathcal{H}^3(\mathbb{R}^2)^3} \leqslant C\varepsilon^{3/2}$.
One way to simplify this problem could be with the help of an extension theorem. We use a special case of Theorem 2.5.7 in [38] that states the following:

Lemma 7.1.2 (Extension Lemma)
For arbitrary functions $\boldsymbol{\phi}^{(k),\pm} \in H^{3-k}(\mathbb{R})^3$, $k \in \{0,1,2\}$, there exists a function $\boldsymbol{\Phi} \in \mathcal{H}^3(\mathbb{R}^2)^3$ with $\lim_{x_1\to 0^\pm} \partial_{x_1}^k \boldsymbol{\Phi}^\pm(x_1,x_2) = \boldsymbol{\phi}^{(k),\pm}(x_2)$, $k \in \{0,1,2\}$ and

$$\|\boldsymbol{\Phi}\|_{\mathcal{H}^3(\mathbb{R}^2)^3} \leqslant C\sum_{k=0}^{2}\left(\left\|\boldsymbol{\phi}^{(k),+}\right\|_{H^{3-k}(\mathbb{R})^3} + \left\|\boldsymbol{\phi}^{(k),-}\right\|_{H^{3-k}(\mathbb{R})^3}\right). \tag{7.1.3}$$

PROOF: One combines the results of Theorem 2.5.7 in [38] for $n = 2$, $m = 3$ and $s = 0$ with the analogue result for the left half-space. The estimate follows from the proof of Theorem 2.5.7 in [38] where one uses that $\|f\|_{H^{m_1}(\mathbb{R})} \leqslant \|f\|_{H^{m_2}(\mathbb{R})}$ for $m_1 \leqslant m_2$ and $f \in H^{m_2}(\mathbb{R})$. □
To apply Lemma 7.1.2 to our problem, note that the nonlinear compatibility conditions have only to be satisfied on the interface Γ_2. Instead of finding a function $\boldsymbol{\Phi}(x_1,x_2)$ that satisfies the nonlinear compatibility conditions, we only have to find functions $\boldsymbol{\phi}^{(k),\pm}(x_2)$, $k \in \{0,1,2\}$ and extend them to $\mathbb{R}^2_\pm$. To find the equations which $\boldsymbol{\phi}^{(k),\pm}(x_2)$ have to satisfy we simply replace $\partial_{x_1}^k\boldsymbol{\Phi}(x_1,x_2)$ in the nonlinear compatibility conditions by $\boldsymbol{\phi}^{(k),\pm}(x_2)$, e.g. (7.1.2) will give us the equation

$$\begin{aligned} &m_{1,+}\varepsilon^2\partial_{X_2}A(\varepsilon x_2)\mathrm{e}^{\mathrm{i}k_0x_2} + \partial_{x_2}\phi_1^{(0),+} - \phi_2^{(1),+} \\ &\quad = m_{1,-}\varepsilon^2\partial_{X_2}A(\varepsilon x_2)\mathrm{e}^{\mathrm{i}k_0x_2} + \partial_{x_2}\phi_1^{(0),-} - \phi_2^{(1),-}, \end{aligned}$$

with $m_{1,\pm}(\boldsymbol{x}) := \lim_{x_1\to 0^\pm} m_1(x_1,x_2)$ for $\boldsymbol{x} \in \Gamma_2$.
Hence, the open question is how to find suitable $\boldsymbol{\phi}^{(k),\pm}(x_2)$ that satisfy the equations corresponding to the nonlinear compatibility conditions. If additionally

$$\sum_{k=0}^{2}\left(\left\|\boldsymbol{\phi}^{(k),+}\right\|_{H^{3-k}(\mathbb{R})^3} + \left\|\boldsymbol{\phi}^{(k),-}\right\|_{H^{3-k}(\mathbb{R})^3}\right) \leqslant C\varepsilon^{3/2},$$

then $\|\boldsymbol{\Phi}\|_{\mathcal{H}^3(\mathbb{R}^2)^3} \leqslant C\varepsilon^{3/2}$ by (7.1.3).
Note that for this method (4.0.5) and (4.0.6) will be in general not fulfilled. Additional conditions on $\phi_1^{(0),\pm}, \phi_2^{(0),\pm}$ are necessary to satisfy $\left[\!\left[\mathcal{D}_1\left(\boldsymbol{U}_E^{(0)}\right)\right]\!\right]_{2\mathrm{D}} = \varrho_\Gamma$. Moreover, a more sophisticated extension theorem would be needed to deal with $\nabla\cdot\boldsymbol{\mathcal{D}}\left(\boldsymbol{U}_E^{(0),\pm}\right) = \varrho_0$ on $\mathbb{R}^2\backslash\Gamma_2$.

7.2. Maxwell Conditions for $\boldsymbol{U}^{(\mathbf{0})}$

In this section, we try to satisfy the conditions

$$\partial_{x_1}\mathcal{D}_1\left(\boldsymbol{U}_E^{(0),\pm}\right) + \partial_{x_2}\mathcal{D}_2\left(\boldsymbol{U}_E^{(0),\pm}\right) = \varrho_0 \tag{7.2.1}$$

and

$$\left[\!\left[\mathcal{D}_1\left(\boldsymbol{U}_E^{(0)}\right)\right]\!\right]_{2\mathrm{D}} = \varrho_\Gamma \tag{7.2.2}$$

for $\boldsymbol{U}^{(0)} = \boldsymbol{U}^{(0)}_{\mathrm{ans}} + \boldsymbol{\Phi}$ and $\|\boldsymbol{\Phi}\|_{\mathcal{H}^3(\mathbb{R}^2)^3} \leqslant C\varepsilon^{3/2}$.

Remark 7.2.1

For the case in which ϱ_0 and ϱ_Γ are not prescribed we could simply define

$$\varrho_0 := \partial_{x_1}\mathcal{D}_1\left(\boldsymbol{U}_{\mathrm{ans},E}^{(0),\pm}\right) + \partial_{x_2}\mathcal{D}_2\left(\boldsymbol{U}_{\mathrm{ans},E}^{(0),\pm}\right)$$

and

$$\varrho_\Gamma := \left[\!\left[\mathcal{D}_1\left(\boldsymbol{U}_{\mathrm{ans},E}^{(0)}\right)\right]\!\right]_{2\mathrm{D}}.$$

Then the conditions (7.2.1), (7.2.2) *would be satisfied for $\boldsymbol{U}^{(0)} := \boldsymbol{U}^{(0)}_{\mathrm{ans}}$.*
Since $\boldsymbol{U}_{\mathrm{ans}}$ satisfies (7.2.1), (7.2.2) *for the linear displacement field $\boldsymbol{\mathcal{D}}_{\mathrm{lin}} = \epsilon_1\boldsymbol{\mathcal{E}}$ exactly, one can easily show that $\left\|\nabla\cdot\boldsymbol{\mathcal{D}}\left(\boldsymbol{U}_{\mathrm{ans},E}^{(0)}\right)\right\|_{L^2(\mathbb{R}^2)} \leqslant C\varepsilon^{3/2}$ and $\left[\!\left[\mathcal{D}_1\left(\boldsymbol{U}_{\mathrm{ans},E}^{(0)}\right)\right]\!\right]_{2\mathrm{D}}$ is of order ε^3, see Section 4.1.2. Therefore, ϱ_0 and ϱ_Γ would be small.*

For $\varrho_0 = 0$ and $\varrho_\Gamma = 0$ but for a larger class of nonlinearities, this problem was considered in [25]. There, initial data of the form

$$\boldsymbol{U}^{(0)} = \boldsymbol{U}^{(0)}_{\mathrm{ans}} + (\partial_{x_1}\phi, \partial_{x_2}\phi, 0)^\top$$

with a correction function $\phi : \mathbb{R}^2 \to \mathbb{R}$ was found by using minimization techniques. Let us now present this approach.

7.2.1. Quasilinear Transmission Problem

We consider the more general quasilinear transmission problem

$$\begin{cases} -\nabla \cdot \boldsymbol{a}(\boldsymbol{x}, \nabla\phi) = b(\boldsymbol{x}) & \text{in} \quad \mathbb{R}^2_\pm, \quad (7.2.3a) \\ [\![(\boldsymbol{a}(\boldsymbol{x}, \nabla\phi) + \tilde{\epsilon}_1 \boldsymbol{U}_0) \cdot \boldsymbol{e}_1]\!]_{2D} = 0, & \quad (7.2.3b) \end{cases}$$

with

$$\boldsymbol{a}(\boldsymbol{x}, \nabla\phi) := \epsilon_f \boldsymbol{f}(\boldsymbol{U}_0 + \nabla\phi) + \tilde{\epsilon}_1 \nabla\phi,$$

$\tilde{\epsilon}_1, \epsilon_f : \mathbb{R}^2 \to \mathbb{R}_+$ and

$$b := \nabla \cdot (\tilde{\epsilon}_1 \boldsymbol{U}_0),$$

and where $\boldsymbol{U}_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is a given vector field which satisfies the linear interface condition $[\![\tilde{\epsilon}_1 \boldsymbol{U}_0 \cdot \boldsymbol{e}_1]\!]_{2D} = 0$. Here $\boldsymbol{e}_1$ is the first unit vector. We aim to prove the existence of a solution ϕ and establish an estimate for $\nabla\phi$.
Before we can state the main result of [25] we have to collect some definitions and assumptions.

Assumptions

We assume the following conditions on $\tilde{\epsilon}_1$, ϵ_f, $\boldsymbol{f}$ and $\boldsymbol{U}_0$:

i) $\tilde{\epsilon}_1, \epsilon_f \in L^\infty(\mathbb{R}^2)$ and there exists a constant d such that $\tilde{\epsilon}_1(\boldsymbol{x}), \epsilon_f(\boldsymbol{x}) \geqslant d > 0$ for almost all $\boldsymbol{x} \in \mathbb{R}^2$;

ii) there exist $p > 1$ and $F \in C^1(\mathbb{R}^2)$ convex such that

$$F(0) = 0, \quad \boldsymbol{f} = \nabla F \quad \text{and} \quad F(\boldsymbol{v}) \geqslant \mu_p |\boldsymbol{v}|^{p+1}$$

for some $\mu_p > 0$;

iii) there exist $1 < \alpha \leqslant p$ and constants $0 < \lambda_p \leqslant \Lambda_p$ and $\Lambda_\alpha \geqslant 0$ such that for all $\boldsymbol{v} \in \mathbb{R}^2$
- $|\boldsymbol{f}(\boldsymbol{v})| \leqslant \Lambda_p |\boldsymbol{v}|^p + \Lambda_a |\boldsymbol{v}|^\alpha$;
- $\boldsymbol{f}(\boldsymbol{v}) \cdot \boldsymbol{v} > \lambda_p |\boldsymbol{v}|^{p+1}$;

iv) $\boldsymbol{U}_0 \in L^2(\mathbb{R}^2)^2 \cap L^{p+1}(\mathbb{R}^2)^2$;

v) $b := \nabla \cdot (\epsilon_1 U_0) \in L^2(\mathbb{R}^2) \cap L^1(\log, \mathbb{R}^2)$, where

$$L^1(\log, \mathbb{R}^2) := \left\{ \varphi \in L^1_{\text{loc}}(\mathbb{R}^2) \,\middle|\, \|\varphi\|_{L^1(\log, \mathbb{R}^2)} := \int_{\mathbb{R}}^2 \log(2 + |\boldsymbol{x}|) |\varphi(\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} < \infty \right\}.$$

vi) $\boldsymbol{U}_0$ satisfies the transmission condition $[\![\tilde{\epsilon}_1 \boldsymbol{U}_0 \cdot \boldsymbol{e}_1]\!]_{2D} = 0$;

vii) $\phi = 0$ is not a solution of the transmission problem (7.2.3a), (7.2.3b).

We can now define the energy functional corresponding to (7.2.3a) as

$$\mathcal{J}(\phi) := \int_{\mathbb{R}^n} \epsilon_f F(\boldsymbol{U}_0 + \nabla\phi)\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^n} \widetilde{\epsilon}_1 \frac{1}{2}|\nabla\phi|^2\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^n} \widetilde{\epsilon}_1 \boldsymbol{U}_0 \cdot \nabla\phi\,\mathrm{d}\boldsymbol{x}.$$

Indeed, we claim that the Euler-Lagrange equation associated with $\mathcal{J}$ is (7.2.3a). For any $\eta \in C_c^\infty(\mathbb{R}^2)$ we get

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\mathcal{J}(\phi + \delta\eta)|_{\delta=0} = \int_{\mathbb{R}^n} \epsilon_f \boldsymbol{f}(\boldsymbol{U}_0 + \nabla\phi)\cdot\nabla\eta\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d} \widetilde{\epsilon}_1(\boldsymbol{U}_0 + \nabla\phi)\cdot\nabla\eta\,\mathrm{d}\boldsymbol{x}.$$

Assumptions ii) and iii) suggest the following function space for ϕ in which the functional $\mathcal{J}$ is well-defined:

$$\mathfrak{D}_{2,p+1}(\mathbb{R}^2) := D_0^{1,2}(\mathbb{R}^2) \cap D_0^{1,p+1}(\mathbb{R}^2),$$

where for $q \geqslant 1$

$$D_0^{1,q}(\mathbb{R}^2) := \overline{C_c^\infty(\mathbb{R}^2)}^{|\cdot|_{1,q}} \qquad \text{with the norm} \qquad |u|_{1,q} := \|\nabla u\|_{L^q(\mathbb{R}^2)}$$

is the homogeneous Sobolev space. The norm on $\mathfrak{D}_{2,p+1}$ is defined as $\|\cdot\|_{\mathfrak{D}} := |\cdot|_{1,2} + |\cdot|_{1,p+1}$. Now critical points of $\mathcal{J}$ are the weak solutions of (7.2.3a) in the following sense:

Definition 7.2.2 (Weak Solution of the Transmission Problem)
We say that $\phi \in \mathfrak{D}_{2,p+1}(\mathbb{R}^2)$ is a weak solution of problem (7.2.3a), (7.2.3b) *if*

$$\int_{\mathbb{R}^2} \epsilon_f \boldsymbol{f}(\boldsymbol{U}_0 + \nabla\phi)\cdot\nabla\eta\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^2} \widetilde{\epsilon}_1(\boldsymbol{U}_0 + \nabla\phi)\cdot\nabla\eta = 0, \qquad \forall\,\eta \in \mathfrak{D}_{2,p+1}(\mathbb{R}^2).$$

Theorem 7.2.3 (Existence Theorem for the Transmission Problem)
Let $\widetilde{\epsilon}_1, \epsilon_f, \boldsymbol{f}, \boldsymbol{U}_0$ satisfy Assumptions i)–vii). Then there exists a non-trivial minimum ϕ of the functional $\mathcal{J}$ in $\mathfrak{D}_{2,p+1}(\mathbb{R}^2)$ and there holds

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla\phi|^2\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^2} |\boldsymbol{U}_0 + \nabla\phi|^{p-1}|\nabla\phi|^2\,\mathrm{d}\boldsymbol{x} \\ \leqslant C\left(\|\boldsymbol{U}_0\|_{L^{p+1}(\mathbb{R}^2)^2}^{p+1} + \|\boldsymbol{U}_0\|_{L^{\alpha+1}(\mathbb{R}^2)^2}^{\alpha+1} + \|b\|_{L^2(\mathbb{R}^2)}^2 + \|b\|_{L^1(\log,\mathbb{R}^2)}^2\right),\end{aligned} \tag{7.2.4}$$

where the constant C depends only on Λ_p, λ_p, Λ_a, d, $\|\widetilde{\epsilon}_1\|_{L^\infty(\mathbb{R}^2)}$ and $\|\epsilon_f\|_{L^\infty(\mathbb{R}^2)}$. Moreover, ϕ is a weak solution of (7.2.3a), (7.2.3b).

PROOF: The proof is carried out in detail in [25] for the n-dimensional case. Here we give just a sketch of the proof.
By means of Assumptions i)–vii) one may show that $\mathcal{J}$ is well-defined and coercive on the reflexive Banach space $\mathfrak{D}_{2,p+1}(\mathbb{R}^2)$. Hence, the existence of ϕ follows from the direct method of the calculus of variations, see e.g. [34].
Estimate (7.2.4) follows from the assumptions and some technical involved estimates.

□

7.2.2. Application to the Maxwell Problem

The transmission problem (7.2.3a), (7.2.3b) corresponds to (7.2.1), (7.2.2) with $\varrho_0 = 0$, $\varrho_\Gamma = 0$, $\boldsymbol{U}^{(0)} = \boldsymbol{U}^{(0)}_{\text{ans}} + (\partial_{x_1}\phi, \partial_{x_2}\phi, 0)^\top$ if we set

$$\boldsymbol{U}_0 = \left(U^{(0)}_{\text{ans},1}, U^{(0)}_{\text{ans},2}\right)^\top, \quad \widetilde{\epsilon}_1(\boldsymbol{x}) = \epsilon_1(x_1), \quad \epsilon_f(\boldsymbol{x}) = \epsilon_3(x_1), \quad \boldsymbol{f}(\boldsymbol{v}) = |\boldsymbol{v}|^2\boldsymbol{v}. \tag{7.2.5}$$

It is easy to see that for our Maxwell problem most of the Assumptions i)–vii) from Section 7.2.1 are satisfied. Indeed, the regularity assumptions are satisfied since ϵ_1, ϵ_3, $\boldsymbol{U}^{(0)}_{\text{ans}}$ satisfy (A1), (A6) and (4.2.10). For our nonlinearity all assumptions are satisfied with $\alpha = p = 3$. Simply notice that with $F(\boldsymbol{v}) := \frac{1}{4}|\boldsymbol{v}|^4$ we have $\nabla F(\boldsymbol{v}) = |\boldsymbol{v}|^2\boldsymbol{v}$. The last two Assumptions vi)–vii) are also satisfied, see Remark 7.2.1.
Only for the assumption that ϵ_f has to be positive we have to modify our Assumption (A6). Assume that there are constants $\epsilon^\pm_{3,m}$, $\epsilon^\pm_{3,M}$ such that

$$\epsilon_3^\pm \in C^3(\mathbb{R}_\pm) \cap W^{3,\infty}(\mathbb{R}_\pm), \quad 0 < \epsilon^\pm_{3,m} \leqslant \epsilon_3^\pm(x_1) \leqslant \epsilon^\pm_{3,M}, \qquad \forall\, x_1 \in \mathbb{R}\backslash\{0\}. \tag{A6*}$$

Now Theorem 7.2.3 yields the existence of a function ϕ such that $\boldsymbol{U}^{(0)}$ satisfies (7.2.1) and (7.2.2) and

$$\|\nabla\phi\|^2_{L^2(\mathbb{R}^2)^2} \leqslant C\left(\|\boldsymbol{U}_0\|^4_{L^4(\mathbb{R}^2)^2} + \|b\|^2_{L^2(\mathbb{R}^2)} + \|b\|^2_{L^1(\log,\mathbb{R}^2)}\right). \tag{7.2.6}$$

For $\nabla\phi$ to be a meaningful correction term of $\boldsymbol{U}_0$, its L^2-norm should be at least $\mathcal{O}(\varepsilon^{1/2})$ or in the optimal case $\mathcal{O}(\varepsilon^{3/2})$, see (6.3.2) and Remark 6.3.2. Let us therefore estimate the norms on the right-hand side of (7.2.6), see [25, Proposition 5.1].
By the definition of $\boldsymbol{U}_{\text{ans}}$ and with $\widetilde{\boldsymbol{m}} := (m_1, m_2)^\top$ we have

$$\int_{\mathbb{R}^2} |\boldsymbol{U}_0|^4\,\mathrm{d}\boldsymbol{x} = \varepsilon^4 \int_{\mathbb{R}} |A^{(0)}(\varepsilon x_2)|^4\,\mathrm{d}x_2 \int_{\mathbb{R}} |\widetilde{\boldsymbol{m}}(x_1)|^4\,dx_1 \leqslant \varepsilon^3 \left\|A^{(0)}\right\|^4_{L^4(\mathbb{R})} \|\boldsymbol{m}\|^4_{L^4(\mathbb{R})^3}.$$

Note that we again lose one power of ε due to the scaling of A. By Sobolev embedding we know that $A^{(0)} \in L^4(\mathbb{R})$ and $\boldsymbol{m} \in L^4(\mathbb{R})^3$ and we conclude

$$\|\boldsymbol{U}_0\|^4_{L^4(\mathbb{R}^2)^2} = \mathcal{O}(\varepsilon^3). \tag{7.2.7}$$

For $b = \nabla\cdot(\epsilon_1\boldsymbol{U}_0)$, we get

$$b(x_1,x_2) = \left(\varepsilon A^{(0)}(\varepsilon x_2)(\partial_{x_1}(\epsilon_1(x_1)m_1(x_1)) + \epsilon_1(x_1)\mathrm{i}k_0 m_2) + \varepsilon^2\epsilon_1\partial_{X_2}A^{(0)}(\varepsilon x_2)m_2\right)\mathrm{e}^{\mathrm{i}k_0x_2} + \text{c.c.}$$

Since $\boldsymbol{m}$ is a solution of the linear eigenvalue problem, the terms to order ε vanish, see Remark 4.1.5, and hence

$$b(x_1,x_2) = \varepsilon^2\epsilon_1(x_1)A^{(0)}(\varepsilon x_2)m_2(x_1)\mathrm{e}^{\mathrm{i}k_0x_2} + \text{c.c.},$$

which implies

$$\|b\|^2_{L^2(\mathbb{R}^2)} = \mathcal{O}(\varepsilon^3) \tag{7.2.8}$$

since $m_2 \in L^2(\mathbb{R})$ and $A^{(0)} \in L^2(\mathbb{R})$.

For the last term in (7.2.6), we first note the simple equality

$$\log(s+t) = \log(s) + \log\left(1 + \frac{t}{s}\right), \qquad s,t \in \mathbb{R}_+,$$

to obtain, for a suitable constant $c > 0$,

$$\begin{aligned}\log(2+|\boldsymbol{x}|) &\leqslant \log(2 + c|x_1| + c|x_2|) = \log(2+c|x_1|) + \log\left(1 + \frac{c|x_2|}{2+c|x_1|}\right)\\ &\leqslant \log(2+c|x_1|) + \log(2+c|x_2|).\end{aligned}$$

Hence,

$$\begin{aligned}\|b\|_{L^1(\log,\mathbb{R})} &\leqslant \varepsilon^2 \|\epsilon_1\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \left|\partial_{X_2} A^{(0)}(\varepsilon x_2)\right| |m_2(x_1)| \log(2+|\boldsymbol{x}|)\,\mathrm{d}\boldsymbol{x}\\ &\leqslant C\varepsilon^2 \left(\|m_2\|_{L^1(\log,\mathbb{R})} \int_{\mathbb{R}} \left|\partial_{X_2} A^{(0)}(\varepsilon x_2)\right|\,\mathrm{d}x_2 + \|m_2\|_{L^1(\mathbb{R})} \int_{\mathbb{R}} \left|\partial_{X_2} A^{(0)}(\varepsilon x_2)\right| \log(2+c|x_2|)\,\mathrm{d}x_2\right)\\ &\leqslant C\varepsilon^2 \|m_2\|_{L^1(\log,\mathbb{R})} \int_{\mathbb{R}} \left|\partial_{X_2} A^{(0)}(\varepsilon x_2)\right| (1 + \log(2+c|x_2|))\,\mathrm{d}x_2\\ &\leqslant C\varepsilon \|m_2\|_{L^1(\log,\mathbb{R})} \int_{\mathbb{R}} \left|\partial_{X_2} A^{(0)}(y)\right| \log\left(\mathrm{e} + c\frac{|y|}{\varepsilon}\right)\,\mathrm{d}y,\end{aligned}$$

where in the last inequality we use the variable transformation $y = \varepsilon x_2$. Finally, choosing any $\gamma > 0$ one may estimate $\log(\mathrm{e} + ct) \leqslant C(1 + t^\gamma)$ for all $t \geqslant 0$. Therefore,

$$\|b\|_{L^1(\log,\mathbb{R})} \leqslant C \|m_2\|_{L^1(\log,\mathbb{R})} \left(\varepsilon \|\partial_{X_2} A\|_{L^1(\mathbb{R})} + \varepsilon^{1-\gamma} \int_{\mathbb{R}} |\partial_{X_2} A(y)||y|^\gamma\,\mathrm{d}y\right) = \mathcal{O}(\varepsilon^{1-\gamma}), \tag{7.2.9}$$

provided $m_2 \in L^1(\log,\mathbb{R})$ and

$$\partial_{X_2} A \in L^1(\gamma,\mathbb{R}) := \left\{\varphi \in L^1(\mathbb{R}) \,\middle|\, \int_{\mathbb{R}} |\varphi(x)|(1+|x|^\gamma)\,\mathrm{d}x < \infty\right\}.$$

Since we are mostly interested in localized wave packet solutions, the conditions $m_2 \in L^1(\log,\mathbb{R})$ and $\partial_{X_2} A \in L^1(\gamma,\mathbb{R})$ are not very restrictive. Indeed, see Figure 3.2 (b) where we see an example for an eigenfunction $\boldsymbol{m}$ that is exponentially decaying.

In summary, from estimate (7.2.6) one infers by (7.2.7), (7.2.8) and (7.2.9)

$$\|\nabla\phi\|_{L^2(\mathbb{R}^2)^2} \leqslant C\left(\varepsilon^{3/2} + \varepsilon^{1-\gamma}\right), \tag{7.2.10}$$

where $\gamma > 0$ is arbitrary and the constant C does not depend on ε. Hence, choosing $\gamma \in \left(0, \frac{1}{2}\right)$ one obtains an estimate of order $\mathcal{O}(\varepsilon^{1/2})$ and $\|\nabla\phi\|_{L^2(\mathbb{R}^2)^2} = \mathcal{O}(\varepsilon^{1-\gamma})$.

Finally, note that $\boldsymbol{U}^{(0)} = \boldsymbol{U}^{(0)}_{\text{ans}} + (\partial_{x_1}\phi, \partial_{x_2}\phi, 0)^\top$ satisfies the interface conditions

$$\left[\!\left[\mathcal{D}_1\left(\boldsymbol{U}_E^{(0)}\right)\right]\!\right]_{2\text{D}} = \left[\!\left[U_2^{(0)}\right]\!\right]_{2\text{D}} = \left[\!\left[U_3^{(0)}\right]\!\right]_{2\text{D}} = 0.$$

Indeed, since $\left[\!\left[U^{(0)}_{\text{ans},2}\right]\!\right]_{2\text{D}} = \left[\!\left[U^{(0)}_{\text{ans},3}\right]\!\right]_{2\text{D}} = 0$, see Section 3.2, we only have to check if $\partial_{x_2}\phi$ is continuous at $x_1 = 0$. From $\phi \in \mathfrak{D}_{2,4}(\mathbb{R}^2) \subset D_0^{1,4}(\mathbb{R}^2)$ it follows that $\phi \in W^{1,4}_{\text{loc}}(\mathbb{R}^2)$, see e.g. Lemma II.6.1 in [32]. By Sobolev embedding this implies that $\phi \in C^{0,1/2}_{\text{loc}}(\mathbb{R}^2)$. On the half-spaces we even have the improved regularity $\phi^\pm \in C^{1,1/2}_{\text{loc}}\left(\mathbb{R}^2_\pm\right)$, see [20]. Hence, ϕ is continuous across the interface and every tangential derivative with respect to the interface is continuous too, see e.g. [83, Section 173-175]. Note however, that the nonlinear compatibility conditions of higher order will in general not be satisfied.

Remark 7.2.4

Note that we only estimated the L^2-norm of $\nabla\phi$. In Theorem 2.2 and Remark 12 of [25] it is shown that estimates analogous to (7.2.10) *in the $\mathcal{H}^m$-norm are possible, when the coefficients satisfy additional regularity assumptions. However, mainly because of the loss of powers in ε caused by the estimate on the logarithmic term, the estimates one can prove for the higher-order derivatives of $\nabla\phi$ are not able to provide* (6.3.2).

In Section 7.2.3 we will present a numerical method to calculate $\nabla\phi$. The numerical tests suggest that a correction with $\|\nabla\phi\|_{L^2(\mathbb{R}^2)^2} \leqslant C\varepsilon^{3/2}$ is possible.

7.2.3. Numerical Calculation of the Correction $\nabla\phi$

In this section we present a numerical method to calculate a solution ϕ of (7.2.3a), (7.2.3b) with $[\![\partial_{x_2}\phi]\!]_{2\text{D}} = 0$ for the Maxwell setting (7.2.5), i.e. $\boldsymbol{U}^{(0)} = \boldsymbol{U}^{(0)}_{\text{ans}} + (\partial_{x_1}\phi, \partial_{x_2}\phi, 0)^\top$ satisfies (7.2.1), (7.2.2) and $\left[\!\left[U_2^{(0)}\right]\!\right]_{2\text{D}} = 0$. We use the finite element method in combination with a fixed-point iteration.

Moving the nonlinear term to the right-hand side, one needs to solve

$$\begin{cases} -\nabla \cdot (\epsilon_1 \nabla\phi) = \mathfrak{f}(\phi) & \text{in} \quad \mathbb{R}^2_\pm, \\ [\![\epsilon_1 \partial_{x_1}\phi]\!]_{2\text{D}} = \mathfrak{h}(\phi), & \\ [\![\partial_{x_2}\phi]\!]_{2\text{D}} = 0, & \end{cases} \tag{7.2.11}$$

with the ϕ-dependent functions

$$\begin{aligned} \mathfrak{f}(\phi) &:= \nabla \cdot \left(\epsilon_1 \boldsymbol{U}_0 + \epsilon_3 |\boldsymbol{U}_0 + \nabla\phi|^2 (\boldsymbol{U}_0 + \nabla\phi)\right), \\ \mathfrak{h}(\phi) &:= -\left[\!\left[\epsilon_3 |\boldsymbol{U}_0 + \nabla\phi|^2 (U_{0,1} + \partial_{x_1}\phi)\right]\!\right]_{2\text{D}}. \end{aligned}$$

To find a solution of (7.2.11), we rewrite the problem as a system of two coupled Neumann boundary value problems, in which we have to determine the functions $\phi : \mathbb{R}^2_\pm \to \mathbb{R}$ and

$\mathfrak{g} : \Gamma \to \mathbb{R}$ such that

$$\begin{cases} -\nabla \cdot (\epsilon_1 \nabla \phi) = \mathfrak{f}(\phi) & \text{in} \quad \mathbb{R}^2_\pm, \\ (\epsilon_1 \partial_{x_1} \phi)_- = \mathfrak{g}, \\ (\epsilon_1 \partial_{x_1} \phi)_+ = \mathfrak{h}(\phi) + \mathfrak{g}, \\ [\![\partial_{x_2} \phi]\!]_{2\mathrm{D}} = 0, \end{cases} \tag{7.2.12}$$

where we use $u_\pm(\boldsymbol{x}) = \lim_{x_1 \to 0^\pm} u(x_1, x_2)$, for $\boldsymbol{x} \in \Gamma_2$. Note that a solution ϕ of (7.2.12) is also a solution of (7.2.11). We will use the freedom in the choice of $\mathfrak{g}$ to satisfy the second interface condition $[\![\partial_{x_2} \phi]\!]_{2\mathrm{D}} = 0$. Let us now describe how to approximate the solution of the nonlinear problem (7.2.12) with the help of a fixed-point iteration. We select an initial guess ϕ_0 and solve

$$\begin{cases} -\nabla \cdot (\epsilon_1 \nabla \phi_{n+1}) = \mathfrak{f}(\phi_n) & \text{in} \quad \mathbb{R}^2_\pm, \\ (\epsilon_1 \partial_{x_1} \phi_{n+1})_- = \mathfrak{g}_{n+1}, \\ (\epsilon_1 \partial_{x_1} \phi_{n+1})_+ = \mathfrak{h}(\phi_n) + \mathfrak{g}_{n+1}, \\ [\![\partial_{x_2} \phi_{n+1}]\!]_{2\mathrm{D}} = 0 \end{cases}$$

iteratively for $n \geqslant 0$. The weak formulation of the problem is given by

$$\begin{cases} \int_{\mathbb{R}^2_+} \epsilon_1 \nabla \phi_{n+1} \cdot \nabla \eta \, \mathrm{d}\boldsymbol{x} + \int_\Gamma \mathfrak{g}_{n+1} \eta \, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^2_+} \mathfrak{f}(\phi_n) \eta \, \mathrm{d}\boldsymbol{x} - \int_\Gamma \mathfrak{h}(\phi_n) \eta \, \mathrm{d}\boldsymbol{x}, & \eta \in H^1(\mathbb{R}^2_+), \\ \int_{\mathbb{R}^2_-} \epsilon_1 \nabla \phi_{n+1} \cdot \nabla \eta \, \mathrm{d}\boldsymbol{x} - \int_\Gamma \mathfrak{g}_{n+1} \eta \, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^2_-} \mathfrak{f}(\phi_n) \eta \, \mathrm{d}\boldsymbol{x}, & \eta \in H^1(\mathbb{R}^2_-), \\ [\![\partial_{x_2} \phi_{n+1}]\!]_{2\mathrm{D}} = 0. \end{cases} \tag{7.2.13}$$

To solve (7.2.13) numerically, we use the finite element method, see e.g. [3, 39].
First, we replace $\mathbb{R}^2_\pm$ and Γ_2 by suitable bounded domains $\Omega_\pm \subset \overline{\mathbb{R}^2_\pm}$, $\tilde{\Gamma} := \overline{\Omega}_+ \cap \overline{\Omega}_- \subset \Gamma_2$, respectively, and add homogeneous Neumann boundary conditions at $\partial\Omega_\pm \backslash \tilde{\Gamma}$.
Furthermore, we substitute $H^1\left(\mathbb{R}^2_\pm\right)$ with the following N-dimensional subspaces $\mathcal{V}_\pm := \operatorname{span}\{\eta_k^\pm \mid k \in \{1, \ldots, N\}\}$, $N \in \mathbb{N}$, where the shape functions $\eta_k^\pm \in H^1(\Omega_\pm)$ are the standard piecewise linear hat functions, which are linearly independent. Then we look for solutions of the form

$$\phi_{n+1}(\boldsymbol{x}) = \begin{cases} \sum_{k=1}^N \Phi^+_{n+1,k} \, \eta_k^+(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_+, \\ \sum_{k=1}^N \Phi^-_{n+1,k} \, \eta_k^-(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_-, \end{cases}$$

$$\mathfrak{g}_{n+1}(\boldsymbol{x}) = \sum_{k=1}^N G_{n+1,k} \, \eta_k^+(\boldsymbol{x})|_{\tilde{\Gamma}},$$

where the coefficients $\Phi^{\pm}_{n+1,k}, G_{n+1,k} \in \mathbb{R}$ are the solutions of the following system:

$$\begin{cases} \sum_{k=1}^{N} \left(\Phi^{+}_{n+1,k} \int_{\Omega_+} \epsilon_1 \nabla \eta^{+}_{j} \cdot \nabla \eta^{+}_{k} \, \mathrm{d}\boldsymbol{x} + G_{n+1,k} \int_{\widetilde{\Gamma}} \eta^{+}_{j} \eta^{+}_{k} \, \mathrm{d}\boldsymbol{x} \right) \\ \qquad = \int_{\Omega_+} \mathfrak{f}(\phi_n) \eta^{+}_{j} \, \mathrm{d}\boldsymbol{x} - \int_{\widetilde{\Gamma}} \mathfrak{h}(\phi_n) \eta^{+}_{j} \, \mathrm{d}\boldsymbol{x}, & j \in \{1, \ldots, N\}, \\ \sum_{k=1}^{N} \left(\Phi^{-}_{n+1,k} \int_{\Omega_-} \epsilon_1 \nabla \eta^{-}_{j} \cdot \nabla \eta^{-}_{k} \, \mathrm{d}\boldsymbol{x} - G_{n+1,k} \int_{\widetilde{\Gamma}} \eta^{-}_{j} \eta^{+}_{k} \, \mathrm{d}\boldsymbol{x} \right) \\ \qquad = \int_{\Omega_-} \mathfrak{f}(\phi_n) \eta^{-}_{j} \, \mathrm{d}\boldsymbol{x}, & j \in \{1, \ldots, N\}, \\ \sum_{k=1}^{N} \Phi^{-}_{n+1,k} \partial_{x_2} \eta^{-}_{k} |_{\widetilde{\Gamma}} - \sum_{j=1}^{N} \Phi^{+}_{n+1,j} \partial_{x_2} \eta^{+}_{j} |_{\widetilde{\Gamma}} = 0. \end{cases} \tag{7.2.14}$$

Remark 7.2.5

Note that only $N_\Gamma < N$ shape functions η^{+}_{k} are not trivial on $\widetilde{\Gamma}$ and that the discretization divides $\widetilde{\Gamma}$ in $(N_\Gamma - 1)$ line segments. From the last equation in (7.2.14) *we therefore get $(N_\Gamma - 1)$ equations we have to satisfy. The extra degree of freedom left will be used below to satisfy a compatibility condition. Due to the Neumann boundary conditions and the fact that $\mathfrak{f}$ and $\mathfrak{h}$ depend only on the gradient of ϕ, the solution is unique only up to an additive constant. To get uniqueness we additionally demand $\int_{\Omega_\pm} \phi_{n+1} \, \mathrm{d}\boldsymbol{x} = 0$. Therefore, we extend the finite element formulation with*

$$\sum_{k=1}^{N} \Phi^{\pm}_{n+1,k} \int_{\Omega_\pm} \eta^{\pm}_{k} \, \mathrm{d}\boldsymbol{x} = 0.$$

In the implementation we will incorporate these zero mean value conditions with the help of Lagrange multipliers. This increases the discretization matrix by two rows and columns, see e.g. [48, Chapter 4.8].

Additionally, note that $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ have to satisfy certain compatibility conditions. Indeed, Gauss's theorem implies that

$$\int_{\Omega_\pm} \mathfrak{f}^{\pm}(\phi^{\pm}_{n}) \, \mathrm{d}\boldsymbol{x} = -\int_{\Omega_\pm} \nabla \cdot \left(\epsilon^{\pm}_{1} \nabla \phi^{\pm}_{n+1} \right) \, \mathrm{d}\boldsymbol{x} = \pm \int_{\widetilde{\Gamma}} (\epsilon_1 \partial_{x_1} \phi_{n+1})_{\pm} \, \mathrm{d}\boldsymbol{x}$$

and hence

$$\int_{\Omega_+} \mathfrak{f}^{+}(\phi^{+}_{n}) \, \mathrm{d}\boldsymbol{x} = \int_{\widetilde{\Gamma}} (\mathfrak{h}(\phi_n) + \mathfrak{g}_{n+1}) \, \mathrm{d}\boldsymbol{x}, \tag{7.2.15}$$

$$\int_{\Omega_-} \mathfrak{f}^{-}(\phi^{-}_{n}) \, \mathrm{d}\boldsymbol{x} = -\int_{\widetilde{\Gamma}} \mathfrak{g}_{n+1} \, \mathrm{d}\boldsymbol{x}. \tag{7.2.16}$$

We now use the structure of $\mathfrak{f}$ *and Gauss's theorem to show that*

$$\begin{aligned}\int_{\Omega_-} \mathfrak{f}^-(\phi_n^-)\,\mathrm{d}\boldsymbol{x} + \int_{\Omega_+} \mathfrak{f}^+(\phi_n^+)\,\mathrm{d}\boldsymbol{x} &= \int_{\tilde{\Gamma}} \left(\epsilon_1 U_{0,1} + \epsilon_3 |\boldsymbol{U}_0 + \nabla\phi|^2 (U_{0,1} + \partial_{x_1}\phi)\right)_- \mathrm{d}\boldsymbol{x} \\ &\quad - \int_{\tilde{\Gamma}} \left(\epsilon_1 U_{0,1} + \epsilon_3 |\boldsymbol{U}_0 + \nabla\phi|^2 (U_{0,1} + \partial_{x_1}\phi)\right)_+ \mathrm{d}\boldsymbol{x} \\ &= \int_{\tilde{\Gamma}} \mathfrak{h}(\phi_n)\,\mathrm{d}\boldsymbol{x},\end{aligned}$$

where in the last step we used that $[\![\epsilon_1 U_{0,1}]\!]_{2D} = [\![\epsilon_1 U_{\mathrm{ans},1}]\!]_{2D} = 0$*. Therefore, the sum of* (7.2.15) *and* (7.2.16) *is satisfied. To fulfill both conditions we have to add an additional condition to our finite element method, e.g.*

$$\sum_k G_{n+1,k} \int_{\tilde{\Gamma}} \eta_k^+(x)\,\mathrm{d}\boldsymbol{x} = -\int_{\Omega_-} \mathfrak{f}^-(\phi_n^-)\,\mathrm{d}\boldsymbol{x}.$$

Note that we still have one degree of freedom left in our finite element formulation to satisfy this condition.

Let

$$\left(\Phi^-_{n+1,1}, \ldots, \Phi^-_{n+1,N}, \Phi^+_{n+1,1}, \ldots, \Phi^+_{n+1,N}, \lambda_1, \lambda_2, G_{n+1,1}, \ldots, G_{n+1,N}\right)^\top$$

be the vector of the unknowns, where λ_1, λ_2 *are the mentioned Lagrange multipliers. The structure of the discretization matrix of the finite element method can then be seen in Figure 7.1. From the top left to the bottom right we see the discretization of the differential equations, the zero mean value condition, the interface condition and finally the compatibility condition.*

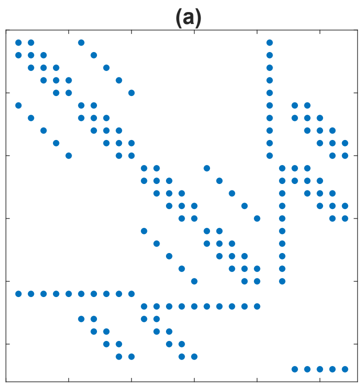

Figure 7.1.: (a) Schematic structure of the discretization matrix of the finite element method.

Let us now consider two examples. First, let us study an example where all the assumptions of Section 7.2.1 are satisfied. This example was also studied in [25].

Example 7.2.6 (Numerical Calculation of $\nabla\phi$ (positive ϵ_3))
Let $\epsilon_1(x_1) = 1\chi_{\mathbb{R}_-} + (1+\mathrm{e}^{-x_1})\chi_{\mathbb{R}_+}$ as in Example 3.4.1. We now want to select values for $(k_0, \omega(k_0))$ such that the eigenfunction has a strong localization. We select $k_0 \approx 3.435$ and $\omega(k_0) = 3$ and the corresponding eigenfunction $\boldsymbol{m} := \boldsymbol{w}(k_0)$. For the nonlinearity we choose $\epsilon_3 \equiv 1$ positive and select a strongly localized initial value for the envelope $A^{(0)}(x_2) = \mathrm{e}^{-5\cdot 10^6 \varepsilon^2 x_2^2}$. For the discretization we select $\overline{\Omega}_- = [-6,0]\times[-6,6]$ and $\overline{\Omega}_+ = [0,6]\times[-6,6]$ and choose a regular triangulation of step size h together with standard hat functions for $\eta_k^\pm$.
For the fixed-point iteration we start with $\phi_0 \equiv 0$ as an initial guess. Let us first check the convergence of the discretization in h and in the iteration n. For Figure 7.2 (a) we fixed $\varepsilon = 3\cdot 10^{-4}$ and calculated $\left\|\nabla\cdot\boldsymbol{\mathcal{D}}\left(\boldsymbol{U}_E^{(0)}\right)\right\|_{L^2(\mathbb{R}^2)}$ for different step sizes h ranging from 0.25 to 0.005. For Figure 7.2 (b) we also fixed $h = 0.005$ and calculated the L^2-norm of the residual

$$\mathrm{Res}_n := -\nabla\cdot(\epsilon_1\nabla\phi_n) - \mathfrak{f}(\phi_n)$$

in each step of the fixed-point iteration. We see the numerical convergence in both plots.

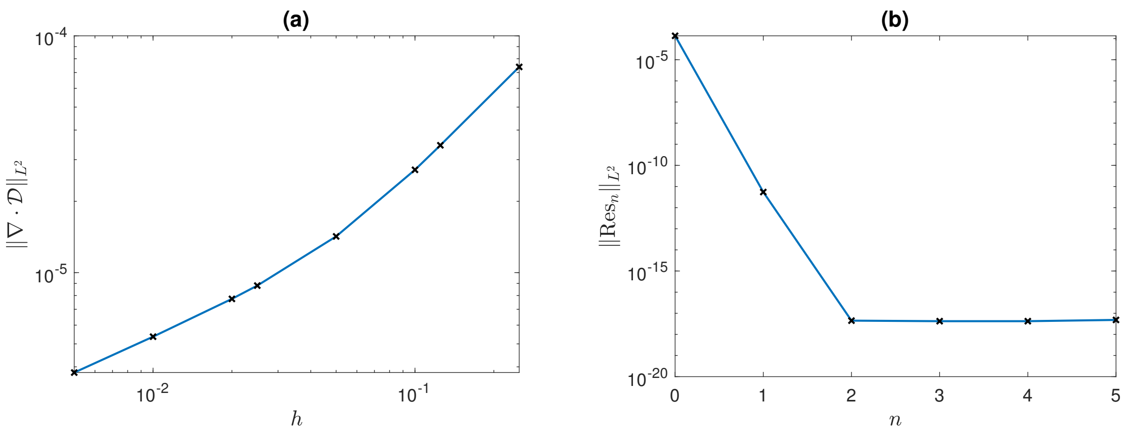

Figure 7.2.: Example 7.2.6: (a) $\left\|\nabla\cdot\boldsymbol{\mathcal{D}}\left(\boldsymbol{U}_E^{(0)}\right)\right\|_{L^2(\mathbb{R}^2)}$ in dependence on the step size h.
(b) $\|\mathrm{Res}_n\|_{L^2(\mathbb{R}^2)}$ for the first five steps of the iteration.

Finally, we study the ε-convergence of $\|\nabla\phi\|_{L^2(\mathbb{R}^2)^2}$. For the fixed step size $h = 0.005$ and ε ranging from 10^{-4} to 10^{-3} we obtain the convergence rate $\frac{3}{2}$, see Figure 7.3 (a).

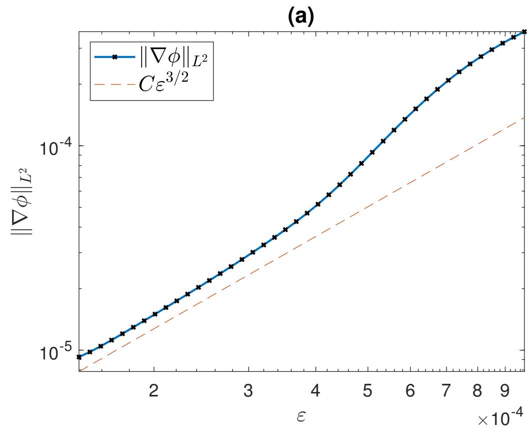

Figure 7.3.: Example 7.2.6: (a) $\|\nabla\phi\|_{L^2(\mathbb{R}^2)^2}$ in dependence of ε.

For a second example we use the same setting as in Example 3.4.1 and Example 6.3.3.

Example 7.2.7 (Numerical Calculation of $\nabla\phi$ (negative ϵ_3))
Let again $\epsilon_1(x_1) = 1\chi_{\mathbb{R}_-} + (1 + \mathrm{e}^{-x_1})\chi_{\mathbb{R}_+}$. *For* $k_0 = 0.5$ *we calculate* $\boldsymbol{m}$, *see Figure 3.2, choose* $\epsilon_3 \equiv -1$ *and set* $A^{(0)}(\varepsilon x_2)$ *as in Example 6.3.3. For the discretization we select* $\overline{\Omega}_- = [-100, 0] \times [-400, 400]$ *and* $\overline{\Omega}_+ = [0, 100] \times [-400, 400]$ *and choose a regular triangulation of step size* $h = 0.1$.
We can now study the ε*-convergence of* $\|\nabla\phi\|_{L^2(\mathbb{R}^2)^2}$ *numerically. For* ε *ranging from* 0.004 *to* 0.1 *we again obtain the desired rate of convergence, see Figure 7.4. Note that this example did not satisfy Assumption* (A6*)*, nevertheless the numerical convergence rate is* $\frac{3}{2}$.

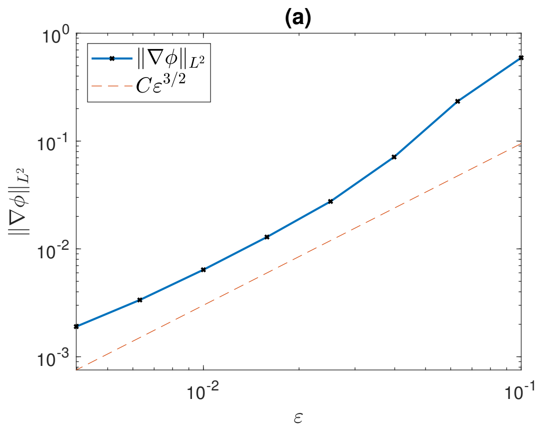

Figure 7.4.: Example 7.2.7: (a) $\|\nabla\phi\|_{L^2(\mathbb{R}^2)^2}$ in dependence of ε.

Finally, for $\varepsilon = 0.01$ *Figures 7.5* (a) *and* (b) *show the first components of the computed solutions* $\nabla\phi$ *and* $\boldsymbol{U}^{(0)}$*, respectively. To calculate an approximative solution of* (7.2.14) *we used the generalized minimum residual method in form of the Matlab function "gmrs".*

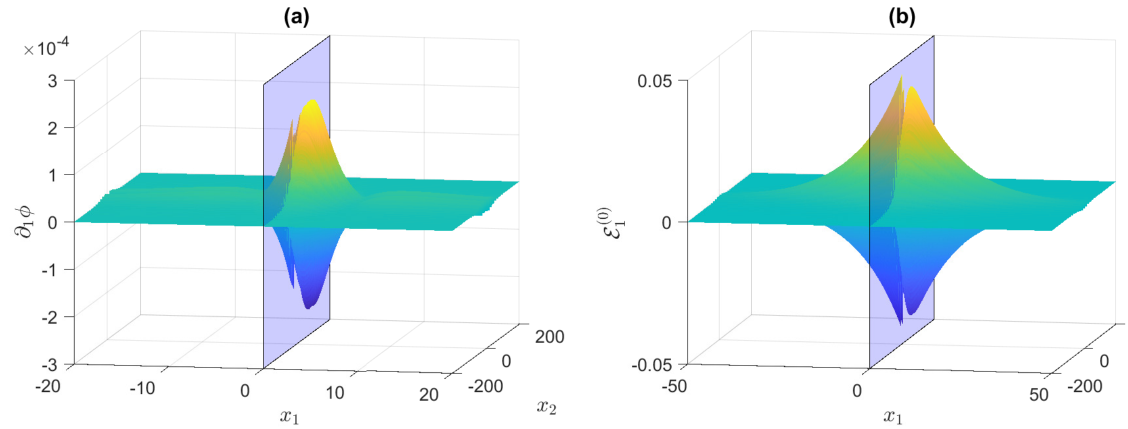

Figure 7.5.: Example 7.2.7: (a) $\partial_{x_1}\phi$ for $\varepsilon = 0.01$. (b) $U_1^{(0)} = U_{\text{ans},1}^{(0)} + \partial_{x_1}\phi$ for $\varepsilon = 0.01$. Note the different scales in (a) and (b).

8. Conclusion and Outlook

Let us summarize the results of this thesis and give an outlook on possible future research topics.

We studied the two-dimensional Maxwell problem with Kerr nonlinearity for transverse magnetic modes at the interface of two homogeneous dielectrics with instantaneous material response and formally constructed an asymptotic wave packet solution $\boldsymbol{U}_{\text{ans}}$ with the help of the method of amplitude equations. Our analysis showed that the corresponding amplitude equation is given by a nonlinear Schrödinger equation.
In Theorem 6.3.1 we rigorously proved the approximation properties of $\boldsymbol{U}_{\text{ans}}$. We showed that under certain assumption on the initial values $\boldsymbol{U}^{(0)}$, there exists a solution $\boldsymbol{U}$ of the Maxwell problem (4.0.2) – (4.0.6) such that

$$\|\boldsymbol{U} - \boldsymbol{U}_{\text{ans}}\|_{\mathcal{G}^3(\mathbb{R}^2 \times (0, T_0\varepsilon^{-2}))^3} \leqslant C\varepsilon^{\frac{3}{2}-\delta}.$$

To this end, we extended an existing local existence result for Maxwell's equations to the long time interval $[0, T_0\varepsilon^{-2}]$, by employing an involved bootstrapping argument for small initial data.
Additionally, we developed techniques to study the linear Maxwell problem analytically and numerically and provided first ideas to study the open problem of finding suitable initial values $\boldsymbol{U}^{(0)}$.
To our knowledge this is the first time that approximative solutions for the time dependent, multidimensional Maxwell problem with Kerr nonlinearity were rigorously studied and no reduction to the simpler scalar or time-harmonic setting was deployed.

In future research the open questions presented in Chapter 7 are of utmost interest. Here new techniques are necessary to improve the results of Section 7.2 and to involve the compatibility conditions of higher order.
From a numerical point of view, a validation of the approximation result would be of interest. Here a numerical solution of the Maxwell problem has to be calculated, which can be rather difficult. One promising method for quasilinear wave-type equations involving a discontinuous Galerkin method and a leapfrog scheme is presented in [50].
An adaptation of the presented methods to different Maxwell problems is also intriguing. Here one could study different geometries, e.g. Maxwell's equations in three dimensions, curved interfaces or multiple layers of different materials, or study different models for the

displacement field as discussed in Remark 2.2.1. Adapting the presented proof to these new formulations of the Maxwell problem will be challenging. Our analytical and numerical methods for the analysis of the linear Maxwell problem are heavily dependent on properties of ϵ_1 and it would not be trivial to transfer them to different settings. The derivation of the amplitude equation and the construction of the extended ansatz $\boldsymbol{U}_{\text{ext}}$ can also be challenging. For displacement fields with complex valued material functions ϵ_1 or displacement fields that involve convolutions in time, e.g.

$$\widetilde{\boldsymbol{\mathcal{D}}}(\boldsymbol{x},t) = \epsilon_0 \left(\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) + \int_{-\infty}^{\infty} \widetilde{\chi}_1(\boldsymbol{x},t-s)\boldsymbol{\mathcal{E}}(\boldsymbol{x},s)\,\mathrm{d}s + \widetilde{\chi}_3(\boldsymbol{x})(\boldsymbol{\mathcal{E}}(\boldsymbol{x},t)\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{x},t))\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) \right), \quad (8.0.1)$$

one also expects to derive the complex Ginzburg-Landau equation instead of the nonlinear Schrödinger equation as the amplitude equation, see [72] and Appendix A. For the rigorous analysis a well-posedness and local existence result is necessary. For different settings, such as (8.0.1), the local existence theory of Chapter 5 may not be applicable and new results are necessary.

A. Derivation of the Complex Ginzburg-Landau Equation

In this chapter we will formally derive the complex Ginzburg-Landau equation as an amplitude equation for the approximative solution of Maxwell's equations with a displacement field that is non-local in time. Many calculations will be similar to the ones in Chapter 4 and we will therefore only illustrate the main differences. The complex Ginzburg-Landau equation can be derived as an amplitude equation in various different problems, see e.g. [36, 55, 72, 52, 71].

We want to study the two-dimensional Maxwell problem with an interface (2.2.10), (2.2.11) but with the displacement field

$$\boldsymbol{\mathcal{D}}(\boldsymbol{x},t) = \epsilon_0 \left(\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) + \int_{-\infty}^{\infty} \chi_1(\boldsymbol{x}, t-s)\boldsymbol{\mathcal{E}}(\boldsymbol{x},s)\,\mathrm{d}s + \chi_3(\boldsymbol{x},t)(\boldsymbol{\mathcal{E}}(\boldsymbol{x},t)\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{x},t))\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) \right)$$

and the discontinuous susceptibilities

$$\chi_1(\boldsymbol{x},t) = \begin{cases} \chi_1^-(t), & x_1 < 0, \\ \chi_1^+(t), & x_1 > 0, \end{cases} \qquad \chi_3(\boldsymbol{x},t) = \begin{cases} \chi_3^-, & x_1 < 0, \\ \chi_3^+, & x_1 > 0, \end{cases}$$

where $\chi_1^\pm : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ are sufficiently smooth and $\chi_3^\pm \in \mathbb{R}$. Due to causality we also assume $\chi_1(t) = 0$ for $t < 0$.

Remark A.1
For surface plasmon polaritons the interface between a metal and a dielectric material is of interest. To model the properties of the metal one often uses the Drude model

$$\widehat{\chi}(\omega) = -\frac{\omega_p^2}{\omega^2 + \mathrm{i}\gamma\omega},$$

where $\omega_p, \gamma \in \mathbb{R}$ and $\widehat{\chi}_1$ is the Fourier transform of χ_1.
For some types of materials other models may be appropriate. A discussion of other models can be found in [65, 72, 61], e.g. the Lorentz model

$$\widehat{\chi}(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 + \mathrm{i}\gamma\omega},$$

where $\omega_0, \omega_p, \gamma \in \mathbb{R}$.
Note that for the Drude and the Lorentz model $\chi(t) = 0$ *for* $t < 0$, $\chi(t)$ *is exponentially decaying for* $t > 0$ *and* $\overline{\widehat{\chi}_1(-\omega)} = \widehat{\chi}(\overline{\omega})$. *These properties will be necessary for the following analysis.*

We again start with the analysis of the linear problem. With the ansatz

$$\mathcal{E}(x_1, x_2, t) = \begin{pmatrix} \phi_1(x_1, k, \omega) \\ \phi_2(x_1, k, \omega) \\ \phi_3(x_1, k, \omega) \end{pmatrix} \mathrm{e}^{\mathrm{i}(kx_2 - \omega t)} + \text{c.c.}, \quad \mathcal{H}(x_1, x_2, t) = \begin{pmatrix} \psi_1(x_1, k, \omega) \\ \psi_2(x_1, k, \omega) \\ \psi_3(x_1, k, \omega) \end{pmatrix} \mathrm{e}^{\mathrm{i}(kx_2 - \omega t)} + \text{c.c.}$$

for some $k \in \mathbb{R}$ and $\omega \in \mathbb{C}$ we get for the linear part of the displacement field

$$\begin{aligned} \mathcal{D}_{\text{lin}}(\boldsymbol{x}, t) &= \epsilon_0 \left(\mathcal{E}(\boldsymbol{x}, t) + \int_{-\infty}^{\infty} \chi_1(\boldsymbol{x}, s) \mathcal{E}(\boldsymbol{x}, t - s) \, \mathrm{d}s \right) \\ &= \epsilon_0 \left(1 + \sqrt{2\pi} \, \widehat{\chi}_1(\boldsymbol{x}, -\omega) \right) \boldsymbol{\phi}(\boldsymbol{x}, t) \mathrm{e}^{\mathrm{i}(kx_2 - \omega t)} + \epsilon_0 \left(1 + \sqrt{2\pi} \, \widehat{\chi}_1(\boldsymbol{x}, \overline{\omega}) \right) \overline{\boldsymbol{\phi}}(\boldsymbol{x}, t) \mathrm{e}^{-\mathrm{i}(kx_2 - \overline{\omega} t)} \\ &= \epsilon_0 \left(1 + \sqrt{2\pi} \, \widehat{\chi}_1(\boldsymbol{x}, -\omega) \right) \mathcal{E}(\boldsymbol{x}, t), \end{aligned}$$

where we used that $\overline{\widehat{\chi}_1(-\omega)} = \widehat{\chi}(\overline{\omega})$. Note that

$$\epsilon_1(x_1, \omega) := \begin{cases} \epsilon_1^-(\omega) = \epsilon_0 \left(1 + \sqrt{2\pi} \, \widehat{\chi}_1^-(-\omega) \right), & x_1 < 0, \\ \epsilon_1^+(\omega) = \epsilon_0 \left(1 + \sqrt{2\pi} \, \widehat{\chi}_1^+(-\omega) \right), & x_1 > 0 \end{cases}$$

is constant in x_1 on both sides of the interface. We can therefore repeat the same calculations as in Example 3.1.4 and get explicit formulas for the functions ϕ_j, ψ_j, see (3.1.4), where the dispersion relation

$$k^2 = \omega^2 \mu_0 \frac{\epsilon_1^+(\omega) \epsilon_1^-(\omega)}{\epsilon_1^+(\omega) + \epsilon_1^-(\omega)}, \tag{A.1}$$

and the condition

$$\operatorname{Re} \left(-\mathrm{i} \sqrt{\mu_0 \epsilon_1^{\pm}(\omega) \omega^2 - k^2} \right) > 0 \tag{A.2}$$

have to be satisfied, such that non-trivial, integrable solutions, that satisfy the interface conditions, can exist. Note that only TM-modes are possible and that we combine the three non-trivial components in one vector $\boldsymbol{w}(k) := (\phi_1(k), \phi_2(k), \psi_3(k))^\top$. In contrast to the main part of this thesis we now have complex valued functions $\omega(k)$, $\boldsymbol{w}(k)$ and $\epsilon_1(\omega)$.
The linear eigenvalue problem is similar to (3.1.10) given by

$$L(k) \boldsymbol{w}(x_1) + \omega \Lambda(\omega) \boldsymbol{w}(x_1) = \boldsymbol{0}, \qquad x_1 \in \mathbb{R} \backslash \{0\}, \tag{A.3}$$

with

$$L(k)\boldsymbol{w} := \begin{pmatrix} kw_3 \\ \mathrm{i}\partial_{x_1} w_3 \\ kw_1 + \mathrm{i}\partial_{x_1} w_2 \end{pmatrix}, \qquad \Lambda(\omega)\boldsymbol{w} := \begin{pmatrix} \epsilon_1(x_1,\omega)w_1 \\ \epsilon_1(x_1,\omega)w_2 \\ \mu_0 w_3 \end{pmatrix},$$

and $\mathrm{D}\left(L(k)\right)$, $\mathrm{D}\left(\Lambda(\omega)\right)$ as in (3.2.1). Note that the operator Λ is now dependent on ω and that $L(k) + \omega\Lambda(\omega)$ is no longer self-adjoint, since $\epsilon_1(\omega), \omega$ are complex.
The adjoint problem is given by

$$\left(L(k) + \omega\Lambda(\omega)\right)^* \boldsymbol{v}(x_1) = L(k)\boldsymbol{v}(x_1) + \overline{\omega}\overline{\Lambda}(\omega)\boldsymbol{v}(x_1) = \mathbf{0}, \qquad x_1 \in \mathbb{R}\backslash\{0\}.$$

For a solution $\boldsymbol{v}$ of the adjoint problem it is easy to see that $\boldsymbol{w} = (\overline{v}_1, -\overline{v}_2, \overline{v}_3)^\top$ is a solution of (A.3). Since solutions of (A.3) can only exist when (A.1) and (A.2) are satisfied, the same must be true for the adjoint problem.
Let us now study the inhomogeneous problem

$$L(k)\boldsymbol{w}(x_1) + \omega\Lambda(\omega)\boldsymbol{w}(x_1) = \boldsymbol{f}, \qquad x_1 \in \mathbb{R}\backslash\{0\}.$$

As in Section 3.3 we can use the exponential dichotomy to show that $L(k) + \omega\Lambda(\omega)$ is a Fredholm operator when (A.2) is satisfied. Now the closed range theorem implies that

$$\mathrm{R}\left(L(k) + \omega\Lambda(\omega)\right) = \mathrm{N}\left(\left(L(k) + \omega\Lambda(\omega)\right)^*\right)^\perp. \tag{A.4}$$

Note that we can use the variation of constants formula to explicitly calculate solutions of the inhomogeneous problem.
Let us now construct the asymptotic solution. We fix $k = k_0 > 0$ and determine $\omega(k_0)$ with the help of (A.1). Similar to [72] we thereby assume that k_0 can be chosen such that for a small $\varepsilon > 0$ and a constant $\alpha \in \mathbb{R}$ we have

$$\mathrm{Im}\left(\omega(k_0)\right) = \alpha\varepsilon^2, \quad \mathrm{Im}\left(\partial_k\omega(k_0)\right) = 0.$$

Remark A.2
This form of $\omega(k_0)$ will allow us to derive the complex Ginzburg-Landau equation. Typically, one chooses $\alpha < 0$, which results in a damping effect. Note that the asymptotic parameter ε is coupled to the eigenvalue $\omega(k_0)$. We therefore need a physical setting that allows for small enough ε such that the asymptotic analysis is meaningful.

For $\nu_0 := \mathrm{Re}\left(\omega(k_0)\right)$ we additionally assume that

$$\begin{aligned} &\text{(A.1) does not hold for } (k,\omega) = (3k_0, 3\nu_0), \\ &\text{(A.2) holds for } (k,\omega) = (k_0, \omega(k_0)) \text{ and } (k,\omega) = (3k_0, 3\nu_0). \end{aligned} \tag{A.5}$$

Note that for small enough ε condition (A.2) also holds for $(k,\omega) = (3k_0, \omega(3k_0))$.
These assumptions give us the existence of an eigenfunction $\boldsymbol{m}(x_1) := \boldsymbol{w}(x_1)$ and are needed

for the treatment of the higher harmonics later on. We then set $\nu_1 := \partial_k \omega(k_0)$, $\nu_2 := \partial_k^2 \omega(k_0)$ and make the ansatz

$$\boldsymbol{U}_{\text{ans}}(\boldsymbol{x},t) := \begin{pmatrix} \mathcal{E}_{\text{ans},1}(\boldsymbol{x},t) \\ \mathcal{E}_{\text{ans},2}(\boldsymbol{x},t) \\ \mathcal{H}_{\text{ans},3}(\boldsymbol{x},t) \end{pmatrix} := \varepsilon A(\varepsilon(x_2-\nu_1 t), \varepsilon^2 t)\boldsymbol{m}(x_1)F_1 + \text{c.c.}, \tag{A.6}$$

where as before $F_1 := \mathrm{e}^{\mathrm{i}(k_0 x_2 - \nu_0 t)}$ and $A = A(X_2, T)$ is a complex envelope with the variables $X_2 := \varepsilon(x_2 - \nu_1 t)$ and $T := \varepsilon^2 t$.
We now apply the Fourier transform between x_2 and k and get analogously to Section 4.1.1

$$\widehat{\boldsymbol{U}}_{\text{ans}}(x_1,k,t) := \begin{pmatrix} \widehat{\mathcal{E}}_{\text{ans},1}(x_1,k,t) \\ \widehat{\mathcal{E}}_{\text{ans},2}(x_1,k,t) \\ \widehat{\mathcal{H}}_{\text{ans},3}(x_1,k,t) \end{pmatrix} := \widehat{A}(K,T)\boldsymbol{m}(x_1)E_1 + \widehat{\text{c.c.}},$$

with $K := \frac{k-k_0}{\varepsilon}$, $E_1 := \mathrm{e}^{-\mathrm{i}(\nu_0 + (k-k_0)\nu_1)t}$.
We can now proceed as in Section 4.1.1 by taking the Taylor expansions of $\omega(k)$, $\boldsymbol{w}(k)$ and also $\widehat{\chi}_1(-\omega(k))$:

$$\begin{aligned}
\omega(k) &= \omega(k_0 + \varepsilon K) = \nu_0 + \varepsilon K \nu_1 + \frac{1}{2}\varepsilon^2 K^2 \nu_2 + \varepsilon^2 \mathrm{i}\alpha + \mathcal{O}(\varepsilon^3), \\
\boldsymbol{w}(k) &= \boldsymbol{w}(k_0 + \varepsilon K) = \boldsymbol{m} + \varepsilon K \partial_k \boldsymbol{w}(k_0) + \frac{1}{2}\varepsilon^2 K^2 \partial_k^2 \boldsymbol{w}(k_0) + \mathcal{O}(\varepsilon^3), \\
\widehat{\chi}_1(-\omega(k_0)) &= \widehat{\chi}_1(-\nu_0 - \varepsilon^2 \mathrm{i}\alpha) = \widehat{\chi}_1(-\nu_0) - \varepsilon^2 \mathrm{i}\alpha \partial_\omega \widehat{\chi}_1(-\nu_0) + \mathcal{O}(\varepsilon^3), \\
\widehat{\chi}_1(-\omega(k)) &= \widehat{\chi}_1(-\omega(k_0 + \varepsilon K)) \\
&= \widehat{\chi}_1(-\nu_0) - \varepsilon K \nu_1 \partial_\omega \widehat{\chi}_1(-\nu_0) - \varepsilon^2 \mathrm{i}\alpha \partial_\omega \widehat{\chi}_1(-\nu_0) \\
&\quad - \frac{1}{2}\varepsilon^2 K^2 \nu_2 \partial_\omega \widehat{\chi}_1(-\nu_0) + \frac{1}{2}\varepsilon^2 K^2 \nu_1^2 \partial_\omega^2 \widehat{\chi}_1(-\nu_0) + \mathcal{O}(\varepsilon^3).
\end{aligned}$$

Compared to Section 4.1.1 we note the additional term $\varepsilon^2 \mathrm{i}\alpha$ in the expansion of $\omega(k)$. The Taylor expansion of the operator $L(k)$ is the same as before, but for $\Lambda(\omega)$ we get the expansion

$$\Lambda(\omega) = \Lambda_0 + \varepsilon K \Lambda_1 + \frac{1}{2}\varepsilon^2 K^2 \Lambda_2 + \varepsilon^2 \mathcal{A} + \mathcal{O}(\varepsilon^3),$$

with the operators Λ_0, Λ_1, Λ_2, $\mathcal{A}$ defined as

$$\Lambda_0 \boldsymbol{m} := \begin{pmatrix} \epsilon_0 \left(1 + \sqrt{2\pi}\,\widehat{\chi}_1(-\nu_0)\right) m_1 \\ \epsilon_0 \left(1 + \sqrt{2\pi}\,\widehat{\chi}_1(-\nu_0)\right) m_2 \\ \mu_0 m_3 \end{pmatrix}, \qquad \Lambda_1 \boldsymbol{m} := -\epsilon_0 \sqrt{2\pi}\,\nu_1 \partial_\omega \widehat{\chi}_1(-\nu_0) \begin{pmatrix} m_1 \\ m_2 \\ 0 \end{pmatrix},$$

$$\Lambda_2 \boldsymbol{m} := \epsilon_0 \sqrt{2\pi} \left(-\nu_2 \partial_\omega \widehat{\chi}_1(-\nu_0) + \nu_1^2 \partial_\omega^2 \widehat{\chi}_1(-\nu_0)\right) \begin{pmatrix} m_1 \\ m_2 \\ 0 \end{pmatrix},$$

$$\mathcal{A} \boldsymbol{m} := -\epsilon_0 \sqrt{2\pi}\,\mathrm{i}\alpha \partial_\omega \widehat{\chi}_1(-\nu_0) \begin{pmatrix} m_1 \\ m_2 \\ 0 \end{pmatrix}.$$

From the Taylor expansion of (A.3) we therefore get the three equations

$$\mathbf{0} = (L_0 + \nu_0 \Lambda_0)\boldsymbol{m}, \tag{A.7}$$

$$\mathbf{0} = K(L_1 + \nu_0 \Lambda_1 + \nu_1 \Lambda_0)\boldsymbol{m} + K(L_0 + \nu_0 \Lambda_0)\partial_k \boldsymbol{w}(k_0), \tag{A.8}$$

$$\begin{aligned} \mathbf{0} &= \frac{1}{2} K^2 (\nu_0 \Lambda_2 + 2\nu_1 \Lambda_1 + \nu_2 \Lambda_0)\boldsymbol{m} + (\nu_0 \mathcal{A} + \mathrm{i}\alpha \Lambda_0)\boldsymbol{m} \\ &\quad + \frac{1}{2} K^2 (2L_1 + 2\nu_0 \Lambda_1 + 2\nu_1 \Lambda_0)\partial_k \boldsymbol{w}(k_0) + \frac{1}{2} K^2 (L_0 + \nu_0 \Lambda_0)\partial_k^2 \boldsymbol{w}(k_0). \end{aligned} \tag{A.9}$$

To get a formally small enough residual

$$\widehat{\mathbf{Res}}(\boldsymbol{U}_{\mathrm{ans}}) := \begin{pmatrix} \partial_t \widehat{\mathcal{D}}_1(\boldsymbol{U}_{\mathrm{ans},E}) - \mathrm{i}k \widehat{U}_{\mathrm{ans},3} \\ \partial_t \widehat{\mathcal{D}}_2(\boldsymbol{U}_{\mathrm{ans},E}) + \partial_{x_1} \widehat{U}_{\mathrm{ans},3} \\ -\mathrm{i}k \widehat{U}_{\mathrm{ans},1} + \partial_{x_1} \widehat{U}_{\mathrm{ans},2} + \mu_0 \partial_t \widehat{U}_{\mathrm{ans},3} \end{pmatrix}$$

we have to modify the ansatz as in Section 4.1.1:

$$\begin{aligned} \widehat{\boldsymbol{U}}_{\mathrm{mod}}(x_1, k, t) &:= \widehat{A}(K, T) \left(\boldsymbol{m}(x_1) + \varepsilon K \partial_k \boldsymbol{w}(x_1, k_0) + \frac{1}{2} \varepsilon^2 K^2 \partial_k^2 \boldsymbol{w}(x_1, k_0) \right) E_1 \\ &\quad + 2\pi \varepsilon^2 \left(\widehat{A} *_K \widehat{A} *_K \widehat{A} \right) (\widetilde{K}, T) \boldsymbol{h}(x_1) E_3 + \widehat{\mathrm{c.c.}}, \end{aligned}$$

where $E_3 := \mathrm{e}^{-\mathrm{i}(3\nu_0 + (k - 3k_0)\nu_1)t}$.

The only difference in the analysis of the residual $\widehat{\mathbf{Res}}(\boldsymbol{U}_{\mathrm{mod}})$ comes from the convolution term. We will treat this term with techniques described in [72]. First, we note that

$$\int_{-\infty}^{\infty} \chi_1(s) s \mathrm{e}^{\mathrm{i}\nu_0 s}\, \mathrm{d}s = -\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}\nu_0} \int_{-\infty}^{\infty} \chi_1(s) \mathrm{e}^{\mathrm{i}\nu_0 s}\, \mathrm{d}s = -\mathrm{i}\sqrt{2\pi}\, \frac{\mathrm{d}}{\mathrm{d}\nu_0}\, \widehat{\chi}_1(-\nu_0) = \mathrm{i}\sqrt{2\pi}\, \partial_\omega \widehat{\chi}_1(-\nu_0),$$

$$\int_{-\infty}^{\infty} \chi_1(s) s^2 \mathrm{e}^{\mathrm{i}\nu_0 s}\, \mathrm{d}s = -\frac{\mathrm{d}^2}{\mathrm{d}\nu_0^2} \int_{-\infty}^{\infty} \chi_1(s) \mathrm{e}^{\mathrm{i}\nu_0 s}\, \mathrm{d}s = -\sqrt{2\pi}\, \frac{\mathrm{d}^2}{\mathrm{d}\nu_0^2}\, \widehat{\chi}_1(-\nu_0) = -\sqrt{2\pi}\, \partial_\omega^2 \widehat{\chi}_1(-\nu_0).$$

Second, we use the Taylor expansion of $\widehat{A}(K, \varepsilon^2(t-s))\mathrm{e}^{\varepsilon \mathrm{i} K \nu_1 s}$ in s around the point $s = 0$:

$$\widehat{A}(K,T) + \varepsilon \mathrm{i} K \nu_1 s \widehat{A}(K,T) - \varepsilon^2 s \partial_T \widehat{A}(K,T) - \varepsilon^2 \frac{1}{2} K^2 \nu_1^2 s^2 \widehat{A}(K,T) + \mathcal{O}(\varepsilon^3).$$

With the above equations we conclude that

$$\begin{aligned}
&\mathcal{F}(\chi_1 *_t \varepsilon A F_1)(x_1,k,t) \\
&= (2\pi)^{-1/2} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_1(x_1,s) \varepsilon A \left(\varepsilon(x_2 - \nu_1(t-s)), \varepsilon^2(t-s)\right) \mathrm{e}^{\mathrm{i}(k_0 x_2 - \nu_0(t-s))} \mathrm{e}^{-\mathrm{i} k x_2} \, \mathrm{d}x_2 \mathrm{d}s \\
&= E_1 \int_{\mathbb{R}} \chi_1(s) \widehat{A}\left(K, \varepsilon^2(t-s)\right) \mathrm{e}^{\varepsilon \mathrm{i} K \nu_1 s} \mathrm{e}^{\mathrm{i}\nu_0 s} \, \mathrm{d}s \qquad (A.10) \\
&= E_1 \sqrt{2\pi} \left(\widehat{\chi}_1(-\nu_0) - \varepsilon K \nu_1 \partial_\omega \widehat{\chi}_1(-\nu_0) + \varepsilon^2 \frac{1}{2} K^2 \nu_1^2 \partial_\omega^2 \widehat{\chi}_1(-\nu_0) \right) \widehat{A} \\
&- \varepsilon^2 \mathrm{i} E_1 \sqrt{2\pi} \, \partial_\omega \widehat{\chi}_1(-\nu_0) \partial_T \widehat{A} + \mathcal{O}(\varepsilon^3).
\end{aligned}$$

Note that for exponentially decaying $\widehat{\chi}_1$ and bounded $\widehat{A}$ the integrand in the third line of (A.10) is localized around $s = 0$ and the Taylor expansion at $s = 0$ gives us a suitable approximation.

For the nonlinear correction term we proceed similarly. By using the Taylor expansion

$$\varepsilon^2 \left(\widehat{A} *_K \widehat{A} *_K \widehat{A} \right) \left(\widetilde{K}, \varepsilon^2(t-s) \right) \mathrm{e}^{\varepsilon \mathrm{i} \widetilde{K} \nu_1 s} = \varepsilon^2 \left(\widehat{A} *_K \widehat{A} *_K \widehat{A} \right) (\widetilde{K}, T) + \mathcal{O}(\varepsilon^3)$$

one derives

$$\begin{aligned}
&\mathcal{F}(\chi_1 *_t \varepsilon^3 A^3 F_1^3)(x_1,k,t) \\
&= 2\pi \varepsilon^2 E_3 \int_{\mathbb{R}} \chi_1(s) \left(\widehat{A} *_K \widehat{A} *_K \widehat{A} \right) \left(\widetilde{K}, \varepsilon^2(t-s) \right) \mathrm{e}^{\varepsilon \mathrm{i} \widetilde{K} \nu_1 s} \mathrm{e}^{\mathrm{i} 3 \nu_0 s} \, \mathrm{d}s \qquad (A.11) \\
&= (2\pi)^{3/2} \varepsilon^2 E_3 \widehat{\chi}_1(-3\nu_0) \left(\widehat{A} *_K \widehat{A} *_K \widehat{A} \right) (\widetilde{K}, T) + \mathcal{O}(\varepsilon^3).
\end{aligned}$$

Let us now study $\widehat{\mathbf{Res}}(\boldsymbol{U}_{\mathrm{mod}})$. In comparison to the calculations of Section 4.1.1 we now have to deal with the expansion of $\Lambda(\omega)$. Here we note that the terms of order ε^0 and ε^1 in the Taylor expansion of $\widehat{\chi}_1(-\omega(k))$ also appear in (A.10). By using (A.7) and (A.8) it is therefore easy to see that the terms of order ε^0 and ε^1 in the residual vanish.

Before we write down the residual to order ε^2 we remove the higher harmonics. Since $\epsilon_3 := \epsilon_0 \chi_3$ is a special case of the functions studied in Section 4.1.1, the right-hand side in the equation for $\boldsymbol{h}$ will be the same as before. With (A.11) we then get

$$(L(3k_0) + 3\nu_0 \Lambda(3\nu_0))\boldsymbol{h} = -3\nu_0 \epsilon_3 \begin{pmatrix} m_1^3 + m_1 m_2^2 \\ m_2^3 + m_2 m_1^2 \\ 0 \end{pmatrix}.$$

Assumption (A.5) and the analysis of the adjoint operator imply that

$$\mathrm{N}\left((L(3k_0)+3\nu_0\Lambda(3\nu_0))^*\right)=\mathrm{N}\left(L(3k_0)+3\nu_0\Lambda(3\nu_0)\right)=\{\mathbf{0}\}.$$

The existence of $\boldsymbol{h}$ then follows from (A.4).
The first component of $\widehat{\mathbf{Res}}(\boldsymbol{U}_{\mathrm{mod}})$ is given by

$$\begin{aligned}
&-\mathrm{i}\frac{1}{2}K^2E_1\widehat{A}(L_0+\nu_0+\Lambda_0)\partial_k^2 w_1(k_0)\\
&-\mathrm{i}\frac{1}{2}K^2E_1\widehat{A}(2L_1+2\nu_0\Lambda_1+2\nu_1\Lambda_0)\partial_k w_1(k_0)\\
&-\mathrm{i}\frac{1}{2}K^2E_1\widehat{A}\epsilon_0\left(\nu_0\nu_1^2\sqrt{2\pi}\,\partial_\omega^2\widehat{\chi}(-\nu_0)-2\sqrt{2\pi}\,\partial_\omega\nu_1^2\widehat{\chi}_1(-\nu_0)\right)m_1\\
&+E_1\partial_T\widehat{A}\epsilon_0\left(1+\sqrt{2\pi}\,\widehat{\chi}_1(-\nu_0)-\nu_0\sqrt{2\pi}\,\partial_\omega\widehat{\chi}_1(-\nu_0)\right)m_1\\
&-2\pi\mathrm{i}\epsilon_3\nu_0E_1\left(\widehat{A}*_K\widehat{\overline{A}}*_K\widehat{A}\right)\left(3|m_1|^2m_1+2|m_2|^2m_1+m_2^2\overline{m}_1\right)+\widehat{\mathrm{c.c.}}
\end{aligned}\tag{A.12}$$

The second component follows from (A.12) by switching the indices of $\partial_k^2 w_1(k_0)$, $\partial_k w_1(k_0)$, m_1, m_2 from 1 to 2 and vice versa. The third component is given by

$$\mu_0E_1\partial_T\widehat{A}m_3+\widehat{\mathrm{c.c.}}$$

We use (A.9) to rewrite these terms as

$$\begin{aligned}
G\widehat{A}:=&\,E_1\left(\partial_T\widehat{A}-\alpha\widehat{A}+\mathrm{i}\frac{1}{2}K^2\nu_2\widehat{A}\right)\begin{pmatrix}\left(\epsilon_0\left(1+\sqrt{2\pi}\,\widehat{\chi}_1(-\nu_0)\right)-\epsilon_0\nu_0\sqrt{2\pi}\,\partial_\omega\widehat{\chi}_1(-\nu_0)\right)m_1\\ \left(\epsilon_0\left(1+\sqrt{2\pi}\,\widehat{\chi}_1(-\nu_0)\right)-\epsilon_0\nu_0\sqrt{2\pi}\,\partial_\omega\widehat{\chi}_1(-\nu_0)\right)m_2\\ \mu_0m_3\end{pmatrix}\\
&-2\pi\mathrm{i}\epsilon_3\nu_0E_1\left(\widehat{A}*_K\widehat{\overline{A}}*_K\widehat{A}\right)\begin{pmatrix}3|m_1|^2m_1+2|m_2|^2m_1+m_2^2\overline{m}_1\\ 3|m_2|^2m_2+2|m_1|^2m_2+m_1^2\overline{m}_2\\ 0\end{pmatrix}.
\end{aligned}$$

We now select $\widehat{A}$ such that $G\widehat{A}$ is zero on a linear subspace. Inspired by the calculation of Section 4.1 we choose the subspace as

$$\mathrm{N}\left((L(k_0)+\nu_0\Lambda(\nu_0))^*\right)=\operatorname{span}\left\{\widetilde{\boldsymbol{m}}:=(\overline{m}_1,-\overline{m}_2,\overline{m}_3)^\top\right\}.$$

To determine the equation for $\widehat{A}$ we set

$$0=\langle G\widehat{A},\widetilde{\boldsymbol{m}}\rangle_{L^2(\mathbb{R})^3}=I_1\left(\mathrm{i}\partial_T\widehat{A}-\mathrm{i}\alpha\widehat{A}-\frac{1}{2}K^2\nu_2\widehat{A}\right)+2\pi I_2\left(\widehat{A}*_K\widehat{\overline{A}}*_K\widehat{A}\right),\tag{A.13}$$

where

$$I_1 := \int_{-\infty}^{\infty} \left(\left(\epsilon_0 \left(1 + \sqrt{2\pi}\,\widehat{\chi}_1(-\nu_0) \right) - \epsilon_0 \nu_0 \sqrt{2\pi}\,\partial_\omega \widehat{\chi}_1(-\nu_0) \right) \left(|m_1|^2 - |m_2|^2 \right) + \mu_0 |m_3|^2 \right) \, \mathrm{d}x_1,$$

$$I_2 := \nu_0 \int_{-\infty}^{\infty} \epsilon_3 \left(3|m_1|^4 - 3|m_2|^4 + m_2^2 \overline{m}_1^2 - m_1^2 \overline{m}_2^2 \right) \mathrm{d}x_1.$$

By applying the inverse Fourier transform to (A.13) we arrive at the following complex Ginzburg-Landau equation for A

$$\mathrm{i}\partial_T A = -\frac{1}{2}\nu_2 \partial_{X_2}^2 A + \mathrm{i}\alpha A - \frac{I_2}{I_1}|A|^2 A.$$

Remark A.3

As mentioned in Section 2.3, the formal derivation of an amplitude equation such that the residual is small is not enough to prove the approximation properties of $\boldsymbol{U}_{\mathrm{ans}}$. For the rigorous analysis new techniques are necessary since the results of Chapter 5 cannot be directly transferred to Maxwell's equations with a non-local displacement field.

List of Function Spaces

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Ich erkläre an Eides statt, dass ich die Arbeit mit dem Titel

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Publikationsliste

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